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MULTIPOINT BOUNDARY-VALUE PROBLEMS WITH IMPULSE EFFECTS

МНОГОТОЧКОВІ КРАЙОВІ ЗАДАЧІ З ІМПУЛЬСНОЮ ДІЄЮ

By using pseudo-inverse matrices, existence and uniqueness conditions are obtained for solutions of linear and weakly linear boundary-value problems for impulsive ordinary differential equations.

The case where the dimension of a differential system and the dimension of boundary conditions do not coincide is considered.

На основі псевдообернених матриць одержані умови існування та єдиності розв'язків лінійних та слабконелінійних крайових задач для звичайних диференціальних рівнянь з імпульсною дією. Розглянуто випадок, коли розмірність диференціальних систем і розмірність крайових умов не співпадають.

1. Introduction. We consider the linear matrix differential equation

$$\dot{x} = A(t)+f(t), \quad t\in [a,b], \quad t \neq v_j, \quad j = \overline{1,p},$$
$$v_j\in (a,b), \quad a = v_0 < v_1 < \ldots < v_p < v_{p+1} = b, \tag{1}$$

where $A(t)$ is an $(n\times n)$-matrix with elements continuous in $[a,b]$, $f(t)$ is a first-order discontinuous n-vector-function for $t=v_j$, $j=\overline{1,p}$, i.e.,

$$f(t) = f_j(t), \quad t\in \,]v_{j-1}, v_j], \quad f(a) = f_1(a), \quad j = \overline{1,p+1}.$$

We consider the problem of finding a first-order discontinuous n-vector-function $x(t)=x_j(t)$, $t\in\,]v_{j-1}, v_j]$ which is a solution of (1) and satisfies the boundary conditions

$$\sum_{j=1}^{p+1} l_j(x_j) = h, \tag{2}$$

where h is an arbitrary vector from R^m. The linear operators $l_j\in C([v_{j-1}, v_j];\ R^m)$ are represented by Stieltjes integrals

$$l_j(x_j) = \int\limits_{v_{j-1}}^{v_j} [dy_j(s)]\, C_j(s)x_j(s), \quad j = \overline{1,p+1}, \tag{3}$$

where $y_j(s) = \operatorname{diag}\left[y_{11}^j(s)\ldots y_{mm}^j(s)\right]$ are $(m\times m)$-diagonal matrices, the elements of which are functions of bounded variation in $[v_{j-1}, v_j]$. The elements of the $(m\times n)$-matrices $C_j(s)$ are continuous functions in $[v_{j-1}, v_j]$, $i = \overline{1,p+1}$.

In this paper, the existence and uniqueness of a solution of (1) – (3) and the corresponding perturbed problem are proved. The most currently used fixed point theorem in the theory of ordinary differential equations refers to a mapping F of a Banach space.

Multipoint boundary-value problems are discussed in [1] with difference boundary conditions, and in [3] from the viewpoint of the theory of generalized functions. In [4 – 6], by using a generalized Green's operator, system (1) with division boundary and impulsive conditions is considered. System (1), (2) and the perturbed problem are considered in [7] without an impulse effect in the boundary conditions.

2. The differential system. The general solution of (1) has the form

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$$x(t) = X(t)X^{-1}(s)x(s) + \int_s^t X(t)X^{-1}(u)f(u)\,du, \quad s, t \in [a, b], \tag{4}$$

where $X(t)$ is a fundamental matrix of solutions of $\dot{x} = A(t)x$ with $X(a) = E_n$ (E_n is the unit $(n \times n)$-matrix). But $f(t)$ is a discontinuous function. Solution (4) has the representation

$$x_j(t) = X(t)X^{-1}(v_{j-1})x_j(v_{j-1}) + \int_{v_{j-1}}^t X(t)X^{-1}(u)f_j(t)\,du,$$
$$t \in \,]v_{j-1}, v_j], \quad j = \overline{1, p+1}. \tag{5}$$

We introduce the following notation: the $(m \times (p+1)n)$-matrix $M = [M_1\ M_2 \ldots M_{p+1}]$, where

$$M_k = l_k(X) = \int_{v_{k-1}}^{v_k} [dy_k(s)]\, C_k(s)X(s), \quad k = \overline{1, p+1}$$

are $(m \times n)$-matrices; the matrices

$$B_j(s) = \int_s^{v_j} [dy_j(u)]\, C_j(u)X(u), \quad \bar{l}(f) = \sum_{j=1}^{p+1} \int_{v_{j-1}}^{v_j} B_j(s)X^{-1}(s)f_j(t)\,ds. \tag{6}$$

Substituting (5) in the boundary conditions (2), (3), we obtain an algebraic system with a solution $x_j(v_{j-1})$, $j = \overline{1, p+1}$,

$$M[X^{-1}(v_0)x_1(v_0)\ X^{-1}(v_1)x_2(v_1) \ldots X^{-1}(v_p)x_{p+1}(v_p)]^T = h - \bar{l}(f). \tag{7}$$

If M^- is an arbitrary $((p+1)n \times m)$-half-inverse matrix of M [2], then the square $((p+1)n \times (p+1)n)$-matrix $P_M = E - M^-M$ is a projector of $R^{(p+1)n}$ into the null space $N(M)$ of M, i.e. $P_M : R^{(p+1)n} \to N(M)$. We represent the projector matrix P_M in a hypermatrix form $P_M = [P_M^1\ P_M^2 \ldots P_M^{p+1}]^T$, where P_M^j, $j = \overline{1, p+1}$ are $(n \times (p+1)n)$-matrices.

System (7) has a solution if and only if

$$(E_m - MM^-)(h - \bar{l}(f)) = 0. \tag{8}$$

The general solution of (7) is

$$x_j(v_{j-1}) = X(v_{j-1})P_M^j w + X(v_{j-1})N_j(h - \bar{l}(f)), \tag{9}$$

where w is an arbitrary $(p+1)n \times 1$ vector, the elements of which are numbers and N_j, $j = \overline{1, p+1}$ are $(n \times m)$-matrices from the hypermatrix form of $M^- = [N_1 N_2 \ldots N_{p+1}]^T$.

Now we substitute (9) in (5) and obtain the general solution $x^0(t, w)$ of (1) – (3)

$$\begin{cases} x^0(t,w) = x_j^0(t,w),\ j = \overline{1, p+1},\ t \in \,]v_{j-1}, v_j], \\ x_j^0(t,w) = X(t)P_M^j w + X(t)N_j(h - \bar{l}(f)) + \int_{v_{j-1}}^{v_j} X(t)X^{-1}(u)f_j(u)\,du \end{cases}. \tag{10}$$

Theorem 1. *A necessary and sufficient condition in order that problem* (1) – (3) *have solutions is that* (8) *holds for any half-inverse matrix* M^-. *Solutions are given by* (10).

3. A perturbed system. Let Ω be the domain

$$\Omega \equiv \{(t, x, \varepsilon) \mid t \in [a, b], \ \|x\| \le \rho, \ \ \varepsilon \in [0, \varepsilon_0]\}.$$

We consider the perturbed system

$$\dot{x} = A(t)x + f(t) + \varepsilon g(t, x, \varepsilon), \quad t \ne v_i, \quad t \in [a, b] \tag{11}$$

with the boundary conditions (2), (3). Besides, $g(t, x, \varepsilon)$ is a function from Ω into R^n, which is first-order discontinuous for $t = v_i$, $i = \overline{1, p}$, continuous with respect to x, has a continuous derivative with respect to ε, and satisfies the Lipschitz condition

$$\| g(t, x, \varepsilon) - g(t, y, \varepsilon) \| \le L(\rho, \varepsilon_0) \| x - y \| \tag{12}$$

uniformly with respect to t, ε. The Lipschitz constant is a nonnegative, nondecreasing function of ρ and ε_0 such that $L(\rho, \varepsilon_0) \to 0$ as $\rho \to 0$, $\varepsilon_0 \to 0$.

We substitute $f + \varepsilon g$ for f in (6), (7) and, since $\bar{l}$ is a linear operator, we obtain

$$M [X^{-1}(v_0) x_1(v_0) \ X^{-1}(v_1) x_2(v_1) \dots X^{-1}(v_p) x_{p+1}(v_p)]^T = h - \bar{l}(f) - \varepsilon \bar{l}(g). \tag{13}$$

Depending on the dimension of the matrix M and rank M, we consider two cases.

3.1. $m = (p+1)n$, $\det M \ne 0$, $M^{-1} = [L_1 \, L_2 \dots L_{p+1}]^T$, L_i – $(n \times m)$-matrices. In this case, system (13) has a unique solution

$$x_j(v_{j-1}) = X(v_{j-1}) L_j (h - \bar{l}(f) - \varepsilon \bar{l}(g)), \quad j = \overline{1, p+1}. \tag{14}$$

By means of (5), we find the system of integral equations and determine the solution of (11), (2), (3)

$$x(t) = x_j(t), \quad t \in]v_{j-1}, v_j], \quad j = \overline{1, p+1},$$
$$x_j(t) = \bar{x}_j(t) - \varepsilon X(t) L_j \bar{l}(g) + \varepsilon \int_{v_{j-1}}^{t} X(t) X^{-1}(u) g_j(u, x_j, \varepsilon) du, \tag{15}$$

where

$$\bar{x}(t) = \bar{x}_j(t), \quad t \in]v_{j-1}, v_j], \quad j = \overline{1, p+1},$$
$$\bar{x}_j(t) = X(t) L_j (h - \bar{l}(f)) + \int_{v_{j-1}}^{t} X(t) X^{-1}(u) f_j(u) du. \tag{16}$$

Let $Q[a, b]$ be a Banach space of a first-order discontinuous function $x(t)$ for $t = v_i$, $i = \overline{1, p}$, with a norm $\|x\| = \max\limits_{j = \overline{1, p}} \max\limits_{t \in [v_{j-1}, v_j]} |x_j(t)|$. Then the existence of constants q_i, $i = \overline{1, 8}$ and $t \in [a, b]$ is fulfilled

$$\|A(t)\| \le q_1, \quad \|X(t)\| \le q_2, \quad \|X^{-1}(t)\| \le q_3, \quad \|M^{-1}\| \le q_4,$$
$$\|f(t)\| \le q_5, \quad \|h\| \le q_6, \quad \|B_j(s)\| \le q_7 \ \text{at} \ s \in [a, b], \ j = \overline{1, p+1}, \tag{17}$$
$$\|\bar{l}(f)\| \le q_8, \quad \|g(t, x, \varepsilon)\| \le M(\varepsilon) \ \text{at} \ (t, x, \varepsilon) \in \Omega.$$

Let F be an operator acting from the ball $\|x\| \le \rho$ into Q and let it have the form

$$F(x) = \bar{x}(t) - \varepsilon X(t) L \bar{l}(g(s, x, \varepsilon)) + \varepsilon (\tilde{L} g(u, x, \varepsilon))(t), \tag{18}$$

where

$$\tilde{L} = (\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_{p+1})^T, \quad (\tilde{L}_j\varphi)(t) = \int\limits_{v_{j-1}}^{t} X(t)X^{-1}(u)\varphi(u)\,du, \quad t \in]v_{j-1}, v_j].$$

If $x^1(t)$ and $x^2(t)$ belong to the space $Q[a, b]$, then, by virtue of (17) and (18), we get $\|F(x^1) - F(x^2)\| \le \varepsilon q L(\rho, \varepsilon_0)\|x^1 - x^2\|$, where $q = q_2 q_3(q_4 q_7 + 1)(b - a)$. But $0 < \varepsilon << 1$. Then there exists a constant $\theta \in (0, 1)$ such that $\varepsilon q L(\rho, \varepsilon_0) \le \theta$. Consequently,

$$\|F(x^1) - F(x^2)\| \le \theta\|x^1 - x^2\|. \tag{19}$$

Inequality (19) shows that the nonlinear operator (18) is contractive for $t \in [a, b]$, $\|x\| \le \rho$ and $\varepsilon \in (0, \varepsilon_0]$. Then F has a unique fixed point $x^* = F(x^*)$, i.e., there exists a unique solution $x = x^*(t) \in Q[a, b]$ of system (15). This solution satisfies the boundary conditions (2), (3).

For $\bar{x}(t)$ of (16) and $\bar{l}(g(s, 0, \varepsilon))$ of (6), we obtain the estimates

$$\|\bar{x}(t)\| \le q_2 m_1, \tag{20}$$

where $m_1 = \max(q_4 q_6, q_4 q_8, q_3 q_5(b - a))$ and

$$\|\bar{l}(g(s, 0, \varepsilon))\| \le M(\varepsilon) m_2 q_7, \tag{21}$$

where $m_2 = q_3(b - a)$.

We evaluate $\|F(0)\|$ by means of (18), (20), and (21)

$$\|F(0)\| \le q_2(m_1 + \varepsilon M(\varepsilon) m_2(q_4 q_7 + 1)).$$

If $\varepsilon \in (0, \varepsilon_0]$, then there exists a constant K such that $q_2(m_1 + \varepsilon M(\varepsilon) \times \times m_2(q_4 q_7 + 1))$ is not greater than K, i.e.,

$$\|F(0)\| \le K. \tag{22}$$

We choose ρ such that $K \le \rho(1 - \theta)$.

Consequently, the nonlinear operator F satisfies the requirements of the fixed point theorem [8], according to which the fixed point x^* is a unique solution of (11), (2), (3) and is obtained by the iterative process $x^k(t)$

$$\begin{aligned} x^0(t) &= 0, \\ x^{k+1}(t) &= \bar{x}(t) - \varepsilon X(t) L \bar{l}(g(s, x^k, \varepsilon)) + \varepsilon(\tilde{L} g(u, x^k, \varepsilon))(t), \end{aligned} \tag{23}$$

such that

$$\|x^*(t) - x^k(t)\| \le \theta^k \rho. \tag{24}$$

Theorem 2. *If* $m = (p + 1)n$ *and* M *is a nonsingular matrix, then there exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$, *the perturbed boundary-value problem* (11), (2), (3) *has a unique first-order solution* $x^*(t)$ *discontinuous for* $t = v_i$, $i = \overline{1, p}$. *This solution is obtained by means of the iterative formula* (23) *such that estimate* (24) *is satisfied.*

3.2. $m = (p + 1)n$ and $\det M = 0$.

Let M^- be an arbitrary half an inverse matrix of M. System (13) has a solution if and only if

$$(E_m - MM^-)(h - \bar{l}(f) - \varepsilon \bar{l}(g)) = 0. \tag{25}$$

According to the notation of Section 2, we obtain the general solution of (13)

$$x_j(v_{j-1}) = X(v_{j-1})P_M^j c + X(v_{j-1})N_j(h - \bar{l}(f) - \varepsilon \bar{l}(g)), \quad j = \overline{1, p+1}, \tag{26}$$

where c is a $(p+1)n \times 1$ arbitrary numerical vector.

Substituting (26) in (5), we find a system of integral equations for determining $x_j(t)$

$$x(t) = x_j(t), \quad t \in]v_{j-1}, v_j], \quad j = \overline{1, p+1},$$
$$x_j(t) = X(t)P_M^j c + \bar{\bar{x}}(t) - \varepsilon X(t) N_j \bar{l}(g) + \varepsilon \int_{v_{j-1}}^{t} X(t)X^{-1}(u) g_j(u, x_j, \varepsilon) du, \tag{27}$$

where $\bar{\bar{x}}(t)$ are obtained from $\bar{x}(t)$ in (16) by substituting N_j for L_j.

If condition (8) is satisfied, then, for $\varepsilon = 0$, the boundary-value problem (11), (2), (3) has a general solution $x^0(t, w)$. Let $x(t, c, \varepsilon)$ be the general solution of (11), (2), (3). Then condition (25) is fulfilled and the nonlinear function $g(s, x(s, c, \varepsilon), \varepsilon)$ satisfies the condition

$$(E_m - MM^-)\, \bar{l}(g(s, x(s, c, \varepsilon), \varepsilon)) = 0. \tag{28}$$

Let $x(t, c, \varepsilon)$, for $\varepsilon = 0$, be reduced to the solution $x^0(t, w)$. Then w satisfies the condition

$$(E_m - MM^-)\, \bar{l}(g(s, x^0(s, w), 0)) = 0. \tag{29}$$

Theorem 3. *Let the vector* w *satisfy system* (29) *and let the Jacobian of* (28) *with respect to* c *be nonvanishing at the point* $\varepsilon = 0$. *Then there exists a number* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$, *system* (11), (2), (3) *has a unique solution.*

Proof. Let w be a unique solution of (29). Then for a sufficiently small μ and c such that $\| c - w \| \le \mu$, the mapping

$$J(x) = X(t)P_M c + \bar{\bar{x}}(t) - \varepsilon X(t) N\, l(g(s, x, \varepsilon)) + \varepsilon(\tilde{L} g(u, x, \varepsilon))(t)$$

has a unique fixed point $x^*(t)$ for $\| x \| \le \rho$. The proof of this proposition is very similar to the proof of Theorem 2.

Consequently, the integral system (27) has a unique solution of class $Q[a, b]$ and is obtained by means of an iterative formula. We substitute this solution in (28). But the Jacobian of (28) with respect to c is nonvanishing; therefore, by means of the implicit function theorem, we obtain the vector-function $c = c(\varepsilon)$ such that $c(0) = w$. We substitute $c = c(\varepsilon)$ in (27) and obtain the required solution of (11), (2), (3).

Remark. The case $m \ne (p+1)n$ is considered by analogy with subsection 3.2.

1. *Conti R.* Recent trends in the theory of boundary-value problems for ordinary differential equations // Bol. UMI.–1967.–**22**, № 3.–P. 135–178.
2. *Generalized* inverses and applications / Ed. M. Z. Nashed.–New York etc.: Acad. press, 1967. – 1054 p.
3. *Halanay A., Wexler D.* Teoria calitativa a sistemelor cu impulsuri.–Bucurecti: Ed. Acad. Republ. Romania, 1968.–308 p.
4. *Самойленко А. М., Бойчук А. А.* Линейные нетеровые краевые задачи для дифференциальных систем с импульсным воздействием // Укр. мат. журн.–1992.–**44**, № 4.–С.564–568.
5. *Каранджулов Л. И.* Структура общего решения краевых задач обыкновенных дифференциальных уравнений с импульсным воздействием при помощи полуобратных матриц // Там же.–1993.–**45**, № 5.– С. 616–625.
6. *Самойленко А. М., Перестюк Н. А.* Дифференциальные уравнения с импульсным воздействием.–Киев: Выща шк., 1987.–288 с.
7. *Зубов В. И.* К теории существования решений краевых задач для систем дифференциальных уравнений // Дифференц. уравнения.–1970.–**6**, № 4.– С. 629–631.
8. *Хартман Ф.* Обыкновенные дифференциальные уравнения.–М.: Мир, 1970.–720 с.

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