

THE BBGKY HIERARCHY AND ITS EVOLUTION OPERATOR FOR ONE CLASS OF INTEGRABLE SYSTEMS

ІЄРАРХІЯ ББГКІ ТА Її ЕВОЛЮЦІЙНИЙ ОПЕРАТОР ДЛЯ ОДНОГО КЛАСУ ІНТЕГРОВНИХ СИСТЕМ

The abstract BBGKY hierarchy and its formal evolution operator are defined. Existence of the latter is established in a special Banach space for a system of charged particles with Chern – Simons interaction, regularized on small and large distances. An analog of the high-temperature expansion of equilibrium statistical mechanics is applied.

Визначені абстрактна ієрархія ББГКІ та її формальний еволюційний оператор. Встановлено існування цього оператора в спеціальному банаховому просторі для інтегрованої системи заряджених частинок із взаємодією Черна–Саймонса, яка регуляризована на малих та великих відстанях. При цьому використовується аналог високотемпературного кластерного розкладу рівноважної статистичної механіки.

Introduction. The states of systems of infinite number of particles in classical statistical mechanics can be described by the sequence of positive functions (distribution or correlation of functions) $\{\rho(P_m, X_m)\}_{m \geq 1}$, where $P_m = (p_1, \dots, p_m)$, $X_m = (x_1, \dots, x_m)$ are the momenta and position vectors of m d -dimensional particles, respectively [1–3]. These functions determine a probability measure on a space of locally finite infinite particle configurations in $2d$ -dimensional space [2] if they satisfy certain compatibility conditions. Evolution of the states are governed by the BBGKY hierarchy [1–3]

$$\frac{d}{dt} \rho_t = \hat{L} \rho_t.$$

The hierarchy plays a remarkable role in deriving kinetic equations in statistical physics [1]. The hierarchy is usually derived from the Liouville equation for the n -particle distribution function in different ensembles. The most straightforward derivation is fulfilled in the grand canonical ensemble, in which a number of particles is not fixed. It suggests the following formal algebraic structure of the evolution operator of the hierarchy

$$\hat{U}^t = e^d e^{tL} e^{-d},$$

where

$$(dF)(P_m, X_m) = \int F(P_{m+1}, X_{m+1}) dp_{m+1} dx_{m+1},$$

$$(LF)(P_m, X_m) = (L_m F)(P_m, X_m),$$

$$L_m = \sum_{j=1}^m \left(\tilde{\partial}_j H \right) (P_m, X_m) \partial_j - (\partial_j H) (P_m, X_m) \tilde{\partial}_j,$$

H is a Hamiltonian, $\tilde{\partial}_j$, ∂_j are the partial derivatives in momentum and position vector of the j -th particle, respectively, and the integration is performed over $2d$ -dimensional space. This means that $\rho_t = \hat{U}^t \rho_0$ and

$$\hat{L} = L + [d, L] + \frac{1}{2} [d, [d, L] + \dots]$$

Hence, the Cauchy problem for the hierarchy can be reduced to the problem of prescribing rigorous meanings to the evolution operator and its generator, given in the last equality, in Banach spaces.

This approach was initiated in [3] for usual systems of particles interacting through hard-core potentials.

We shall deal with Banach spaces which are symmetric tensor algebra B_v

$$B_v = \bigoplus (\otimes B)^n, \quad \|F\| = \max_n e^{v(n)} \|F\|_{(\otimes B)^n}$$

over Banach spaces $B = L^1(R^d) \otimes L^\infty(R^d)$, $B = L^1(R^d) \otimes L^1(R^d)$.

We shall denote the corresponding Banach tensor algebra by B_v^∞ , B_v^1 .

Dense infinite particle systems are described by the space B_v^∞ , $v(n) = -n \ln \xi$. Gibbs (equilibrium) correlation functions in ordinary systems belong to the space with this condition [2, 3]. Let $\hat{\chi}_\Lambda$ be a projection operator in the Banach spaces

$$(\hat{\chi}_\Lambda F)(P_n, X_n) = \prod_{j=1}^n \chi_\Lambda(x_j) F(P_n, X_n),$$

where $\chi_\Lambda(x)$ is the characteristic function of a compact domain Λ .

The important problem of statistical mechanics is to find a bounded operator U^t in B_v^∞ such that

$$U^t \hat{\chi}_\Lambda = \hat{U}^t \hat{\chi}_\Lambda, \quad \lim_{\Lambda \rightarrow R^d} \|\hat{\chi}_K(U^t F - \hat{U}^t \hat{\chi}_\Lambda F)\| = 0, \quad (1)$$

where K is a compact domain and $F \in B_v^\infty$. We call such an operator the evolution operator of the BBGKY hierarchy or simply the evolution operator in what follows.

Let us consider the case where an n -particle Hamiltonian system is integrable, i.e., there exists an invertible operator Z_{0n} such that

$$e^{tL_n} = Z_{0n} U_{0n}^t Z_{0n}^{-1},$$

where U_{0n}^t is an n -fold tensor product of the evolution operator U_{01}^t of a free particle. Then

$$\hat{U}^t = Z_d U_0^t Z_d^{-1},$$

where $Z_d = e^d Z_0 e^{-d}$, Z_0 , U_0^t are direct sums of the operators Z_{0n} , U_{0n}^t , respectively. Hence, the problem of construction of U^t is reduced to the problem of construction of a bounded invertible operator Z in B_v^∞ such that

$$Z \hat{\chi}_\Lambda = Z_d \hat{\chi}_\Lambda, \quad \lim_{\Lambda \rightarrow R^d} \|\hat{\chi}_K(ZF - Z_d \hat{\chi}_\Lambda F)\| = 0, \quad F \in B_v^\infty. \quad (2)$$

In this paper, we solve the problem for a simple integrable system related to the topological electrodynamics [4-6] in the space B_v^∞ with v satisfying a certain condition, which means that we deal with a dilute infinite particle system.

The Hamiltonian $H(P_n, X_n)$ of our system is given by

$$H(P_n, X_n) = 2^{-1} \sum_{j=1}^n \|p_j - a_j(X_n)\|^2,$$

where $\|p\|$ is a Euclidean norm of a d -dimensional vector p , $a_j(X_n) = -(\partial_j U)(X_n)$,

$$U(X_n) = \sum_{k < \infty} \sum_{k \in (1, \dots, n)} \varphi_k(X_k),$$

φ_k are k -particle "topological" potentials, k is the subset of k elements of the set $(1, \dots, n)$, and $\varphi_k \in C^\infty(R^{dk})$.

For the case of a two-particle interaction, the Hamiltonian can be interpreted as a regularized Hamiltonian of the system of charged particles with Chern–Simons (C–S) interaction (topological electrodynamics [7]).

In spite of the fact that our result concerns dilute systems, it can be a starting point for a rigorous derivation of kinetic equations for C–S charged particle systems. This problem is interesting for physics since quantum topological electrodynamics is believed to describe a high-temperature superconductivity.

Our idea of construction of the operator Z in B_v^∞ is based on the idea of a cluster expansion originated from a study of the Kirkwood–Salsburg operator which is the operator on the right-hand side of the equation of the resolvent type for the Gibbs correlation functions. By the cluster expansion for Z_d , we call the expansion

$$Z_d = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Z_d^{(n)}, \quad Z_d^{(n)} = \sum_{k=0}^n (-1)^{n-k} C_n^k d^k Z_0 d^{m-k}. \quad (3)$$

After a certain redefinition of the right-hand side, we define the operator Z in B_v^∞ and show that it satisfies the basic equalities (2) (the lemma). Then we define the evolution operator in B_v^∞

$$\hat{U}^t = Z U_0^t Z^{-1},$$

which satisfies equality (1) (the theorem, Section 2).

Our paper is organized as follows: In Section 1, we formulate our main results (the lemma, the theorem) and, in Section 2, we give the proofs.

1. The main result. In the system with our Hamiltonian, the functions $p_j - a_j(X_n)$ are the integrals of motion. From this fact and the first part of equations of motion

$$\dot{x}_j = p_j - a_j(X_n), \quad j = 1, \dots, n,$$

we obtain the following solution of the equations:

$$x_j(t) = x_j + t(p_j - a_j(X_n)), \quad p_j(t) = p_j + a_j(X_n(t)) - a_j(X_n),$$

where the sequence $(P_n, X_n) = Q_n$ is the initial data. It is evident that

$$Q_n(t) = Z_n \tilde{U}_{0n}^t Z_n^{-1},$$

where $Z_n Q_n = (P_n - A_n(X_n), X_n)$, $A_n(X_n) = \{a_j(X_n)\}$, $j \in (1, \dots, n)$, $(P_n - A_n)_j = p_j - a_j$, $\tilde{U}_{0n}^t Q_n = (P_n, X_n + t X_n)$. The transformation Z_n induces the action of the operator Z_{0n} on the set of sequences of symmetric functions

$$(Z_{0n} F)(Q_n) = F(Z_n Q_n).$$

It is clear that this operator is bounded in both the spaces B_v^1 , B_v^∞ .

Let us consider the cluster expansion (3) for Z_d in B_v^1 . It follows from the symmetry of functions that

$$Z_d^{(n)} F = \int dQ'_n \sum_{\{k\} \in (1, \dots, n)} (-1)^k D_{Q'_{\{k\}}} Z_0 D_{Q'_{\{n \setminus k\}}} F, \quad (4)$$

where $(D_{Q'_k} F)(Q_n) = F(Q_n, Q'_k)$, $\{n \setminus k\} = (1, \dots, n) \setminus \{k\}$, and the integration is performed over $2dn$ -dimensional space.

It follows from the definition of Z_0 that

$$(Z_d F)(Q_m) = \sum_{n \geq 0} \frac{1}{n!} \int dQ'_n \sum_{\{k\} \in (1, \dots, n)} (-1)^k F(Z_{n+m} Q_m, Q'_n). \quad (5)$$

From equalities (3), (5), we derive equality (5), in which $Z_{n+m}(Q_m, Q'_n)$ appears instead of

$$(Z_{n+m} Q_m, Q'_n) = (P_m - A_{n+m}(X_m, X'_n), X'_m; Q'_n).$$

But equality (5) is obtained then by a trivial change of the momentum variables $P'_n - A_n(X_m, X'_n) \rightarrow P'_n$.

Assume now that all the potentials φ_k are equal to zero if $\|x_j - x_k\| > R$ for some j, k .

Let $B_r(x_j)$ be the ball with radius r centered in x_j , and $B_r(q_j) = R^d \times B_r(q_j)$, $q = (p, x)$,

$$B_r(Q_m) = \bigcup_{j=1}^m B_r(q_j), \quad B_r^n(Q_m) = (\times B_r(Q_m))^n,$$

where \times is the Cartesian product.

Let us also consider the one parameter family of operators Z^α

$$(Z^\alpha F)(Q_m) = \sum_{n \geq 0} \frac{1}{n!} \int_{B_r^n(Q_m)} dQ'_n \sum_{\{k\} \in (1, \dots, n)} (-1)^k F(Z_{m+n}^\alpha Q_m, Q'_n), \quad (6)$$

where $Z_{m+n}^\alpha Q_m = (P_m - \alpha A_{n+m}(X_m, X'_n), X'_m)$.

Lemma. *If*

$$\max_m m^n \exp \{v(m) - v(n+m)\} \leq (n!)^\gamma, \quad \gamma < 1, \quad (7)$$

then Z^α is the group of bounded operators in B_∞^V for which equalities (2) hold and $(Z^\alpha)^{-1} = Z^{-\alpha}$.

Theorem. *If v satisfies the condition of the lemma, then the evolution operator U^t satisfying equalities (1) is given by*

$$U^t = Z U_0^t Z^{-1},$$

where $Z = Z^1$.

Remark. The simplest example of v satisfying (7) is $v = n^2$.

2. Proofs. The proof of the first equality (2) follows from the equality

$$\sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} = 0.$$

Let us prove the boundedness of Z^α .

$$\exp \{v(m)\} \int |(Z^\alpha F)(Q_m)| dP_m \leq$$

$$\begin{aligned} &\leq \sum_{n \geq 0} \frac{2^n}{n!} \int_{B_r(Q_m)} dX' \int |F(Q_m, Q'_n)| dP_m dP'_n \leq \\ &\leq \|F\|_v \sum_{n \geq 0} \frac{2^n}{n!} (|B_r| m)^n \exp\{v(m) - v(n+m)\} \leq \\ &\leq \|F\|_v \sum_{n \geq 0} (n!)^{1-\gamma} (2|B_r|)^n \leq \infty, \end{aligned}$$

where $|B_r|$ is the volume of the ball with radius R . The norm of the operators Z^α is given by the sum in the last inequality. Let us prove the group property of Z^α . Let $K^{(r)} = x: |x-x'| \leq r, x' \in K$, and let K be a compact domain in a d -dimensional space. Then it is not difficult to check that

$$\hat{\chi}_K Z^\alpha = \hat{\chi}_K Z^\alpha \hat{\chi}_\Lambda, \quad K^{(r)} \subset \Lambda. \tag{8}$$

This equality yields

$$\hat{\chi}_K (Z^\alpha Z^\beta \hat{\chi}_\Lambda = Z^\alpha_d Z^\beta_d = Z^{\alpha+\beta}_d \hat{\chi}_\Lambda = Z^{\alpha+\beta} \hat{\chi}_\Lambda) = \hat{\chi}_K Z^{\alpha+\beta}.$$

But if $K^{(r)} \subset (\Lambda')^r \subset \Lambda$, then

$$\hat{\chi}_K (Z^\alpha Z^\beta \hat{\chi}_\Lambda = Z^\alpha \hat{\chi}_{\Lambda'} Z^\beta \hat{\chi}_\Lambda = Z^\alpha \hat{\chi}_{\Lambda'} Z^\beta = Z^\alpha Z^\beta).$$

These two equalities imply the group property. Equality (8) follows from the evident relation

$$\max_{x \in K} \int_{B_r(x)} (1 - \chi_\Lambda(y)) dy = 0, \quad B_r(x) \in K^{(r)} \in \Lambda.$$

The lemma is proved because equality (8) implies the second equality (2).

Let us proceed to prove the theorem and put $K^{(r)} \subset \Lambda', \Lambda^{(r)} \subset \Lambda'', \Lambda''^{(r)} \subset \Lambda$. It follows from the equalities

$$\hat{\chi}_K Z(1 - \hat{\chi}_{\Lambda'}) = 0, \quad \hat{\chi}_{\Lambda''} Z(1 - \hat{\chi}_\Lambda) = 0,$$

that

$$\hat{\chi}_K (U^t(1 - \hat{\chi}_\Lambda)) = Z \hat{\chi}_{\Lambda'} U^t_0(1 - \hat{\chi}_{\Lambda''}) Z^{-1}(1 - \hat{\chi}_\Lambda).$$

Now let us enlarge Λ, Λ'' to the whole space. Then the theorem will be proved if we prove the following proposition.

Proposition. U^t_0 is a group of isometric operators in B^∞_v and

$$\lim_{\Lambda \rightarrow R^d} \|\hat{\chi}_K U^t_0(1 - \hat{\chi}_\Lambda) F\|_v = 0.$$

Proof. The set of functions $\Psi(Q_m) = \sum_j \psi_j(P_m) \phi_j(X_m)$, where ψ_j has a compact support, is a dense set in $L^1(R^{dm}) \otimes L^\infty(R^{dm})$. Let K be a union of hypercubes K^0_s . Assume that the function Ψ_j is a sum of the characteristic functions χ_s of the hypercubes. Then

$$\begin{aligned}\Psi(Q_m) &= \sum_s \chi_s(P_m) \varphi_{(s)}(X_m) = \sum_s \chi_s(P_m) [\varphi_{(s)}^+(X_m) - \varphi_{(s)}^-(X_m)] = \\ &= \Psi_+(Q_m) - \Psi_-(Q_m),\end{aligned}$$

where $\varphi_{(s)}^{+(-)} = \max(0, +(-)\varphi_{(s)})$. The set of such functions is also a dense set. As a result,

$$\left| (U_0^t \Psi)(Q_m) \right| \leq \sum_s \chi_s(P_m) \|\varphi_{(s)}\|,$$

where $\|\varphi_{(s)}\| = \|\varphi_{(s)}^+\| + \|\varphi_{(s)}^-\|$. Hence, the following inequality holds on the dense set in $L^1 \otimes L^\infty$: $\|U_0^t \Psi\| \leq \|\Psi\|$, and it holds on the whole space. It follows from the group property that the semigroup is isometric. Now we have to estimate the integral

$$\chi_K(X_n) \int (1 - \chi_\Lambda(X_n + tP_n)) F(Q_n) dP_n.$$

Let us split the integral into two parts: the integral over the domain

$$\Lambda_0 = P_n: |P_n| \leq p_0$$

and its complement Λ^c . For given ε , we can find p_0 such that

$$\max_{X_n \in K^n} \int_{\Lambda_0^c} |F(Q_n)| dP_n \leq \varepsilon.$$

We can also choose a domain Λ , depending on ε , such that $x_j + tP_j \in \Lambda$ if $|P_n| \leq p_0$. The integral taken over Λ is equal to zero. Thus, for arbitrary small ε we can find a domain Λ such that the estimated integral is less than ε . The proposition is proved.

1. *Bogolyubov N. N.* Collected papers. In three volumes. – Kiev: Naukova dumka, 1970. – 1. – 521 p.
2. *Ruelle D.* Statistical mechanics. Rigorous results. – New York: W. A. Benjamin, 1969. – 219 p.
3. *Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.* Mathematical foundations of classical statistical mechanics. – Kiev: Naukova dumka, 1985. – 262 p.
4. *Jackiw R., Pi So-Young.* Classical and quantal non-relativistic Chern–Simons theory. – Preprint BU–HEP–90–11.
5. *Lykken J. D., Sonnenschein J., Wess N.* The theory of anionic superconductivity. A review. – (Preprint / TAUP–1858–91).
6. *Fradkin E.* Field theories of condensed matter systems. – Addison-Wesley Publishing Company.
7. *Skrypnik W. I.* Infinite particle Hamiltonian dynamics of Chern–Simons type. – (Preprint / Dublin Institute of advanced studies, DIAS–STR–91–11).

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