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A. Messaoud, A. Rahali (Univ. Sfax, Tunisia)

ANOTHER PROOF FOR THE CONTINUITY OF THE LIPSMAN MAPPING

ЩЕ ОДНЕ ДОВЕДЕННЯ НЕПЕРЕРВНОСТІ ВІДОБРАЖЕННЯ ЛІПСМАНА

We consider the semidirect product $G = K \ltimes V$ where K is a connected compact Lie group acting by automorphisms on a finite dimensional real vector space V equipped with an inner product $\langle , \rangle$. By $\widehat{G}$ we denote the unitary dual of G and by $\mathfrak{g}^\ddagger/G$ the space of admissible coadjoint orbits, where $\mathfrak{g}$ is the Lie algebra of G. It was pointed out by Lipsman that the correspondence between $\widehat{G}$ and $\mathfrak{g}^\ddagger/G$ is bijective. Under some assumption on G, we give another proof for the continuity of the orbit mapping (Lipsman mapping)

$$\Theta : \mathfrak{g}^\ddagger/G \longrightarrow \widehat{G}.$$

Розглядається напівпрямий добуток $G = K \ltimes V$, де K — зв'язна компактна група Лі автоморфізмів, що діють на скінченновимірному дійсному векторному просторі V із внутрішнім добутком $\langle , \rangle$. Нехай $\widehat{G}$ — унітарний дуал G, а $\mathfrak{g}^\ddagger/G$ — простір допустимих коспряжених орбіт, де $\mathfrak{g}$ — алгебра Лі для G. Ліпсман зазначив, що відповідність між $\widehat{G}$ та $\mathfrak{g}^\ddagger/G$ є бієкцією. При деяких припущеннях на G ми пропонуємо нове доведення неперервності відображення орбіт (відображення Ліпсмана)

$$\Theta : \mathfrak{g}^\ddagger/G \longrightarrow \widehat{G}.$$

1. Introduction. Let G be a second countable locally compact group and $\widehat{G}$ the unitary dual of G, i.e., the set of all equivalence classes of irreducible unitary representations of G. It is well-known that $\widehat{G}$ equipped with the Fell topology [6]. The description of the dual topology is a good candidate for some aspects of harmonic analysis on G (see, for example, [4, 5]). For a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = \exp(\mathfrak{g})$, its dual space $\widehat{G}$ is homeomorphic to the space of coadjoint orbits $\mathfrak{g}^*/G$ through the Kirillov mapping (see [8]). In the context of semidirect products $G = K \ltimes N$ of compact connected Lie group K acting on simply connected nilpotent Lie group N, then it was pointed out by Lipsman in [9], that we have again an orbit picture of the dual space of G. The unitary dual space of Euclidean motion groups is homeomorphic to the admissible coadjoint orbits [5]. This result was generalized in [4], for a class of Cartan motion groups.

In this paper, we consider the semidirect product $G = K \ltimes V$, where K is a connected compact Lie group acting by automorphisms on a finite dimensional real vector space V equipped with an inner product $\langle , \rangle$. In the spirit of the orbit method due to Kirillov, R. Lipsman established a bijection between a class of coadjoint orbits of G and the unitary dual $\widehat{G}$. For every admissible linear form ψ of the Lie algebra $\mathfrak{g}$ of G, we can construct an irreducible unitary representation π_ψ by holomorphic induction and according to Lipsman (see [9]), every irreducible representation of G arises in this manner. Then we get a map from the set $\mathfrak{g}^\ddagger$ of the admissible linear forms onto the dual space $\widehat{G}$ of G. Note that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' are on the same G-orbit, finally we obtain a bijection between the space $\mathfrak{g}^\ddagger/G$ of admissible coadjoint orbits and the unitary dual $\widehat{G}$.

Definition 1. *Let G be a (real) Lie group, $\mathfrak{g}$ its Lie algebra and*

$$\exp : \mathfrak{g} \longrightarrow G$$

its exponential map. We say that G is exponential if $\exp(\mathfrak{g}) = G$.

Now, we give our main result in this paper, which is another proof for the continuity of the orbit mapping (see [11]):

Theorem 1. *We assume that G is exponential. Then the orbit mapping*

$$\Theta\colon \mathfrak{g}^{\ddagger}/G \longrightarrow \widehat{G}$$

is continuous.

This paper is organized as follows. Section 2 is devoted to the description of the unitary dual $\widehat{G}$ of G. Section 3 deals with the space of admissible coadjoint orbits $\mathfrak{g}^{\ddagger}/G$ of G. Theorem 1 is proved in Section 4.

2. Dual spaces of semidirect product. Throughout this paper, K will denote a connected compact Lie group acting by automorphisms on a finite dimensional vector space $(V, \langle , \rangle)$. We write $k.v$ and $A.v$ (resp., $k.\ell$ and $A.\ell$) for the result of applying elements $k \in K$ and $A \in \mathfrak{k} := \mathrm{Lie}\,(K)$ to $v \in V$ (resp., to $\ell \in V^*$).

Now, one can form the semidirect product $G := K \ltimes V$ which so-called generalized motion groups. As a set $G = K \times V$ and the multiplication in this group is given by

$$(k, v)(h, u) = (kh, v + k.u) \quad \forall (k, v), \quad (h, u) \in G.$$

The Lie algebra of G is $\mathfrak{g} = \mathfrak{k} \oplus V$ (as a vector space) and the Lie algebra structure is given by the bracket

$$[(A, a), (B, b)] = ([A, B], A.b - B.a) \quad \forall (A, a), \quad (B, b) \in \mathfrak{g}.$$

Under the identification of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{k}^* \oplus V^*$, we can express the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ as $F(A, a) = f(A) + \ell(a)$ for all $F = (f, \ell) \in \mathfrak{g}^*$ and $(A, a) \in \mathfrak{g}$. The adjoint representation Ad_G and coadjoint representation Ad_G^* of G are given, respectively, by the following relations:

$$\mathrm{Ad}_G(k, v)(A, a) = (\mathrm{Ad}_K(k)A, k.a - \mathrm{Ad}_K(k)A.v) \quad \forall (k, v) \in G, \quad (A, a) \in \mathfrak{g},$$

$$\mathrm{Ad}_G^*(k, v)(f, \ell) = (\mathrm{Ad}_K^*(k)f + k.\ell \odot v, k.\ell) \quad \forall (k, v) \in G, \quad (f, \ell) \in \mathfrak{g}^*,$$

where $\ell \odot v$ is the element of $\mathfrak{k}^*$ defined by

$$\ell \odot v(A) = \ell(A.v) = -(A.\ell)(v) \quad \forall A \in \mathfrak{k}, \quad \ell \in V^*, \quad v \in V.$$

Note that the map $\odot\colon V^* \times V \longrightarrow \mathfrak{k}^*$ defined by $(\ell \odot v)(A) = \ell(A.v)$, $v \in V$, $A \in \mathfrak{k}$ satisfies a fundamental equivariance property

$$\mathrm{Ad}_K^*(k)(\ell \odot v) = (k.\ell) \odot (k.v), \quad k \in K.$$

Therefore, the coadjoint orbit of G passing through $(f, \ell) \in \mathfrak{g}^*$ is given by

$$\mathcal{O}_{(f,\ell)}^G = \Big\{ \Big(Ad_K^*(k)f + k.\ell \odot v, k.\ell \Big), k \in K, v \in V \Big\}.$$

For $\ell \in V^*$, we define $K_\ell := \{k \in K;\ k.\ell = \ell\}$ the isotropy subgroup of ℓ in K and the Lie algebra of K_ℓ is given by the vector space $\mathfrak{k}_\ell = \{A \in \mathfrak{k};\ A.\ell = 0\}$. Let $\imath_\ell\colon \mathfrak{k}_\ell \hookrightarrow \mathfrak{k}$ be the injection map, then $\imath_\ell^*\colon \mathfrak{k}^* \longrightarrow \mathfrak{k}_\ell^*$ is the projection map and we have

$$\mathfrak{k}_\ell^\circ = \mathrm{Ker}\,(\imath_\ell^*), \tag{1}$$

where $\mathfrak{k}_\ell^\circ$ is the annihilator of $\mathfrak{k}_\ell$. If we define the linear map $h_\ell\colon \mathfrak{k} \longrightarrow V^*$ by

$$h_\ell(A) := -A.\ell \quad \forall A \in \mathfrak{k},$$

then we have $\mathfrak{k}_\ell = \operatorname{Ker}(h_\ell)$. The dual $h_\ell^*\colon V \longrightarrow \mathfrak{k}^*$ of h_ℓ is given by the relation $h_\ell^*(v)(A) = = h_\ell(A)(v) = -(A.\ell)(v)$, and so $h_\ell^*(v) = \ell \odot v$ $\forall \ell \in V^*$, $\forall v \in V$ (for more details see [3]).

The following is a useful lemma from [3], giving a characterization of the annihilator $\mathfrak{k}_\ell^\circ$ in terms of the linear map h_ℓ.

Lemma 1. *Using the previous notations, then we have the equality*

$$\mathfrak{k}_\ell^\circ = \operatorname{Im}(h_\ell^*).$$

Here we recall briefly the description of the unitary dual of G via Mackey's little group theory (see [10]). For every non-zero linear form ℓ on V, we denote by χ_ℓ the unitary character of the vector Lie group V given by $\chi_\ell = e^{i\ell}$. Let ρ be an irreducible unitary representation of K_ℓ on some Hilbert space $\mathcal{H}_\rho$. The map

$$\rho \otimes \chi_\ell\colon (k, v) \longmapsto e^{i\ell(v)}\rho(k)$$

is a representation of the Lie group $K_\ell \ltimes V$ such that one induce up so as to get a unitary representation of G. We denote by $\mathcal{H}_{(\rho,\ell)} := L^2(K, \mathcal{H}_\rho)^\rho$ the subspace of $L^2(K, \mathcal{H}_\rho)$ consisting of all the maps ξ which satisfy the covariance condition

$$\xi(kh) = \rho(h^{-1})\xi(k) \quad \forall k \in K, \quad h \in K_\ell.$$

The induced representation

$$\pi_{(\rho,\ell)} := \operatorname{Ind}_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell)$$

is defined on $\mathcal{H}_{(\rho,\ell)}$ by

$$\pi_{(\rho,\ell)}(k, v)\xi(h) = e^{i\ell(h^{-1}.v)}\xi(k^{-1}h),$$

where $(k, v) \in G, h \in K$ and $\xi \in \mathcal{H}_{(\rho,\ell)}$. By Mackey's theory we can say that the induced representation $\pi_{(\rho,\ell)}$ is irreducible and every infinite dimensional irreducible unitary representation of G is equivalent to one of $\pi_{(\rho,\ell)}$. Moreover, tow representations $\pi_{(\rho,\ell)}$ and $\pi_{(\rho',\ell')}$ are equivalent if and only if ℓ and $\ell^{'}$ are contained in the same K-orbit and the representation ρ and $\rho^{'}$ are equivalent under the identification of the conjugate subgroups K_ℓ and $K_{\ell'}$. All irreducible representations of G which are not trivial on the normal subgroup V, are obtained by this manner. On the other hand, we denote also by τ the extension of every unitary irreducible representation τ of K on G, which simply defined by $\tau(k, v) := \tau(k)$ for $k \in K$ and $v \in V$. Let Ω be a K-orbit in V^*. We fix $\ell \in \Omega$ and we define the subset $\widehat{G}(\Omega)$ of $\widehat{G}$ by

$$\widehat{G}(\Omega) = \Big\{\operatorname{Ind}_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell); \rho \in \widehat{K_\ell}\Big\}.$$

Then we conclude that

$$\widehat{G} = \widehat{K} \bigcup \Big(\bigcup_{\Omega \in \Lambda} \widehat{G}(\Omega)\Big),$$

where Λ is the set of the nontrivial orbits in V^*/K.

In the remainder of this paper, we shall assume that G is exponential, i.e., K_ℓ is connected for all $\ell \in V^*$. Let ρ_μ be an irreducible representation of K_ℓ with highest weight μ. For simplicity, we shall write $\pi_{(\mu,\ell)}$ instead of $\pi_{(\rho_\mu,\ell)}$ and $\mathcal{H}_{(\mu,\ell)}$ instead of $\mathcal{H}_{(\rho_\mu,\ell)}$.

We close this section by presenting two results which are being used in the description of the dual topology of G. These are required for our proof of Theorem 1.

Let N be an Abelian group, and assume that the compact Lie group K acts on the left on N by automorphisms. As sets, the semidirect product $K \ltimes N$ is the Cartesian product $K \times N$ and the group multiplication is given by

$$(k_1, x_1) \cdot (k_2, x_2) = (k_1 k_2, x_1 + k_1 x_2).$$

Let χ be a unitary character of N, and let K_χ be the stabilizer of χ under the action of K on $\widehat{N}$ defined by

$$(k \cdot \chi)(x) = \chi(k^{-1}x).$$

If ρ is an element of $\widehat{K_\chi}$, then the triple $(\chi, (K_\chi, \rho))$ is called a cataloguing triple. From the notations of [2], we denote by $\pi(\chi, K_\chi, \rho)$ the induced representation $\mathrm{Ind}_{K_\chi \ltimes N}^{K \ltimes N}(\rho \otimes \chi)$. Referring to [2, p. 187], we have the following proposition.

Proposition 1. *The mapping* $(\chi, (K_\chi, \rho)) \longrightarrow \pi(\chi, K_\chi, \rho)$ *is onto* $\widehat{K \ltimes N}$.

We denote by $\mathcal{A}(K)$ the set of all pairs (K', ρ'), where K' is a closed subgroup of K and ρ' is an irreducible representation of K'. We equip $\mathcal{A}(K)$ with the Fell topology (see [6]). Therefore, every element in $\widehat{K \ltimes N}$ can be catalogued by elements in the topological space $\widehat{N} \times \mathcal{A}(K)$. Larry Baggett has given an abstract description of the topology of the dual space of a semidirect product of a compact group with an Abelian group in terms of the Mackey parameters of the dual space (see [2], Theorem 6.2-A). The following result provides a precise and neat description of the topology of $\widehat{K \ltimes N}$.

Theorem 2. *Let* Y *be a subset of* $\widehat{K \ltimes N}$ *and* π *an element of* $\widehat{K \ltimes N}$. *Then* π *is weakly contained in* Y *if and only if there exist: a cataloguing triple* $(\chi, (K_\chi, \rho))$ *for* π, *an element* (K', ρ') *of* $\mathcal{A}(K)$, *and a net* $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ *of cataloguing triples such that:*

(i) *for each* n, *the irreducible unitary representation* $\pi(\chi_n, K_{\chi_n}, \rho_n)$ *of* $K \ltimes N$ *is an element of* Y;

(ii) *the net* $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ *converges to* $(\chi, (K', \rho'))$;

(iii) K_χ *contains* K', *and the induced representation* $\mathrm{Ind}_{K'}^{K_\chi}(\rho')$ *contains* ρ.

3. Admissible coadjoint orbits of semidirect product. We keep the notations of Section 2. Fix a non-zero linear form $\ell \in V^*$, and we consider an irreducible representation ρ_μ of K_ℓ with highest weight μ. Then the stabilizer G_ψ of $\psi = (\mu, \ell)$ in G is given by

$$G_\psi = \Big\{(k, v) \in G;\ (\mathrm{Ad}_K^*(k)\mu + k.\ell \odot v, k.\ell) = (\mu, \ell)\Big\} =$$

$$= \Big\{(k, v) \in G;\ k \in K_\ell, \mathrm{Ad}_K^*(k)\mu + \ell \odot v = \mu\Big\} =$$

$$= \Big\{(k, v) \in G;\ k \in K_\ell,\ \mathrm{Ad}_K^*(k)\mu = \mu\Big\}$$

since $i_\ell^*(\ell \odot v) = 0$ (see Lemma 1 and the equality (1)). Thus, we have $G_\psi = K_\psi \ltimes V_\psi$, then ψ is aligned (see [9]). A linear form $\psi \in \mathfrak{g}^*$ is called admissible if there exists a unitary character χ of the identity component of G_ψ such that $d\chi = i\psi_{|_{\mathfrak{g}_\psi}}$. According to Lipsman (see [9]), the representation of G obtained by holomorphic induction from (μ, ℓ) is equivalent to the representation $\pi_{(\mu,\ell)}$. Let τ_λ be an irreducible representation of K with highest weight λ, then the representation of G obtained by holomorphic induction from $(\lambda, 0)$ is equivalent to τ_λ. The coadjoint orbit of G through $(\lambda, 0) \in \mathfrak{g}^*$ is denoted by $\mathcal{O}^G_\lambda$. It is clear that $\mathcal{O}^G_\lambda$ is an admissible coadjoint orbit of G. We denote by $\mathfrak{g}^\ddagger \subset \mathfrak{g}^*$ the set of all admissible linear forms on $\mathfrak{g}$. The quotient space $\mathfrak{g}^\ddagger/G$ is called the space of admissible coadjoint orbits of G. Moreover, one can check that $\mathfrak{g}^\ddagger/G$ is the union of the set of all orbits $\mathcal{O}^G_{(\mu,\ell)}$ and the set of all orbits $\mathcal{O}^G_\lambda$.

We conclude this section by recalling needed results. Let L be a closed subgroup of K. By T_K and T_L be maximal tori, respectively, in K and L such that $T_L \subset T_K$. Their corresponding Lie algebras are denoted by $\mathfrak{t}_\mathfrak{k}$ and $\mathfrak{t}_\mathfrak{l}$. We denote by W_K and W_L the Weyl groups of K and L associated, respectively, to the tori T_K and T_L. Notice that every element $\lambda \in P_K$ takes pure imaginary values on $\mathfrak{t}_\mathfrak{k}$, where P_K is the integral weight lattice of T_K. Hence such an element $\lambda \in P_K$ can be considered as an element of $(i\mathfrak{t}_\mathfrak{k})^*$. Let C^+_K be a positive Weyl chamber in $(i\mathfrak{t}_\mathfrak{k})^*$, and we define the set P^+_K of dominant integral weights of T_K by $P^+_K := P_K \cap C^+_K$. For $\lambda \in P^+_K$, denote by $\mathcal{O}^K_\lambda$ the K-coadjoint orbit passing through the vector $-i\lambda$. It was proved by Kostant in [7], that the projection of $\mathcal{O}^K_\lambda$ on $\mathfrak{t}^*_\mathfrak{k}$ is a convex polytope with vertices $-i(w.\lambda)$ for $w \in W_K$, and that is the convex hull of $-i(W_K.\lambda)$. For the same manner, we fix a positive Weyl chamber C^+_L in $\mathfrak{t}^*_\mathfrak{l}$ and we define the set P^+_L of dominant integral weights of T_L.

Also we denote by $\imath^*_\mathfrak{l}$ the $\mathbb{C}$-linear extension of both the natural projection of $\mathfrak{k}^*$ onto $\mathfrak{l}^*$ and the natural projection of $\mathfrak{t}^*_\mathfrak{k}$ onto $\mathfrak{t}^*_\mathfrak{l}$. Consider tow irreducible representations $\tau_\lambda \in \widehat{K}$ and $\rho_\mu \in \widehat{L}$ with respective highest weights $\lambda \in P^+_K$ and $\mu \in P^+_L$. We have the following result.

Lemma 2. *If $\mu = i^*_\mathfrak{l}(s.\lambda)$ with $s \in W_K$, then τ_λ occurs in the induced representation* $\mathrm{Ind}^K_L(\rho_\mu)$.

We refer to [1], for the proof of this lemma.

4. Main results. We shall freely use the notations of the previous sections.

***Remark* 1.** We have the following convergence:

$$\ell_m \longrightarrow \ell,$$

$$K_{\ell_m} \subseteq K_\ell.$$

To study the convergence in the quotient space $\mathfrak{g}^\ddagger/G$, we need to the following result (see [8, p. 135] for the proof).

Lemma 3. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. We denote $\mathfrak{g}^*/G$ the space of coadjoint orbits and by $p_G : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e., a subset V in $\mathfrak{g}^*/G$ is open if and only if $p_G^{-1}(V)$ is open in $\mathfrak{g}^*$. Therefore, a sequence $(\mathcal{O}^G_n)_n$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}^G$ in $\mathfrak{g}^*/G$ if and only if for any $l \in \mathcal{O}^G$, there exist $l_n \in \mathcal{O}^G_n$, $n \in \mathbb{N}$, such that $l = \lim_{n\to+\infty} l_n$.*

Now, we are in position to prove the following propositions.

Proposition 2. *Let $\left(\mathcal{O}^G_{(\mu^m,\ell_m)}\right)_m$ be a sequence in $\mathfrak{g}^\ddagger/G$. If $\left(\mathcal{O}^G_{(\mu^m,\ell_m)}\right)_m$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $\mathfrak{g}^\ddagger/G$, then we have: $(\ell_m)_m$ converges to ℓ and for m large enough, $\rho_\mu \in \mathrm{Ind}^{K_\ell}_{K_{\ell_m}}(\rho_{\mu^m})$.*

Proof. We assume that the sequence of admissible coadjoint orbits $\left(\mathcal{O}^G_{(\mu^m,\ell_m)}\right)_m$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $\mathfrak{g}^\ddagger/G$. By referring to [3], we show that the coadjoint orbit $\mathcal{O}^G_{(\mu,\ell)}$ is always obtained by symplectic induction from the coadjoint orbit $M = \mathcal{O}^H_{(\mu,\ell)}$ of $H := K_\ell \ltimes V$ passing through $(\mu, \ell) \in \mathfrak{k}^*_\ell \oplus V^*$ ($\mathfrak{k}_\ell \ltimes V := \mathrm{Lie}(H)$), i.e.,

$$\mathcal{O}^G_{(\mu,\ell)} = M_{\mathrm{ind}} := J^{-1}_{\widetilde{M}}(0)/H, \tag{2}$$

where $J_{\widetilde{M}}\colon \widetilde{M} = M \times T^*G \longrightarrow \mathfrak{k}_\ell^* \ltimes V^*$ is the momentum map of $\widetilde{M}$ and the zero level set $J_{\widetilde{M}}^{-1}(0)$ is given by

$$J_{\widetilde{M}}^{-1}(0) = \Big\{ \Big((\mathrm{Ad}_K^*(k)\mu, \ell), g, (\mathrm{Ad}_K^*(k)\mu + \ell \odot v, \ell) \Big),\ k \in K_\ell, g \in G, v \in V \Big\}.$$

Let φ_M be the action of H on M, hence H acts on $\widetilde{M} = M \times T^*G$ by $\varphi_{\widetilde{M}}$ as follows:

$$\varphi_{\widetilde{M}}(h)(\alpha, g, f) = \big(\varphi_M(h)(\alpha), gh^{-1}, \mathrm{Ad}_H^*(h) f \big) \tag{3}$$

for all $h \in H, (\alpha, g, f) \in M \times T^*G$. By identifying $\mathfrak{g}^*$ with the left-invariant 1-form on G. Then we can write $T^*G \cong G \times \mathfrak{g}^*$.

Using Lemma 3 and by combining (2) with (3), then there exist sequences $k_m, h_m \in K_{\ell_m}$, $v_m, w_m \in V$, and $g_m \in G$ such that the sequence $(\phi_m)_m$ defined by

$$\phi_m = \varphi_{\widetilde{M}}(k_m, v_m)\Big((\mathrm{Ad}_K^*(h_m)\mu^m, \ell_m), g_m, (\mathrm{Ad}_K^*(h_m)\mu^m + \ell_m \odot w_m, \ell_m) \Big) =$$

$$= \Big(\mathrm{Ad}_K^*(k_m h_m)\mu^m + \imath_{\ell_m}^*(\ell_m \odot v_m), \ell_m \Big), g_m (k_m, v_m)^{-1},$$

$$(\mathrm{Ad}_K^*(k_m h_m)\mu^m + \mathrm{Ad}_K^*(k_m)(\ell_m \odot w_m) + \ell_m \odot v_m, \ell_m) \Big)$$

converges to $\big((\mu, \ell), e_G, (\mu, \ell)\big)$. It follows that

$$\ell_m \longrightarrow \ell$$

and

$$\mathrm{Ad}_K^*(k_m h_m)\mu^m + \imath_{\ell_m}^*(\ell_m \odot v_m) \longrightarrow \mu \tag{4}$$

as $n \longrightarrow +\infty$. By compactness of K we may assume that $(k_m h_m)_m$ converges to $p \in K_{\ell_n} \subset K_\ell$. By using the fact that $\imath_{\ell_m}^*(\ell_m \odot v_m) = 0$, we, from (4), obtain that

$$\mu^m = Ad^*(p^{-1})\mu$$

for m large enough. Furthermore, we known that there exists an element $s \in W_{K_\ell}$ such that $\mathrm{Ad}^*(p^{-1})\mu = s.\mu$. Hence $\mu^m = s.\mu$ for m large enough and we conclude by Lemma 2 that for m large enough, $\rho_\mu \in \mathrm{Ind}_{K_{\ell_m}}^{K_\ell}(\rho_{\mu^m})$.

Proposition 2 is proved.

Proposition 3. *If the sequence* $\big(\mathcal{O}^G_{(\mu^m, \ell_m)}\big)_m$ *converges to* $\mathcal{O}^G_\lambda$ *in* $\mathfrak{g}^\ddagger/G$, *then we have:* $(\ell_m)_m$ *converges to* 0 *and for* m *large enough,* $\tau_\lambda \in \mathrm{Ind}_{K_{\ell_m}}^{K}(\rho_{\mu^m})$.

Proof. We use the notations and proceedings of the proof of the last proposition. Let us assume that the sequence $\big(\mathcal{O}^G_{(\mu^m, \ell_m)}\big)_m$ converges to $\mathcal{O}^G_\lambda$. Then there exist sequences $k_m, h_m \in K_{\ell_m}$, $v_m, w_m \in V$, and $g_m \in G$ such that the sequence $(\Psi_m)_m$ defined by

$$\Psi_m = \varphi_{\widetilde{M}}(k_m, v_m)\Big((\mathrm{Ad}_K^*(h_m)\mu^m, \ell_m), g_m, (\mathrm{Ad}_K^*(h_m)\mu^m + \ell_m \odot w_m, \ell_m) \Big) =$$

$$= \Big(\mathrm{Ad}_K^*(k_m h_m)\mu^m + \imath_{\ell_m}^*(\ell_m \odot v_m), \ell_m \Big), g_m (k_m, v_m)^{-1},$$

$$(\mathrm{Ad}_K^*(k_m h_m)\mu^m + \mathrm{Ad}_K^*(k_m)(\ell_m \odot w_m) + \ell_m \odot v_m, \ell_m) \Big)$$

converges to $\big((\lambda,0),e_G,(\lambda,0)\big)$. From the above facts, we conclude the following convergence:

$$\ell_m \longrightarrow 0, \tag{5}$$

$$\mathrm{Ad}^*(k_m h_m)\mu^m \longrightarrow \lambda. \tag{6}$$

By assumption that the sequence $(k_m h_m)_m$ converges to $p \in K_{\ell_m}$, we obtain, from (6), that $\mu^m = \mathrm{Ad}^*(p^{-1})\lambda$ for m large enough. Hence there exists $w \in W_K$, such that $\mu^m = w.\lambda$ for m large enough. Lemma 2 allows us to derive that $\tau_\lambda \in \mathrm{Ind}_{K_{\ell_m}}^K(\rho_{\mu^m})$ for large m.

Proposition 3 is proved.

Proposition 4. *If $(\mathcal{O}^G_{\lambda^m})_m$ converges to $\mathcal{O}^G_\lambda$ in $\mathfrak{g}^\ddagger/G$, then $\lambda^m = \lambda$ for large m.*

Proof. Suppose that $(\mathcal{O}^G_{\lambda^m})_m$ converges to $\mathcal{O}^G_\lambda$ in $\mathfrak{g}^\ddagger/G$, then there exists $(k_m)_m \subset K$ such that

$$\mathrm{Ad}^*_K(k_m)\lambda^m \longrightarrow \lambda \quad \text{as} \quad m \longrightarrow +\infty.$$

By compactness of K we may assume that $(k_m)_m$ converges to $k \in K$. Then we obtain $\lambda^m = \mathrm{Ad}^*_K(k^{-1})\lambda$ for m large enough. Hence there exists $w \in W_K$ such that $\mathrm{Ad}^*_K(k^{-1}) = w.\lambda$ for m large enough. It follows that $\lambda^m = w.\lambda$ for m large enough. Since the weights λ^m and λ are contained in the set iC^+_K and since each W_K-orbit in $\mathfrak{k}^*$ intersects the closure $\overline{iC^+_K}$ in exactly one point, it follows that $\lambda^m = \lambda$ for m large enough.

Proposition 4 is proved.

Combining the above Propositions 2, 3 and 4 with Baggett's theorem (Theorem 2), we obtain our result (Theorem 1).

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