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K. Wilderotter (Univ. Bonn, Germany)

ON n-WIDTHS OF BOUNDED PERIODIC HOLOMORPHIC FUNCTIONS

The evendimensional Kolmogorov widths d_{2n}, Gel'fand widths d^{2n} and linear widths δ_{2n} of $\tilde{A}$ in $\tilde{L}_q$ and $\tilde{C}$ are determined exactly. It is shown that all three n-widths are equal and a characterization of the widths in terms of Blaschke products is given.

Точно визначені колмогоровські поперечники d_{2n}, поперечники Гельфанда d^{2n} та лінійні поперечники δ_{2n} звуження $\tilde{A}$ в просторах $\tilde{L}_q$ і $\tilde{C}$. Показано, що всі ці n-поперечники рівні та наведено характеристики цих поперечників у термінах добутків Бляшке.

1. Introduction and Statement of Results. The Kolmogorov n-widths of a subset A of a Banach space X is defined by

$$d_n(A, X) = \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|,$$

where X_n runs over all n-dimensional subspaces of X.

The Gel'fand n-width of A in X is defined by

$$d^n(A, X) = \inf_{L^n} \sup_{x \in L^n \cap A} \|x\|,$$

where L^n runs over all subspaces of codimension n.

The linear n-width of A in X is given by

$$\delta_n(A, X) = \inf_{P_n} \sup_{x \in A} \|x - P_n x\|,$$

where P_n varies over all linear operators of rank n, which map X into itself.

Much information on where n-widths may by found in the book of A. Pinkus [1]. In particular, the following fundamental inequality is always valid:

$$d^n(A, X), \quad d_n(A, X) \le \delta_n(A, X). \tag{1}$$

In the present paper we determine the evendimensional n-widths of the following class of analytic functions. Let $S_\beta = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta\}$, and let $\tilde{H}_\beta^\infty$ denote the space of functions f, which are analytic in S_β, real and 2π-periodic on the x-axis, and satisfy $\|f\|_{\tilde{H}_\beta^\infty} := \sup\{|f(z)| : z \in S_\beta\} < \infty$. Let $\tilde{A}$ be the restriction of the real axis of the unit ball of $\tilde{H}_\beta^\infty$. We seek the value of the n-widths of $\tilde{A}$ in the Banach space $\tilde{X}$, where $\tilde{X}$ represents either

$$\tilde{L}_q = \left\{ f\colon \mathbb{R} \to \mathbb{R},\ f \ 2\pi\text{-periodic and } \|f\|_q = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^q dx \right)^{1/q} < \infty \right\},$$

$$1 \le q < \infty,$$

or

$$\tilde{C} = \{ f\colon \mathbb{R} \to \mathbb{R},\ f \ 2\pi\text{-periodic and continuous} \}.$$

We emphasize that all these spaces are real, but not complex vector spaces.

Our approach to periodic functions will consist in transfering the analysis from the strip S_β to the annulus $\Omega_R = \{w \in \mathbb{C}: R < |w| < R^{-1}\}$, where $R = e^{-\beta}$. The transformation $w = e^{iz}$ maps S_β onto Ω_R and the operator $U: f(z) \to g(w) = f((1/i)\ln(w))$ yields an isometry between $\tilde{H}^\infty_\beta$ and $H^\infty(\Omega_R)$, the space of all functions g, which are analitic in Ω_R, real-valued on the unit circle $E = \{w \in \mathbb{C}: |w| = 1\}$, and satisfy $\|g\|_{H^\infty(\Omega_R)} := \sup\{|g(w)|: w \in \Omega_R\} < \infty$. Furthemore U maps the spaces $\tilde{L}_q$ and $\tilde{C}$ isometrically onto the corresponding spaces $L_q(E)$ and $C(E)$, respectively. Denoting by A the unit ball in $H^\infty(\Omega_R)$ and by X either $L_q(E)$, $1 \le q < \infty$, or $C(E)$, we see that the n-widths of $\tilde{A}$ in $\tilde{X}$ are equal to the n-widths of A in X. In the following we will concentrate on the later problem.

In order to determine the n-widths of A in X we generalize a technique of Fisher and Micchelli [2] (see also [3] and [4]). For this purpose we need some more terminology from complex analysis. Let $g(z, \zeta)$ be the Green's function for Ω_R with singularity at ζ, that is, $g(z, \zeta)$ is harmonic and positive on Ω_R except for a logarithmic pole at ζ and $g(z, \zeta) = 0$, if $z \in \partial\Omega_R$. Let $h(z, \zeta)$ be the harmonic conjugate of $g(z, \zeta)$ on Ω_R; h is not single-valued. Its period about $\Gamma^{(1)}$ is given by $2\pi\omega(\zeta)$. Here $\Gamma^{(1)}$ and $\Gamma^{(0)}$ denote the inner and outer boundary of Ω_R, and ω is the harmonic measure of $\Gamma^{(1)}$, i. e. ω is the unique harmonic function in Ω_R with constant boundary values 1 on $\Gamma^{(1)}$ and 0 on $\Gamma^{(0)}$; ω is given explicitly by

$$\omega(\zeta) = \frac{\ln|\zeta| + \ln R}{2\ln R} \tag{2}$$

and possesses the important symmetry property

$$\omega(\zeta) + \omega\left(\bar{\zeta}^{-1}\right) = 1, \quad \zeta \in \Omega_R, \quad \omega(\zeta) = 1/2, \quad \zeta \in E. \tag{3}$$

A Blaschke product B of degree m on Ω_R is a function of the form

$$B(z) = \lambda \exp\left(-\sum_{j=1}^{m} P(z, \zeta_j)\right),$$

where $|\lambda| = 1$ and $\zeta_1, \ldots, \zeta_m$ are points in Ω_R, not necessarily distinct, and $P(z, \zeta) = g(z, \zeta) + ih(z, \zeta)$, is the complex Green's function. In general B is multiple-valued; B is single-valued if and only if

$$\sum_{j=1}^{m} \omega(\zeta_j) \in \mathbb{N}. \tag{4}$$

Finally we denote by

$$\mathbb{B}_{2n} = \{B_{2n}(z) = \exp(-P(z, \zeta_1) - \ldots - P(z, \zeta_{2n})) \text{ with } \zeta_1, \ldots, \zeta_{2n} \in E\}$$

the set of all Blaschke products of degree $2n$, all whose zeros lie on E. After scaling we may assume that $B_{2n}(z)$ is real for all $z \in E$. Furthermore, all functions in $\mathbb{B}_{2n}$ are single-valued in view of (3) and (4).

We are now ready to formulate our first main result.

Theorem 1.

$$d_{2n}(A,X) = d^{2n}(A,X) = \delta_{2n}(A,X) = \inf\{\|B\|_X\colon B \in \mathbb{B}_{2n}\}.$$

As already mentioned, our proof of Theorem 1 is built on the technique of Fisher and Micchelli [2]. Indeed, the upper estimate $\delta_{2n}(A,X) \le \inf\{\|B\|_X\colon B \in \mathbb{B}_{2n}\}$ follows immediately from the cited paper. However, Fisher and Micchelli could only establish a lower bound of the form $\inf\{\|B\|_X\colon B \in \mathbb{B}_{2n+1}\} \le d_{2n}(A,X)$, where $\mathbb{B}_{2n+1}$ denotes the set of single-valued Blaschke products of degree at most $2n+1$. Theorem 1 closes the gap between the lower and upper bound of Fisher and Micchelli under the additional assumption that all arising functions are real-valued on the unit circle E.

In the case $X = C(E)$, we are able to show that the infimum in Theorem 1 is attained by the Blaschke product B^*, whose nodes are equidistant.

Theorem 2. *Set* $\zeta_k^* = \exp(i(2k-1)\pi/2n)$ *for* $k = 1, \ldots, 2n$ *and* $B^*(z) = \exp\left(-P(z, \zeta_1^*) - \ldots - P(z, \zeta_{2n}^*)\right)$. *Then*

$$\|B^*\|_{C(E)} = \inf\{\|B\|_{C(E)}\colon B \in \mathbb{B}_{2n}\}.$$

Finally we determine the asymptotic behavior of $d_{2n}(A, C(E))$ and establish an interesting connection between the widths $d_{2n}(A, C(E))$ and $d_n(A(H^\infty(G)), C[-1,1])$. Here G denotes the interior of the ellipse with foci at the points ∓ 1 and sum of semi-axes $c = e^\beta$, and $H^\infty(G)$ is Hardy space of bounded analytic functions on G with unit ball $A(H^\infty(G))$.

Theorem 3.

$$d_{2n}(A, C(E)) = d_{2n}(\tilde{A}, \tilde{C}) = d_n(A(H^\infty(G)), C[-1,1]).$$

The same equation holds for the linear and Gel'fand widths. Asymptotically we have $d_{2n}(A, C(E)) = 2R^n + O(R^{5n})$.

2. Proof of the Theorems.

Proof of Theorem 1. As mentioned in the introduction, the upper bound $\delta_{2n}(A, X) \le \inf\{\|B\|_X\colon B \in \mathbb{B}_{2n}\}$ follows directly from [2].

In order to establish the lower bound for $d_n(A,X)$ and $d^n(A,X)$, we need the following version of the Pick–Nevanlinna interpolation theorem for the space $H^\infty(\Omega_R, \mathbb{C})$. By definition $H^\infty(\Omega_R, \mathbb{C})$ consists of all complex-valued bounded holomorphic functions on Ω_R. In contrast to $H^\infty(\Omega_R)$, functions in $H^\infty(\Omega_R, \mathbb{C})$ are not necessarily real-valued on the unit circle E.

Theorem 4. *Fix* $2n+1$ *distinct points* $z_0, \ldots, z_{2n}$ *in* E *and let* $t_0, \ldots, t_{2n}$ *be* $2n+1$ *real numbers with* $\sum_{j=0}^{2n} |t_j|^2 = 1$. *Then the vector* $t = (t_0, \ldots, t_{2n})$ *belongs to the unit sphere* S^{2n} *of* $\mathbb{R}^{2n+1}$. *Set* $\rho(t) = \inf\{\|f\|_{H^\infty}\colon f \in H^\infty(\Omega_R, \mathbb{C}),\ f(z_j) = t_j,\ 0 \le j \le 2n\}$. *Then we have:*

(i) ρ *is a continuous function on* S^{2n}.

(ii) There is a unique function $B_t \in H^\infty(\Omega_R, \mathbb{C})$ *with* $\|B_t\| = 1$ *and* $B_t(z_j) = t_j/\rho(t)$, $0 \le j \le 2n$.

(iii) The function B_t *is a single-valued Blaschke product of degree at most* $2n+1$.

(iv) $V: t \to B_t$ *is a continuous mapping from* S^{2n} *into the set* $\mathbb{B}_{2n+1}$ *of single-valued Blaschke products of degree at most* $2n+1$, *when* $\mathbb{B}_{2n+1}$ *is given the topology of uniform convergence on compact subsets of* Ω_R.

For a detailed exposition of the Pick – Nevanlinna theorem we refer to ([3], Chapter 5).

Theorem 4 gurantees the existence of a unique Blaschke product $B_t \in \mathbb{B}_{2n+1}$ interpolating the data $t_j/\rho(t)$ with minimal H^∞-norm. A priori it is possible that B_t is complex-valued on E. However, since the data t are real, the Schwarz reflection principle implies, that the function $\overline{B_t(1/\bar{z})}$ is a minimal interpolant as well. In view of the uniqueness of the minimal interpolant we conclude that $B_t(z) = \overline{B_t(1/\bar{z})}$. Therefore B_t is real-valued on E and its zeros are located symmetrically with respect to E. We numerate the zeros in the following order:

$$z_1,\ 1/\bar{z}_1, \ldots,\ z_l,\ 1/\bar{z}_l,\ z_{l+1}, \ldots,\ z_k,$$

where $z_1, \ldots, z_l \notin E$ and $z_{l+1}, \ldots, z_k \in E$. By (3) we have $\omega(z_j) + \omega(1/\bar{z}_j) = 1$ for $j = 1, \ldots, l$, $\omega(z_j) = 1/2$ for $j = l+1, \ldots, k$. Since B_t is single-valued, we obtain from (4) that

$$\sum_{j=1}^{l} \left(\omega(z_j) + \omega(1/\bar{z}_j)\right) + \sum_{j=l+1}^{k} \omega(z_j) \in \mathbb{N}.$$

This is possible only if $k-l$ is an even number. Consequently B_t possesses always an even number of zeros and in particular the degree of B_t must be less or equal $2n$.

Let us denote by $\hat{\mathbb{B}}_{2n}$ the set of single-valued Blaschke products, which are real-valued on E with an even number of zeros less or equal $2n$, which are located on E or symmetrically with respect to E. As a result of the preceding analysis we obtain a mapping

$$V: S^{2n} \to \hat{\mathbb{B}}_{2n}, \quad t \to B_t;$$

V is a continuous odd mapping, when $\hat{\mathbb{B}}_{2n}$ has the topology of local uniform convergence.

Having established the existence of V, we use now the same technique based on Borsuk's theorem like Fisher and Micchelli [2] to conclude that

$$\inf\{\|B\|_X : B \in \hat{\mathbb{B}}_{2n}\} \le d_{2n}(A, X),\ d^{2n}(A, X).$$

From the last inequality we see that Theorem 1 will be proved, if we manage to show that

$$\inf\{\|B\|_X : B \in \hat{\mathbb{B}}_{2n}\} = \inf\{\|B\|_X : B \in \mathbb{B}_{2n}\}.$$

For this purpose we remark, that, since $\hat{\mathbb{B}}_{2n}$ is compact with respect to local uniform convergence on Ω_R, there exist a $B^* \in \hat{\mathbb{B}}_{2n}$ such that $\|B^*\|_X = \inf\{\|B\|_X : B \in \hat{\mathbb{B}}_{2n}\}$; B^* must possess $2n$ zeros, counted with multiplicities. Indeed, assuming that the number of zeros of B^* were less than $2n$, the Blaschke product

$\exp(-2P(z,1))B^*(z)$ would belong to $\hat{\mathbb{B}}_{2n}$. Since $|\exp(-2P(z,1))| = |\exp(-2g(z,1))| < 1$ on E, we obtain the contradiction $\|\exp(-2P(z,1))B^*(z)\|_X < \|B^*(z)\|_X$. Hence B^* has $2n$ zeros $z_1^*, \ldots, z_{2n}^*$.

We claim that all zeros of B^* lie on E. Again we assume the contrary to be true: Suppose, the zeros z_1^* does not lie on E. Since zeros, which are not located on E, always occur in pairs, $z_2^* = \overline{1/z_1^*}$ is a zero of B^* as well. After rotation we may assume $z_1^*, z_2^* \in (R, 1/R)$. For $z \in E$ we obtain by symmetry that $|\exp(-P(z, z_1^*) - P(z, z_2^*))| = \exp(-2g(z, z_1^*))$. Since for fixed $z \in E$ and variable $\alpha \in (R, 1/R)$ the function $g(z, \alpha)$ attains its maximum in $\alpha = 1$, we conclude that $|\exp(-2P(z, 1))| < |\exp(-P(z, z_1^*) - P(z, z_2^*))|$. Consequently, if we set $B^{**}(z) = \exp\left(-P(z,1) - P(z,1) - P(z, z_3^*) - \ldots - P(z, z_{2n}^*)\right)$, we arrive at the contradiction $\|B^{**}\|_X < \|B^*\|_X$. Hence all zeros of B^* lie on E. This observation completes the proof of Theorem 1.

Proof *of Theorem* 2 is based essentially on the following theorem.

Theorem 5. *Let B_1 and B_2 be two Blaschke products in $\mathbb{B}_{2n}$. Let $\alpha_1, \ldots, \alpha_{2n}$ be $2n$ distinct points on E. If $B_1(\alpha_k) = B_2(\alpha_k)$ for $k = 1, \ldots, 2n$ and $B_1(1/R) = B_2(1/R)$, then $B_1(z) = B_2(z)$ for all $z \in \Omega_R$, i. e. B_1 and B_2 coincide.*

Proof *of Theorem* 5. By the Schwarz reflection principle we can continue both Blaschke products across the boundary $\partial\Omega_R$ onto a domain $\hat{\Omega}$ with $\Omega_R \subset \hat{\Omega}$.

For each $\varepsilon > 0$, and all $z \in \partial\Omega_R$,

$$|(1+\varepsilon)B_1(z)| > |-B_2(z)| = |[(1+\varepsilon)B_1(z) - B_2(z)] - (1+\varepsilon)B_1(z)|.$$

Thus by Rouche's Theorem the functions $(1+\varepsilon)B_1 - B_2$ and $(1+\varepsilon)B_1$ have the same number of zeros in Ω_R, namely $2n$.

The function $(1+\varepsilon)B_1 - B_2$ converges locally uniformly on $\hat{\Omega}$ to $B_1 - B_2$ for $\varepsilon \to 0$. Suppose that $B_1 - B_2$ is not identically zero. Then by a well known result from complex analysis each function $(1+\varepsilon)B_1 - B_2$ has at least $2n+1$ zero in $\hat{\Omega}$ for sufficiently small ε, which converge to the $2n+1$ zeros $\alpha_1, \ldots, \alpha_{2n}$ and $1/R$ of $B_1 - B_2$. Since $(1+\varepsilon)B_1 - B_2$ has exactly $2n$ zeros in Ω_R, one sero z_ε must lie in $\hat{\Omega} \setminus \Omega_R$. Let $(1/R^2)(1/\bar{z}_\varepsilon)$ be the point in Ω_R, which is conjugate to z_ε with respect to the outer boundary of Ω_R. Because of the Schwarz reflection principle the equation $(1+\varepsilon)B_1(z_\varepsilon) - B_2(z_\varepsilon) = 0$ implies the equation

$$(1+\varepsilon)B_2\left((1/R^2)(1/\bar{z}_\varepsilon)\right) - B_1\left((1/R^2)(1/\bar{z}_\varepsilon)\right) = 0.$$

Hence the function $(1+\varepsilon)B_2 - B_1$ possesses the zero $(1/R^2)(1/\bar{z}_\varepsilon)$ in Ω_R, which tends to $1/R$ for $\varepsilon \to 0$. On the other hand, repeating the above analysis once more, we see that $(1+\varepsilon)B_2 - B_1$ has exactly $2n$ zero in Ω_R, which converge to $\alpha_1, \ldots, \alpha_{2n}$. This is a contradiction. Hence $B_1 - B_2$ must be identically zero and Theorem 5 is proved.

In order to prove Theorem 2 we denote by B^* the Blaschke product with equidistant nodes $z_1^*, \ldots, z_{2n}^*$. Let B be an arbitrary other Blaschke product in

$\mathbb{B}_{2n}$. After rotating we may assume that $B^*(1/R) = B(1/R)$. Furthermore both B^* and B are real-valued on E.

The decisive idea of proof relies on the fact that B^* has an extremal alternant consisting of $2n$ points. Therefore the assumption $\|B\|_{C(E)} < \|B^*\|_{C(E)}$ implies that $B^* - B$ has at least $2n$ zeros on E. Since furthermore $B^*(1/R) = B(1/R)$, we obtain $B^* = B$ and the minimal property of B^* is proved.

Proof *of Theorem* 3. In order to determine the asymptotic behavior of $\|B^*\|_{C(E)}$, we transform the domain of definition Ω_R of B^* twice. In the first step we pass over from $B^*(z)$ to $B^*(e^{iw})$; which is a periodic, even and analytic function in the strip S_β with zeros in $w_k^* = (2k-1)\pi/2n$, $k = 1, \dots, 2n$.

In the secon step the change of variables $w \to v = \cos(w)$ maps S_β analytically onto the interior of the ellipse G with foci at the points ∓ 1 and sum of semi-axes $c = e^\beta$. Furthermore, the transformation $F(v) \to f(w) = F(\cos(w))$ yields a one-to-one correspondence between functions F analytic in G and functions f, which are periodic, even and analytic in S_β. Consequently $\Phi^*(v) = B^*(\exp(i \arccos(v)))$ is a well defined analytic function on G with the following properties:

(i) $\Phi^*(v) = 1$ for $v \in \partial G$;

(ii) $\Phi^*\left(\cos((2k-1)\pi/2n)\right) = 0$, $k = 1, \dots, n$.

Hence Φ^* is a Blaschke product of degree n on the simply connected domain G with zeros in the Chebyshev nodes of the interval $[-1, 1]$. Osipenko [5] showed that

$$\|\Phi^*\|_{C[-1,1]} =$$

$$= \inf\{\|\Phi\|_{C[-1,1]} : \Phi \text{ is a Blaschke product on } G \text{ of degree at most } n\} =$$

$$= d_n\left(A(H^\infty(G)), C[-1,1]\right).$$

Furthermore Osipenko [5] gave an explicit formula for $\|\Phi^*\|_{C[-1,1]}$ in terms of elliptic functions. In particular he established the following asymptotic behavior:

$$\|\Phi^*\|_{C[-1,1]} = 2c^{-n} + O(c^{-5n}).$$

Since $c = e^\beta$ and $\beta = \ln(1/R)$, we obtain the statement of Theorem 3 by using the inverse transformation.

The results of this paper are part of the author's dissertation [6].

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