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BOGOLYUBOV AVERAGING AND PROCEDURES OF NORMALIZATION IN NONLINEAR MECHANICS. III*

УСЕРЕДНЕННЯ ЗА БОГОЛЮБОВИМ ТА ПРОЦЕДУРИ НОРМАЛІЗАЦІЇ У НЕЛІНІЙНІЙ МЕХАНІЦІ. III*

This paper is devoted to exposition of asymptotic decomposition technique of normalization in the space of the representation of finite Lie groups. The main topics of the theory necessary for understanding the method are outlined. The models based on Van der Pol equations have been investigated by using the method of asymptotic decomposition in the space of homogeneous polynomials (the space of representation of a general linear group in the plane) and the space of representation of the rotation group in the plane (usual Fourier series). The comparison made shows dramatical decrease of the necessary algebraic manipulations in the second case. Other details of asymptotic decomposition technique of normalization are discussed.

Наведена техніка нормалізації за методом асимптотичної декомпозиції у просторі зображення скінченновимірної групи Лі. Стисло викладені для розуміння методу теоретичні положення. Моделі, що грунтуються на рівнянні Ван дер Поля, вивчені за методом асимптотичної декомпозиції у просторі однорідних поліномів (простір зображення загальної лінійної групи на площині) та у просторі зображення групи обертання на площині (звичайні ряди Фур'є). Проведене порівняння виявляє драматичне зменшення необхідної кількості алгебраїчних обчислень у останньому випадку. Обговорюються також інші деталі техніки нормалізації за методом асимптотичної декомпозиції.

1. Models based on Van der Pol equations. Consider the system of two equations of the first order, which is equivalent to the Van der Pole equation

$$\dot{x}_1' = x_2'; \quad \dot{x}_2' = -x_1' + \varepsilon\left(1 - \dot{x}_2'^2\right)x_2'. \tag{1}$$

The differential operator associated with the system is

$$U_0' = U' + \varepsilon\tilde{U}',$$

where

$$U' = x_2' \frac{\partial}{\partial x_1'} - x_1' \frac{\partial}{\partial x_2'}, \quad \tilde{U}' = \left(x_2' - x_1'^2 x_2' \right) \frac{\partial}{\partial x_2'}.$$

Wrire these operators in the form $U' = \hat{\tilde{x}}_{m_1}' \mathcal{F}\partial'$, $\mathcal{F} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}$. Represent the operator $\tilde{U}$ as the sum

$$\tilde{U}' = \tilde{U}_{\otimes 1}' + \tilde{U}_{\otimes 3}', \quad \tilde{U}_{\otimes i}' \in \mathcal{B}(V_{\otimes i}), \quad i = 1, 3, \tag{2}$$

where

$$\tilde{U}_{\otimes 1}' = \hat{\tilde{x}}_{m_1}' Q_{m_1,1}\partial, \quad \tilde{U}_{\otimes 3}' = \hat{\tilde{x}}_{m_3}' Q_{m_3,1}\partial,$$

$$Q_{m_1,1} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, \quad Q_{m,1}^T = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}.$$

Calculate two approximations in the transformed operator formula (22) [3]

$$U_0' = U' + \varepsilon N_1' + \varepsilon^2 N_2'. \tag{3}$$

Calculate the operators S_1 and S_2, which can be obtained from the equations

$$[U, S_1] = \tilde{U} - \mathrm{pr}\,\tilde{U},$$

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$$[U, S_2] = \left\{ -[\bar{U}, S_1] - 1/2\left[S_1, [U, S_1] \right] \right\} - \mathrm{pr}\{\ldots\} \tag{4}$$

upon the change of variables (20) [3]. Solve these equations in two steps. First, we find S_1:

$$S_1 \equiv S_{\otimes 11} + S_{\otimes 31}, \quad S_{\otimes i1} \in \mathcal{B}(V_{\otimes i}), \quad i = 1, 3,$$

where $S_{\otimes i1} \equiv \hat{x}_{m_i} \Gamma_{1i} \partial$, $i = 1, 3$; Γ_{1i} are the rectangular matrices of the dimensions $m_i \times n$, which are the solutions of the system of independent algebraic equations

$$\mathcal{F}_i \Gamma_{1i} - \Gamma_{1i} \mathcal{F} = Q_{m_i,1} - \mathrm{pr}\, Q_{m_i,1}, \quad \mathcal{F} = \mathcal{A}^T, \quad i = 1, 3. \tag{5}$$

At the second step, we find S_2. We can see that $S_2 \in \mathcal{B}(V_{\otimes 5})$ implies the structure of the right-hand sides of equation (4). We have to find a solution in the form of the sum

$$S_2 = \sum_{i=1}^{5} S_{\otimes i2}, \quad S_{\otimes i1} = \hat{x}_{m_i} \Gamma_{2i} \partial, \quad i = \overline{1,5},$$

where Γ_{2i} are solutions of the system of algebraic equations

$$\mathcal{F}_i \Gamma_{2i} - \Gamma_{2i} \mathcal{F} = Q_{m_i,2} - \mathrm{pr}\, Q_{m_i,2}, \quad i = \overline{1,5}. \tag{6}$$

Conduct the necessary calculations.

The first approximation. Consider equation (5). The matrices $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ of the representation of the operator U in the subspaces $V_{\otimes 1}$, $V_{\otimes 2}$, $V_{\otimes 3}$ are equal to

$$\mathcal{F}_1 = \mathcal{F} = \mathcal{A}^T = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \quad \mathcal{F}_2 = \begin{Vmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 0 \end{Vmatrix},$$

$$\mathcal{F}_2 = \begin{Vmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{Vmatrix},$$

respectively. The matrices $Q_{m_1,1}$, $Q_{m_3,1}$ are defined by the equalities (2).

Pass from equations (5) to the equations in the spaces $\hat{R}^{(m_1,n)}$, $\hat{R}^{(m_2,n)}$

$$G_{\mathcal{F}}^{(i)} \hat{\Gamma}_{i1} = \hat{Q}_{m_i,1} - \mathrm{pr}\, \hat{Q}_{m_i,1}, \tag{7}$$

where

$$G_{\mathcal{F}}^{(i)} = \mathcal{F}_i \otimes \mathcal{E}_2 - \mathcal{E}_{m_i} \otimes \mathcal{A}^T, \quad i = 1, 3.$$

$$G_{\mathcal{F}}^{(1)} = \begin{Vmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{Vmatrix},$$

$$G_{\mathcal{F}}^{(3)} = \begin{Vmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{Vmatrix}.$$

$\hat{\Gamma}_{i1}$, $\hat{Q}_{m_i,1}$ are vector columns composed of rows of matrices Γ_{1i}, Q_{m_i1}.

To find $\mathrm{pr}\,\hat{Q}_{m_i,1}$, we have to find the bases of the subspaces $\hat{N}_{\mathcal{J}}^{(1)}$, $\hat{N}_{\mathcal{J}}^{(3)}$ which are the kernels of the operators $G_{\mathcal{J}}^{(1)}$, $G_{\mathcal{J}}^{(3)}$, given by independent solutions of the equations $G_{\mathcal{J}}^{(i)}\hat{Z} = \bar{0}$, $i = 1, 3$. It is necessary to find the bases of the subspaces $\hat{N}_{\mathcal{J}}^{(1)*}$, $\hat{N}_{\mathcal{J}}^{(3)*}$ which are the kernels of the operators $G_{\mathcal{J}}^{(1)T}$, $G_{\mathcal{J}}^{(3)T}$, given by independent solutions of the equations $G_{\mathcal{J}}^{(i)T}\hat{Z} = 0$, $i = 1, 3$. Represent the results of the calculations

$$\hat{N}_{\mathcal{J}}^{(1)}:\ \hat{Z}_{11} = \mathrm{col}\,\|0, 1, -1, 0\|, \quad \hat{N}_{\mathcal{J}}^{(3)}:\ \hat{Z}_{31} = \mathrm{col}\,\|1, 0, 0, 1, 1, 0, 0, 1\|,$$
$$\hat{Z}_{12} = \mathrm{col}\,\|1, 0, 0, 1\|, \qquad \hat{Z}_{32} = \mathrm{col}\,\|0, -1, 1, 0, 0, -1, 1, 0\|,$$
$$\hat{N}_{\mathcal{J}}^{(1)*}:\ \hat{Z}_{11*} = \mathrm{col}\,\|1, -1, 1, 0\|, \quad \hat{N}_{\mathcal{J}}^{(3)*}:\ \hat{Z}_{31*} = \mathrm{col}\,\|3, 0, 0, 1, 1, 0, 0, 3\|,$$
$$\hat{Z}_{12*} = \mathrm{col}\,\|1, 0, 0, 1\|, \qquad \hat{Z}_{32} = \mathrm{col}\,\|0, 3, -1, 0, 0, 1, -3, 0\|.$$

Find the projection of the vector $\hat{Q}_{m_1,1} = \mathrm{col}\,\|0, 0, 0, 1\|$. Denote the projection $\hat{Q}_{m_1,1}$ by $\mathrm{pr}\,\hat{Q}_{m_1,1N}$: $\hat{Q}_{m_1,1N} = \alpha_{11}\hat{Z}_{11} + \alpha_{11}\hat{Z}_{12}$. Taking into account that the difference $\hat{Q}_{m_1,1} - \hat{Q}_{m_1,1N}$ belongs to the image $T_{\mathcal{J}}^{(1)}$ of the operator $G_{\mathcal{J}}^{(1)}$ and is orthogonal to the space $\hat{N}_{\mathcal{J}}^{(1)}$, we obtain the system of linear algebraic equations for the coefficients α_{11}, α_{12}:

$$\langle \hat{Z}_{11}, \hat{Z}_{11*}\rangle \alpha_{11} + \langle \hat{Z}_{12}, \hat{Z}_{11*}\rangle \alpha_{12} = \langle \hat{Q}_{m_1,1}, \hat{Z}_{11*}\rangle,$$
$$\langle \hat{Z}_{11}, \hat{Z}_{12*}\rangle \alpha_{11} + \langle \hat{Z}_{12}, \hat{Z}_{12*}\rangle \alpha_{12} = \langle \hat{Q}_{m_1,1}, \hat{Z}_{12*}\rangle.$$

The values of the scalar products are equal

$$\langle \hat{Z}_{11}, \hat{Z}_{11*}\rangle = -2, \quad \langle \hat{Z}_{12}, \hat{Z}_{12*}\rangle = 0, \quad \langle \hat{Z}_{11}, \hat{Z}_{12*}\rangle = 0, \quad \langle \hat{Z}_{12}, \hat{Z}_{12*}\rangle = 2,$$
$$\langle \hat{Q}_{m_1,1}, \hat{Z}_{11}\rangle = 0, \quad \langle \hat{Q}_{m_1,1}, \hat{Z}_{12*}\rangle = 1.$$

Finally, after simple calculations, we obtain $\alpha_{11} = 0$, $\alpha_{12} = 1/2$. Hence,

$$\mathrm{pr}\,\hat{Q}_{m_1,1} \equiv \hat{Q}_{m_1,1N} \equiv \mathrm{col}\,\|1/2, 0, 0, 1/2\|. \tag{8}$$

Thus, we have defined the right-hand side of the first equation in system (7)

$$G_{\mathcal{J}}^{(1)}\hat{\Gamma}_{11} = \hat{Q}_{m_1,1T} = \hat{Q}_{m_1,1} - \mathrm{pr}\,\hat{Q}_{m_1,1} = \mathrm{col}\,\|-1/2, 0, 0, 1/2\|,$$
$$\hat{\Gamma}_{11} \equiv \mathrm{col}\,\|0, 1/4, 1/4, 0\|. \tag{9}$$

Pass to the solution of the second equation in system (7). Find the projection of the vector $\hat{Q}_{m_3,1} = \mathrm{col}\,\|0, 0, 0, -1, 0, 0, 0, 0\|$ in the form

$$\mathrm{pr}\,\hat{Q}_{m_3,1} \equiv \hat{Q}_{m_3,1} = \alpha_{31}\hat{Z}_{31} + \alpha_{32}\hat{Z}_{32}$$

from the condition of orthogonality of the difference $\hat{Q}_{m_3,1} - \mathrm{pr}\,\hat{Q}_{m_3,1}$ to subspace $\hat{N}_{\mathcal{J}}^{(3)*}$. This condition also implies the system of linear algebraic equations

$$\alpha_{31}\langle \hat{Z}_{31}, \hat{Z}_{31*}\rangle + \alpha_{32}\langle \hat{Z}_{32}, \hat{Z}_{31*}\rangle = \langle \hat{Q}_{m_3,1}, \hat{Z}_{31*}\rangle,$$
$$\alpha_{31}\langle \hat{Z}_{31}, \hat{Z}_{32*}\rangle + \alpha_{32}\langle \hat{Z}_{32}, \hat{Z}_{32*}\rangle = \langle \hat{Q}_{m_3,1}, \hat{Z}_{32*}\rangle.$$

Here,

$$\langle \hat{Z}_{31}, \hat{Z}_{31*} \rangle = 8, \quad \langle \hat{Z}_{31}, \hat{Z}_{32*} \rangle = 0, \quad \langle \hat{Z}_{32}, \hat{Z}_{31*} \rangle = 0, \quad \langle \hat{Z}_{32}, \hat{Z}_{32*} \rangle = -8,$$
$$\langle \hat{Q}_{m_3,1}, \hat{Z}_{31*} \rangle = -1, \quad \langle \hat{Q}_{m_3,1}, \hat{Z}_{32*} \rangle = 0.$$

After solving system (7), we obtain $\alpha_{31} = 1/8$, $\alpha_{32} = 0$ and, hence,

$$\operatorname{pr} \hat{Q}_{m_3,1} \equiv \hat{Q}_{m_3,1N} = \operatorname{col} \| -1/8, 0, 0, -1/8, -1/8, 0, 0, 1/8 \|. \qquad (10)$$

For the right-hand side $\hat{Q}_{m_3,1T} \equiv \hat{Q}_{m_3,1} - \operatorname{pr} \hat{Q}_{m_3,1}$, we obtain, from the second equation of system (5), $\hat{Q}_{m_3,1T} \equiv \operatorname{col} \| 1/8, 0, 0, -7/8, 1/8, 0, 0, 1/8 \|$.

The basis of the space $T_{\mathcal{F}}^{(3)}$, defined as a solution of the system of the equations $\langle \hat{Z}_{31*}, \mathcal{Y} \rangle = 0$, $\langle \hat{Z}_{32*}, \mathcal{Y} \rangle = 0$, contains six vectors:

$$\hat{\mathcal{Y}}_{21} = \operatorname{col} \| 1, 0, 0, -3, 0, 0, 0, 0 \|, \quad \hat{\mathcal{Y}}_{24} = \operatorname{col} \| 0, 1, 3, 0, 0, 0, 0, 0 \|,$$
$$\hat{\mathcal{Y}}_{22} = \operatorname{col} \| 0, 0, 0, -1, 1, 0, 0, 0 \|, \quad \hat{\mathcal{Y}}_{25} = \operatorname{col} \| 0, 0, 1, 0, 0, 1, 0, 0 \|,$$
$$\hat{\mathcal{Y}}_{23} = \operatorname{col} \| 0, 0, 0, -3, 0, 0, 0, 1 \|, \quad \hat{\mathcal{Y}}_{24} = \operatorname{col} \| 0, 0, -3, 0, 0, 0, 1, 0 \|.$$

Find the solution of the equation $G_{\mathcal{F}}^{(3)} \hat{\Gamma}_{13} = \hat{Q}_{m_3,1T} \equiv \hat{Q}_{m_3,1} - \operatorname{pr} \hat{Q}_{m_3,1}$ in the form of the expansion in the basis of the subspace $T_{\mathcal{F}}^{(3)} \hat{\Gamma}_{13} = \sum_{i=1}^{6} \gamma_{2i} \hat{\mathcal{Y}}_{2i}$. Substituting $\hat{\Gamma}_{13}$, $\hat{Q}_{m_3,1T}$ into the equations given above, we obtain a system of eight equations with six variables

$$\begin{Vmatrix} 0 & 0 & 0 & -4 & -1 & 3 \\ 4 & 1 & 3 & 0 & 0 & 0 \\ 6 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & -1 & -3 \\ 0 & 0 & 0 & 6 & 1 & -9 \\ 6 & -1 & -9 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{Vmatrix} \begin{Vmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{26} \end{Vmatrix} = \begin{Vmatrix} 1/8 \\ 0 \\ 0 \\ -7/8 \\ 1/8 \\ 0 \\ 0 \\ 1/8 \end{Vmatrix}.$$

Multiplying both sides of the given equation by the transposed matrix of the coefficients of this equation, we obtain the system which contains six equations

$$\begin{Vmatrix} 88 & 4 & 84 & 0 & 0 & 0 \\ 4 & 4 & 8 & 0 & 0 & 0 \\ 84 & 8 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 88 & 4 & -84 \\ 0 & 0 & 0 & 4 & 4 & -8 \\ 0 & 0 & 0 & -84 & -8 & 100 \end{Vmatrix} \begin{Vmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{25} \\ \gamma_{26} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 1 \\ 2 \end{Vmatrix}.$$

It is easy to find the solution of this system $\gamma_{21} = \gamma_{22} = \gamma_{23} = 0$, $\gamma_{24} = -5/32$, $\gamma_{25} = 7/32$, $\gamma_{26} = -3/32$. Finally, we have

$$\hat{\Gamma}_{13} \equiv \operatorname{col} \| 0, -5/32, -1/32, 0, 0, 7/32, -3/32, 0 \|. \qquad (11)$$

Summarize everything described above. Taking into account formulae (8) and (10) for the matrices $\operatorname{pr} Q_{m_1,1}$ and $\operatorname{pr} Q_{m_3,1}$, we write the operator U_0 in the first approximation:

$$U_0 = U + \varepsilon N_1,$$

where

$$N_1 = \operatorname{pr}\tilde{U} = N_{\circledast 11} + N_{\circledast 13},$$

$$N_{\circledast 11} = \hat{\hat{x}}_{m_1} Q_{m_1,1N}\,\partial = \frac{1}{2}\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right),$$

$$N_{\circledast 31} = \hat{\hat{x}}_{m_3} Q_{m_3,1N}\,\partial = -\frac{1}{8}\left(\left(x_1^2 + x_2^2\right) x_1 \frac{\partial}{\partial x_1} + \left(x_1^2 + x_2^2\right) x_2 \frac{\partial}{\partial x_2} \right).$$

Taking into account formulae (9), (11) for the matrices $\hat{\Gamma}_{11}$, $\hat{\Gamma}_{13}$, we define the operator of the transformation of the first approximation as

$$S_1 = S_{\circledast 11} + S_{\circledast 31},$$

where

$$S_{\circledast 11} = \hat{\hat{x}}_{m_1} \Gamma_{11}\,\partial = \frac{1}{2}\left(x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right),$$

$$S_{\circledast 31} = \hat{\hat{x}}_{m_1} \Gamma_{13}\,\partial = \left(\frac{1}{32} x_1^2 x_2 - \frac{3}{32} x_2^3 \right) \frac{\partial}{\partial x_1} + \left(-\frac{5}{32} x_1^3 + \frac{7}{32} x_1 x_2^2 \right) \frac{\partial}{\partial x_2}.$$

The centralized system of the first approximation has the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 + \varepsilon \left\{ \frac{1}{2} - \frac{1}{8}\left(x_1^2 + x_2^2\right) \right\} x_1, \\ \frac{dx_2}{dt} &= -x_1 + \varepsilon \left\{ \frac{1}{2} - \frac{1}{8}\left(x_1^2 + x_2^2\right) \right\} x_2. \end{aligned} \tag{12}$$

The second approximation. Represent the expression for the operator on the right-hand side of equation (4)

$$-[\tilde{U}, S_1] - 1/2\,[S_1, [U, S_1]] =$$

$$= \left(-\frac{3}{4} x_2 + \frac{23}{32} x_1^2 x_2 + \frac{9}{32} x_2^3 + \frac{1}{32} x_1^4 x_2 - \frac{9}{32} x_1^2 x_2^5 \right) \frac{\partial}{\partial x_1} +$$

$$+ \left(-\frac{1}{4} x_1 + \frac{3}{32} x_1^3 - \frac{55}{32} x_1 x_2^2 + \frac{5}{32} x_1^5 + \frac{5}{32} x_1^3 x_2^2 + \frac{6}{32} x_1 x_2^4 \right) \frac{\partial}{\partial x_2}.$$

Represent this operator as the sum

$$-[\tilde{U}, S_1] - 1/2\,[S_1, [U, S_1]] \equiv W_{\circledast 21} + W_{\circledast 23} + W_{\circledast 25},$$

where

$$W_{\circledast 21} = \hat{\hat{x}}_{m_1} Q_{m_1,2}\,\partial, \quad W_{\circledast 23} = \hat{\hat{x}}_{m_3} Q_{m_3,2}\,\partial, \quad W_{\circledast 25} = \hat{\hat{x}}_{m_5} Q_{m_5,2}\,\partial,$$

$$Q^T_{m_1,2} = \begin{Vmatrix} 0 & -\frac{3}{4} \\ -\frac{1}{4} & 0 \end{Vmatrix}, \quad Q^T_{m_3,2} = \begin{Vmatrix} 0 & \frac{23}{32} & 0 & \frac{9}{32} \\ \frac{3}{32} & 0 & -\frac{55}{32} & 0 \end{Vmatrix},$$

$$Q^T_{m_5,2} = \begin{Vmatrix} 0 & \frac{1}{32} & 0 & -\frac{9}{32} & 0 & 0 \\ \frac{5}{32} & 0 & \frac{5}{32} & 0 & \frac{6}{32} & 0 \end{Vmatrix}.$$

The right-hand sides are not equal to zero only in three equations (6). We can find the transformation matrices Γ_{21}, Γ_{23}, Γ_{25}:

$$\mathcal{F}_1 \Gamma_{21} - \Gamma_{21} \mathcal{F} = \mathcal{Q}_{m_1,2}, \quad \mathcal{F}_3 \Gamma_{23} - \Gamma_{23} \mathcal{F} = \mathcal{Q}_{m_3,2},$$
$$\mathcal{F}_5 \Gamma_{25} - \Gamma_{25} \mathcal{F} = \mathcal{Q}_{m_5,2}.$$

We give here the final results. Thus, the operator U_0 (3) associated with the centralized system in the second approximation is defined as

$$U_0 = U + \varepsilon\left(\frac{1}{2} - \frac{1}{8}\left(x_1^2 + x_2^2\right)\right)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) + $$
$$+ \varepsilon^2\left(-\frac{1}{4} + \frac{3}{8}\left(x_1^2 + x_2^2\right) - \frac{11}{128}\left(x_1^2 + x_2^2\right)^2\right)\left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right).$$

The centralized system which correspond to this operator has the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 + \varepsilon\left(\frac{1}{2} - \frac{1}{8}\left(x_1^2 + x_2^2\right)\right)x_1 + \\ &\qquad + \varepsilon^2\left(-\frac{1}{4} + \frac{3}{8}\left(x_1^2 + x_2^2\right) - \frac{11}{128}\left(x_1^2 + x_2^2\right)^2\right)x_2, \\ \frac{dx_2}{dt} &= -x_1 + \varepsilon\left(\frac{1}{2} - \frac{1}{8}\left(x_1^2 + x_2^2\right)\right)x_2 - \\ &\qquad - \varepsilon^2\left(-\frac{1}{4} + \frac{3}{8}\left(x_1^2 + x_2^2\right) - \frac{11}{128}\left(x_1^2 + x_2^2\right)^2\right)x_2. \end{aligned} \tag{13}$$

Study the possibility of separating the variables in the centralized system (13). Of the five equations $\mathcal{F}_j \alpha_j = 0$, $j = 1,5$, only the second one has a nontrivial solution $\alpha_2 = \operatorname{col} \| 1, 0, 1 \|$. By immediate calculations, we can see that there is a unique integral $\rho(x) = \overset{2}{\hat{x}}_{m_2} \alpha_2 = x_1^2 + x_2^2$. The operator U over the field R has no eigenfunction in $V_{\circledast 1}$. Thus, we can take the arbitrary integral of the equation $x_1 \partial_1 f + x_2 \partial_2 f = 0$ as the second variable. Since the arbitrary function of x_1 / x_2 is the general integral of this equation, we can take

$$y_1 = \sqrt{x_1^2 + x_2^2}, \quad y_2 = \operatorname{arctg} \frac{x_1}{x_2}. \tag{14}$$

The functions $x_1 = y_1 \sin y_2$, $x_2 = y_1 \cos y_2$, can be taken as the functions inverse to the change (14).

Upon transformation of the variables by formulae (14) the operator U_0 is described by the formula

$$U_0 = \frac{\partial}{\partial y_2} + \varepsilon\left(\frac{1}{2} - \frac{1}{8} y_1^2\right) y_1 \frac{\partial}{\partial y_1} + \varepsilon^2\left(-\frac{1}{4} + \frac{3}{8} y_1^4\right) \frac{\partial}{\partial y_2}$$

and the centralized system takes the form

$$\begin{aligned} \frac{dy_1}{dt} &= \varepsilon\left(\frac{1}{2} - \frac{1}{8} y_1^2\right) y_1, \\ \frac{dy_2}{dt} &= 1 - \varepsilon^2\left(\frac{1}{4} - \frac{3}{8} y_1^2 + \frac{11}{128} y_1^4\right). \end{aligned} \tag{15}$$

Consider solving of system (15). The system describes the motion of an unperturbed system along a circle of radius $R = y_1 = \text{const}$ for $\varepsilon = 0$. To integrate system (15), we use the results of Theorem 2.2 Ch. 2 [2]. The solution of this system can be represented in the form

$$y_1 = \exp t\overline{U}\,\overline{y}_1(\tau), \quad y_2 = \exp t\overline{U}\,\overline{y}_2(\tau), \tag{16}$$

where $\overline{U} = \partial/\partial\overline{y}_2$, $\overline{y}_2(\tau)$, $\overline{y}_2(\tau)$ are solutions of the system of differential equations

$$\frac{d\overline{y}_1}{d\tau} = \left(\frac{1}{2} - \frac{1}{8}\overline{y}_1^2\right)\overline{y}_1,$$
$$\frac{d\overline{y}_2}{d\tau} = -\varepsilon\left(\frac{1}{4} - \frac{3}{8}\overline{y}_1^2 + \frac{11}{128}\overline{y}_1^4\right), \quad \tau = \varepsilon t. \tag{17}$$

By virtue of the structure of the operator $\overline{U}$, the right-hand side of relations (16) can be written in the explicit form

$$y_1 = \overline{y}_1(\tau), \quad y_2 = \overline{y}_2(\tau) + t. \tag{18}$$

System (17) can be easily integrated in quadratures

$$\overline{y}_1 = \frac{2e^{32\tau}c_1}{\sqrt{1+e^{64\tau}c_1}}, \qquad \overline{y}_2 = c_2 - \varepsilon\int_0^\tau\left(\frac{1}{4} - \frac{3}{8}\overline{y}_1^2(\tau) + \frac{11}{128}\overline{y}_1^4(\tau)\right)d\tau,$$

where c_1, c_2 are arbitrary constants.

Denoting

$$v_1(\tau) = -\varepsilon\int_0^\tau\left(\frac{1}{4} - \frac{3}{8}\overline{y}_1^2(\tau) + \frac{11}{128}\overline{y}_1^4(\tau)\right)d\tau$$

and taking into account formulae (18), we can write the solution of the centralized system (13) in terms of the variables x_1, x_2

$$x_1 = \overline{y}_1(\tau)\sin(c_2 + v_1(\tau)), \quad x_2 = \overline{y}_1(\tau)\cos(c_2 + v_1(\tau)).$$

To pass to the solution of the initial equations (1) in the second approximation we have to know the operator S_2. Calculation of S_2 is analogous to that of S_1. The final result is

$$S_2 = S_{\bullet 12} + S_{\bullet 32} + S_{\bullet 52},$$

where

$$S_{\bullet 12} = -0{,}25x_1\frac{\partial}{\partial x_1} + 0{,}25x_2\frac{\partial}{\partial x_2},$$

$$S_{\bullet 32} = \left(-0{,}00764x_1^3 + 0{,}0558x_1x_2^2\right)\frac{\partial}{\partial x_1} + \left(-0{,}4775x_1^2x_2 + 0{,}1482x_2^3\right)\frac{\partial}{\partial x_2},$$

$$S_{\bullet 52} = \left(0.02511x_1^5 + 0{,}02195x_1^3x_2^2 + 0{,}05265x_1x_2^4\right)\frac{\partial}{\partial x_1} - \left(0{,}03785x_1^4x_2 + 0{,}03765x_1^2x_2^3 + 0{,}02493x_2^5\right)\frac{\partial}{\partial x_2}.$$

The variables x' are expressed in terms of the variables x to within ε^2 by formulae

$$x_i' = (1 + \varepsilon S + \varepsilon^2/2!\,S^2)x_i, \quad S = S_1 + \varepsilon S_2, \quad i = 1, 2.$$

or

$$x_1' = x_1 + \varepsilon\left(0{,}25x_2 + 0{,}03125x_1^3 - 0{,}09375x_2^3\right) + \varepsilon^2\left(0{,}25x_1 -\right.$$

$$- 0{,}00764 x_1^3 + 0{,}0558 x_1 x_2^2 + 0{,}02511 x_1^5 + 0{,}02195 x_1^3 x_2^2 + 0{,}05265 x_1 x_2^4 \Big),$$

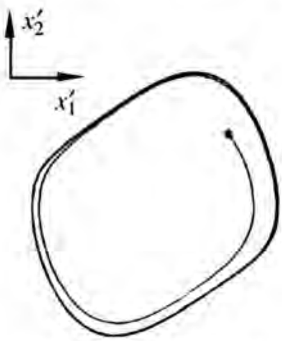
$$x_2' = x_2 + \varepsilon \left(0{,}25 x_2 - 0{,}15625 x_1^2 + 0{,}21875 x_1 x_2^2\right) + \varepsilon^2 \Big(0{,}25 x_2 -$$

$$- 0{,}4775 x_1^2 x_2 + 0{,}1482 x_2^3 - 0{,}03785 x_1^4 x_2 - 0{,}03765 x_1^2 x_2^3 - 0{,}02493 x_2^5 \Big).$$

The preceding analysis is illustrated in Figures 1 – 5. The solution of the Van der Pol system (1) in the phase plane for $\varepsilon = 0{,}5$ is given in Figure 1. The solution of the centralized system (12) of the first approximation in the phase plane obtained from system (1) is given in Figure 2(a). The corresponding transformations are written up to ε (see operator S_1 above)

$$x_1' = x_2 + \varepsilon \left(\frac{1}{4} x_2 + \frac{1}{32} x_1^2 x_2 - \frac{3}{32} x_2^3 \right),$$
$$x_2' = x_2 + \varepsilon \left(\frac{1}{4} x_1 - \frac{5}{32} x_1^3 + \frac{8}{32} x_1 x_2^2 \right), \tag{19}$$

$$x_1 = x_1' - \varepsilon \left(\frac{1}{4} x_2' + \frac{1}{32} x_1'^2 x_2' - \frac{3}{32} x_2'^3 \right),$$
$$x_2 = x_2' - \varepsilon \left(\frac{1}{4} x_1' - \frac{5}{32} x_1'^3 + \frac{7}{32} x_1' x_2'^3 \right). \tag{20}$$



$x_0'(1{,}2;\ 1{,}2),\quad \varepsilon = 0{,}5$

Fig. 1. Solution in the phase plane for the Van der Pol system.

The solution of the system with separated variables

$$\dot{y}_1 = \varepsilon \left(\frac{1}{2} - \frac{1}{8} y_1^2 \right) y_1,$$
$$\dot{y}_2 = 1 \tag{21}$$

is given in Figure 2(b). System (21) is obtained from system (12) with the help of the change of variables

$$x_1 = y_1 \sin y_2, \qquad x_2 = y_1 \cos y_2, \tag{22}$$

$$y_1 = \sqrt{x_1^2 + x_2^2}, \quad y_2 = \operatorname{arctg} \frac{x_1}{x_2}. \tag{23}$$

The initial conditions of system (12) are obtained by recalculating the initial conditions of system (1) by formulae (20). The initial conditions of system (21) are obtained by recalculating the initial conditions of system (12) by formulae (23).

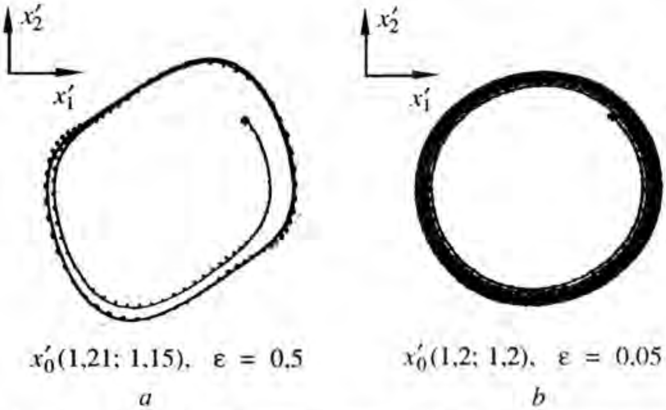


Fig. 3. Approximation of the solution of the Van der Pol system in the phase plane (continuous curves) by the solution of a system with separated variables (dot curves) in the first approximation, which is enumerated to the initial coordinates.

Consider the subspace $T_{\circledast}$ that is the direct sum of subspaces

$$T_{\circledast 1},\ T_{\circledast 2},\ T_{\circledast 3},\ \ldots$$

with the bases

$$f^{(m_1)} = \| x_1, x_2 \|,$$
$$f^{(m_2)} = \| 2x_1x_2,\ x_2^2 - x_1^2 \|, \tag{26}$$
$$f^{(m_3)} = \left\| 3\left(x_1^2 + x_2^2\right)x_1 - 4x_1^3,\ 4x_2^3 - 3\left(x_1^2 + x_2^2\right)x_2 \right\|,$$

....

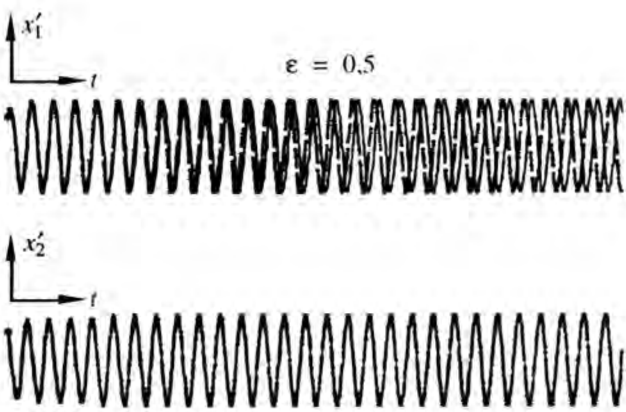
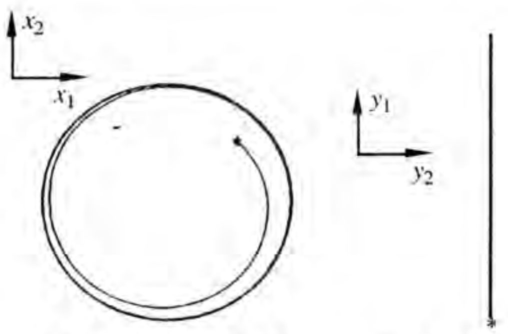


Fig. 4. Approximation of the solution of the Van der Pol system by the solution of a system with separated variables (dot curves) in the first approximation (time dependence).

It is easy to verify that each subspace $T_{\circledast j}$ is mapped by U into itself. To do so, it is sufficient to find representation matrices of U in these subspaces



$x_0(1.11;\ 1.0)$, $\varepsilon = 0.5$ $x_0(0.07;\ 0.04)$, $\varepsilon = 0.5$

a *b*

Fig. 2. Solution in the phase plane for a centralized system the Van der Pol system in the first approximation (a) and a system with separated variables (b).

Approximation of the solutions of the initial system (1) in the phase plane by solutions of the corresponding centralized system of the first approximation is given in Figure 3(a). This approximation is obtained from the phase points of system (21) in Figure 2(b) by using successively transformations (22) and (19).

The structure of Figure 3(b) is identical to that of Figure 3(a), which is received for $\varepsilon = 0.005$. The time dependence of coordinates x_1' and x_2' of exact (continuous curves) and approximate solution (dot curves) for phase Figures 3(a) and 3(b) is shown in Figure 4 and Figure 5, respectively.

2. Asymptotic decomposition in the representation space of finite-dimentional Lie group.

2.1. Formulation of the problem. Consider the nonlinear oscillator

$$\dot{x}_1 = x_2 + \varepsilon\, \tilde{\omega}_1 (x_1, x_2),$$
$$\dot{x}_2 = -x_1 + \varepsilon\, \tilde{\omega}_2 (x_1, x_2),$$

where $\tilde{\omega}_1$, $\tilde{\omega}_2$ are certain polynomials in x_1 and x_2.

All considerations of the asymptotic decomposition algorithm in the space of homogeneous polynomials [4] were based upon the invariance property of the subspaces

$$V_{\circledast 1},\ V_{\circledast 2},\ \ldots \tag{24}$$

with bases

$$\hat{\tilde{x}}_{m_1} = \| x_1, x_2 \|,\ \hat{\tilde{x}}_{m_2} = \| x_1^2, x_1 x_2, x_2^2 \|,\ \ldots$$

with respect to the action of the operator

$$\boldsymbol{U} = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \tag{25}$$

which is associated with the system of zero approximation. The fact of invariance is expressed by the relation

$$\boldsymbol{U}\, \hat{\tilde{x}}_{m_j} = \hat{\tilde{x}}_{m_j}\, \mathcal{F}_j,\quad j = 1, 2, \ldots,$$

where $\mathcal{F}_j$ is the representation matrix of $\boldsymbol{U}$ in the subspace $V_{\circledast j}$.

A natural question arises: Are subspaces (24) unique invariant subspaces in the linear space of homogeneous polynomials? It turns out that they are not.

$$U f^{(m_j)} = f^{(m_j)} \mathcal{F}_j, \quad \mathcal{F}_j = \begin{Vmatrix} 0 & -j \\ j & 0 \end{Vmatrix}. \tag{27}$$

For a better understanding of the structure of the subspace $T_{\otimes}$, let us introduce new variables ρ and φ by the formulae

$$x_1 = \rho \sin\varphi, \quad x_2 = \rho\cos\varphi.$$

In new variables, bases, vectors (26) are wtitten as follows:

$$\hat{\tilde{\varphi}}_{m_k} = \| \rho^k \sin k\varphi, \rho^k \cos k\varphi \|, \quad k = 1, 2, \ldots .$$

So, passing to the $T_{\otimes} \subset T(V)$ means passing from the space of homogeneous polynomials in two variables to the space of trigonometric functions (Fourier series).

Notice two evident advantages of passing from the space $T(V)$ to $T_{\otimes}$. First, matrices $\mathcal{F}_j$ (27) have lower order (it equals 2 for any j). Second, there is a certain "naturality" in passing to polar coordinates. New variables have correct physical interpretations (for instance, ρ is the radius of a circumference, φ is an angle variable, etc.).

The just-described process of choosing a new representation space for the operator U has deep group-theoretical background. Let us consider them in detail.

Consider the set of four linearly independent operators

$$V_{11} = x_1 \frac{\partial}{\partial x_1}, \quad V_{12} = x_1 \frac{\partial}{\partial x_2}, \quad V_{21} = x_2 \frac{\partial}{\partial x_1}, \quad V_{22} = x_2 \frac{\partial}{\partial x_2}, \tag{28}$$

which generate a general linear finite-dimensional Lie algebra $GL(2)$ of order four. From (28), a general linear group $GL(2)$ is restored. To write the elements of this group in the explicit form, let us write its general element as a Lie series

$$x' = \exp V x,$$

where $V = s_{11} V_{11} + s_{12} V_{12} + s_{21} V_{12} + s_{22} V_{22}$, $s_{11}, s_{12}, s_{21}, s_{22}$ are group parameters which range in a neighborhood of zero.

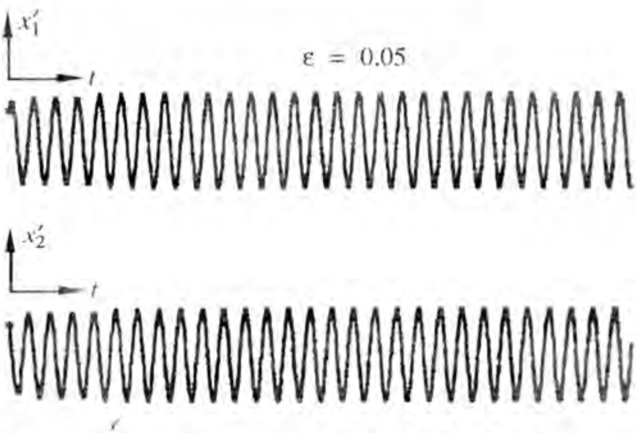


Fig. 5. Approximation of the solution of the Van der Pol system by the solution of a system with separated variables (dot curves) in the first approximation (time dependence).

Let us compute the representation matrix of V in the subspace $V_{\otimes 1}$

$$V\,\hat{\hat{x}}_{m_1} = \hat{\hat{x}}_{m_1}\,\mathcal{F}_1, \quad \mathcal{F}_1 = \begin{Vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{Vmatrix}.$$

Taking into account relations

$$V^k\,\hat{\hat{x}}_{m_1} = \hat{\hat{x}}_{m_1}\,\mathcal{F}_1^k, \quad k = 2, 3, \ldots,$$

we write series (13) in the finite form

$$x' = x\,e^{\mathcal{F}_1(s)}.$$

The matrix $G(s) = e^{\mathcal{F}_1(s)}$ determines a general element of the group $GL(2)$.

In the light of above considerations we may say that the space of homogeneous polynomials $T(V)$ is the representation space for the general linear group $GL(n)$, $n = 2$.

The operator U of the system of zero approximation given by formula (25) generates the circle rotation group $SO(2)$ in the plane. To find the explicit form of the elements of this group, we also make use of a Lie series $x' = \exp(\varphi U)x$. The corresponding computations have already been done in [2, p. 19], so we write only the result

$$\begin{Vmatrix} x_1' \\ x_2' \end{Vmatrix} = \begin{Vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} \qquad (29)$$

Under the action of transformation (29), points of circle with radius 1 are transformed into themselves. This can be easily done by verifying the identity $x_1'^2 + x_2'^2 = x_1^2 + x_2^2$.

Thus, the linear space of trigonometrical functions T_φ is the representation space for the rotation group $SO(2)$ in the plane. Let us denote this space by $T_{SO(2)}$.

Now we can formulate the main difference between the approach of the asymptotic decomposition method and that of the normal form method.

In the normal form method, the representation space for general linear group $GL(n)$ is shosen as a representation space. In the asymptotic decomposition method, the representation space for the subgroup of the same $GL(n)$, which has the solution of the system of zero aproximation as its element, is chosen as a representation space.

So, the normal form method, which makes use of a universal representation space for general linear group, does not consider the true algebraic structure of the system of zero aproximation.

Contrary to that, the asymptotic decomposition method is based essentially on a deep connection between the representation theory for continuous groups and special functions of mathematical physics. This theory has been developed intensively during the last decades (see N. Ya. Vilenkin [5], A. Barut, R. Roczka [1]).

In the asymptotic decomposition method, we get a significant generalization, which consists in possibility for considering in zero approximation an essentially nonlinear system provided its generating Lie algebra is finite-dimensional.

2.2. Group-theoretical properties of the system of zero approximation. In the present subsection, we apply the asymptotic decomposition method to differential systems, the zero approximation of which generates a finite-dimensional Lie group. Usage of a representations space of this group allows one to reduce all algorithms of this method to simple problems of linear algebra.

Consider a system of ordinary differential equations

$$\frac{dx}{dt} = \omega(x). \qquad (30)$$

Suppose the finite-dimensional algebra with the basis operators

$$X_j = \xi_{j1}(x)\frac{\partial}{\partial x_1} + \ldots + \xi_{jn}(x)\frac{\partial}{\partial x_n}, \quad j = \overline{1,h}, \tag{31}$$

is the generating Lie algebra $\mathcal{B}_h$ for the system (see [3]).

The operator

$$U(x) = \omega_1(x)\frac{\partial}{\partial x_1} + \ldots + \omega_n(x)\frac{\partial}{\partial x_n},$$

associated with system (30) belongs to $\mathcal{B}_h$ and, therefore, can be expressed in terms of basis (31):

$$U = c_1 X_1 + \ldots + c_h X_h, \quad c_j \in P, \quad j = \overline{1,h}. \tag{32}$$

The Lie algebra $\mathcal{B}_h$ determines a finite Lie group $\mathcal{G}(\mathcal{B}_h)$, the elements of which can be defined by the convergent Lie series (see [2], Ch. 1)

$$x' = \exp(s_1 X_1 + \ldots + s_h X_h)x, \tag{33}$$

where $s = \| s_1, \ldots, s_h \|$ are parameters of the group varying in a nejghborhood of the point $s = 0$.

Let us use expansion (32) and write the solution of system (30) as a Lie series

$$x = \exp((t - t_0)U(x_0))x_0. \tag{34}$$

Hence, solution (34) of system (30) is an element of the group $\mathcal{G}(\mathcal{B}_h)$ for sufficently small $t - t_0$. This property of the solution of system (30) can be taken as the general characteristic of a given system when studying the influence of perturbations on its behavior.

Further, we shall suppose that the Hilbert space H of representation, which is equal to the direct orthogonal sum of linear subspaces H_ν; $H = \sum_{\nu=1}^{\infty} \oplus H_\nu$ can be set to correspond to $\mathcal{G}(\mathcal{B}_h)$. Every linear subspace H_ν of dimension m_ν over P with the basis consisting of m_ν functions

$$f^{(\nu)} \overset{\text{def}}{=} \| f_{1m_\nu}(x), \ldots, f_{m_\nu m_\nu}(x) \|$$

is invariant under transformations of the group. If $f(x') \in H_\nu$, then after a change of the variable x', according to the formulae (33) $f(\exp(s_1 X_1 + \ldots + s_h X_h)x) \in H_\nu$.

Suppose the space H contains a subspace H_1 of dimension $m_1 = n$. The necessary and sufficient condition for the subspace H_ν to be invariant under the group $\mathcal{G}(\mathcal{B}_h)$ is invariance of subspace H_ν under the basis operators (31) of the algebra $\mathcal{B}_h$, i. e., fulfillment of the conditions:

$$X_j f_{1m_\nu}(x) = c_{j1m_\nu}^{1} f_{1m_\nu} + \ldots + c_{j1m_\nu}^{m_\nu} f_{m_\nu m_\nu},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_j f_{m_\nu m_\nu}(x) = c_{jm_\nu m_\nu}^{1} f_{1m_\nu} + \ldots + c_{jm_\nu m_\nu}^{m_\nu} f_{m_\nu m_\nu}.$$

These relation can be represented in a compact form $X_j f^{(\nu)} = f^{(\nu)} C_j^{(m_\nu)}$, where $C_j^{(m_\nu)}$ is a square matrix of dimension $m_\nu \times m_\nu$. This matrix is called a representation matrix of the operator X_j, $j = \overline{1,h}$, in the invariant subspace H_ν.

2.3. The main algorithm. Reduction of operator equations to systems of linear algebraic equations. Let system (30), which will be called the system of zero approximation, be subjected to small perturbations:

$$\frac{dx'}{dt} = \omega(x') + \varepsilon\tilde{\omega}(x'), \tag{35}$$

where $\tilde{\omega}(x') = \mathrm{col}\,\|\tilde{\omega}_1(x'), \ldots, \tilde{\omega}_n(x')\|$, $\tilde{\omega}_i(x') \in \mathcal{D}(G)$, $i = \overline{1,n}$, ε is a small positive parameter.

Let us make an assumption on properties of coefficients of a perturbed system. Let the basis of the subspace H_1 be given by n functions independent in G

$$v_1(x), \ldots, v_n(x). \tag{36}$$

Denote by $L_1, \ldots, L_n$ differential operators $\partial/\partial v_1, \ldots, \partial/\partial v_n$ represented in variables x. It is easy to see that operator U occuring in

$$U_0 = U + \varepsilon\tilde{U}(x),$$

(here $\tilde{U}(x) = \tilde{\omega}_1(x)\partial/\partial x_1 + \ldots + \tilde{\omega}_n(x)\partial/\partial x_n$), which is associated with system (35), can be represented in the form

$$U = g_1(x)L_1 + \ldots + g_n(x)L_n,$$

where $g_1(x), \ldots, g_n(x) \in H_1$. Let us assume that $\tilde{U}(x)$ admits the expansion $\tilde{U} = h_1(x)L_1 + \ldots + h_n(x)L_n$, where the coefficients $h_1(x), \ldots, h_n(x)$ are represented as a series over the basis H:

$$h_j(x) = \sum_{\nu=1}^{\infty} h_j^{(\nu)}(x), \quad h_j^{(\nu)}(x) \in H_\nu.$$

Denote by $L(H_\nu)$ the linear space of operators of the form

$$L^{(\nu)} = h_1^{(\nu)}(x)L_1 + \ldots + h_n^{(\nu)}(x)L_n, \tag{37}$$

where coefficients $h_1^{(\nu)}, \ldots, h_n^{(\nu)} \in H_\nu$. We also write

$$L^{(\nu)} \in L(H_\nu). \tag{38}$$

The main idea of the asymptotic decomposition algorithm is to transform the perturbed system (35) into a standard one

$$\frac{dx_j}{dt} = \omega_j(x) + \sum_{\nu=1}^{\infty} \varepsilon^\nu q_{\nu j0}(x), \quad j = \overline{1,n}, \tag{39}$$

which is known as a centralized system (see [3]). Transition from system (35) to (39) is fulfilled by exchanging variables in the form of Lie series

$$x'_j = \exp(\varepsilon S)\,x_j, \quad j = \overline{1,n}, \tag{40}$$

where

$$S = S_1 + \varepsilon S_2 + \ldots; \quad S_i = \gamma_{i1}\frac{\partial}{\partial x_1} + \ldots + \gamma_{in}\frac{\partial}{\partial x_n}.$$

Introducing the differential operator

$$N(x) = \sum_{k=1}^{\infty} \varepsilon^{k-1} N_k(x),$$

where

$$N_k = \sum_{j=1}^{n} q_{kj0}\frac{\partial}{\partial x_j},$$

we can write the centralized system in the form

$$\frac{dx_j}{dt} = \omega_j(x) + \varepsilon N(x)x_j, \quad j = \overline{1,n}. \tag{41}$$

Operators $S_1, S_2, \ldots$, occurring in transformation (40) are determined from the system of operator equations

$$[U, S_\nu] = F_\nu - \operatorname{pr} F_\nu, \quad \nu = 1, 2, \ldots, \tag{42}$$

where $\operatorname{pr} F_\nu \equiv N_\nu$ is the projection of operator F_ν on the algebra of a centralizer, which is determined by solutions of the homogeneous equation $[U, S] = 0$.

Next, we solve the system of equations (42) and find $\operatorname{pr} F_\nu$ using the representation space H of group $\mathcal{G}(\mathcal{B}_h)$. Remembering the assumptions made and taking into account the notation (37), (38), we can represent the operators F_ν on the right-hand sides of equations (42) as a sum:

$$F_\nu = \sum_{j=1}^{n} F_\nu^{(j)}, \quad F_\nu^{(j)} \in L(H_j). \tag{43}$$

We find the solutions S_ν of equations (42) as expansions:

$$S_\nu = \sum_{j=1}^{n} S_\nu^{(j)}, \quad S_\nu^{(j)} \in L(H_j). \tag{44}$$

The operator $F_\nu^{(j)}$ can be represented in the following form:

$$F_\nu^{(j)} = f^{(j)} Q_{\nu j} L, \tag{45}$$

where $L = \operatorname{col} \| L_1, \ldots, L_n \|$, $Q_{\nu j}$ is a known matrix of dimension $m_j \times n$ and $f^{(j)}$ is a basis vector of the subspace H_j. Similarly, the operator $S_\nu^{(j)}$ can be represented in the form $S_\nu^{(j)} = f^{(j)} \Gamma_{\nu j} L$, where $\Gamma_{\nu j}$ is a matrix of dimension $m_j \times n$.

Denote by $\mathcal{F}_j$ the representation matrix of operator U in H_j determined by the identity

$$U f^{(j)} = f^{(j)} \mathcal{F}_j. \tag{46}$$

We use the particular notation $\mathcal{F}_j \equiv \mathcal{F}$. Then the following vector relation

$$U = f^{(1)} \mathcal{F} L, \quad f^{(1)} = \| v_1(x), \ldots, v_n(x) \| \tag{47}$$

is valid for the operator U.

Theorem 1. *The solution of the operator equation*

$$[U, S_\nu] = F_\nu \tag{48}$$

is equivalent to the solution of a system of independent linear matrix equations

$$\mathcal{F}_j \Gamma_{\nu j} - \Gamma_{\nu j} \mathcal{F}_j = Q_{\nu j}, \quad j = 1, 2, \ldots. \tag{49}$$

Proof. Let us substitute expressions (43), (44) into equation (48):

$$\sum_{j=1}^{\infty} \left[U, S_\nu^{(j)} \right] = \sum_{j=1}^{n} F_\nu^{(j)}. \tag{50}$$

Taking into account relations (45), (47), let us compute the Poisson bracket

$$\left[U, S_\nu^{(j)} \right] = U f^{(j)} \Gamma_{\nu j} L - S_\nu^{(j)} f^{(1)} \mathcal{F} L.$$

Next, using identity (46) and the identity $S_\nu^{(j)} f^{(1)} \equiv f^{(j)} \Gamma_{\nu j}$, we finally obtain

$$\left[U, S_\nu^{(j)}\right] = f^{(j)} \mathcal{F}_j \Gamma_{\nu j} L - f^{(j)} \Gamma_{\nu j} \mathcal{F} L. \tag{51}$$

Let us substitute expressions (45) and (51) into (50) :

$$\sum_{j=1}^{\infty} \left(f^{(j)} \mathcal{F}_j \Gamma_{\nu j} L - f^{(j)} \Gamma_{\nu j} \mathcal{F} L \right) = \sum_{j=1}^{\infty} f^{(j)} Q_{\nu j} L$$

and equate the coefficients at the vectors $f^{(j)}, L$. As a result, we get (49).

Denote $G_{\mathcal{F}}^{(j)} = \mathcal{F}_j \otimes \mathcal{E}_n - \mathcal{E}_{m_j} \otimes \mathcal{F}^n$, where $\mathcal{E}_{m_j}$, $\mathcal{E}_n$ are unit matrices of dimensions $m_j \times m_j$, $n \times n$. Since we consider the algebras of a centralizer to be of the first degree, $\mathcal{F}$ is a diagonable matrix and, hence, matrices $\mathcal{F}_j$ and $G_{\mathcal{F}}^{(j)}$ are also a diagonable matrices.

The matrix equation (49) is equivalent to the system of linear algebraic equations

$$G_{\mathcal{F}}^{(j)} \hat{\Gamma}_{\nu j} = \hat{Q}_{\nu, j}. \tag{52}$$

Next, we construct the projector $\operatorname{pr} \boldsymbol{F}_\nu$. Consider the homogeneous equation

$$G_{\mathcal{F}}^{(j)} \hat{\Gamma}_{\nu j} = 0, \tag{53}$$

which corresponds to (52), and the conjugate equation

$$G_{\mathcal{F}}^{(j)*} \hat{\Gamma}_{\nu j} = 0, \tag{54}$$

where $G_{\mathcal{F}}^{(j)*} = \mathcal{F}_j^* \otimes \mathcal{E}_n - \mathcal{E}_{m_j} \otimes (\mathcal{F})^{T*}$, is the matrix complex conjugate to $G_{\mathcal{F}}^{(j)}$.

Denote by $\hat{N}_{\mathcal{F}}^{(j)}$, $\hat{N}_{\mathcal{F}}^{(j)*}$ kernels and by $\hat{T}_{\mathcal{F}}^{(j)}$, $\hat{T}_{\mathcal{F}}^{(j)*}$ images of these matrices. The space $\hat{R}^{(m_j, n)}$ can be splitted uniquely into the direct sum of subspaces $\hat{N}_{\mathcal{F}}^{(j)}$, $\hat{T}_{\mathcal{F}}^{(j)*}$. According to this, we represent the right-hand side of equation (52) as a sum:

$$\hat{Q}_{\nu, j} = \hat{Q}_{\nu j N} \dotplus \hat{Q}_{\nu j T}, \quad \hat{Q}_{\nu j N} \in \hat{N}_{\mathcal{F}}^{(j)}, \quad \hat{Q}_{\nu j T} \in \hat{T}_{\mathcal{F}}^{(j)}. \tag{55}$$

In the space $\hat{R}^{(m_j, n)}$, the vector $\hat{Q}_{\nu j N}$ is in correspondence with the rectangular matrix $Q_{\nu j N}$, which in turn determines the differential operator $N_{\nu j} = f^{(j)} Q_{\nu j N} L$.

The constructed operator $\boldsymbol{N}_{\nu j}$ commutes with $\boldsymbol{U}$: $[\boldsymbol{U}, \boldsymbol{N}_{\nu j}] \equiv 0$.

Definition 1. $\operatorname{pr} \boldsymbol{F}_\nu = \sum_{j=1}^{\infty} \boldsymbol{N}_{\nu j}$.

Let us consider the actual computation of the matrix $Q_{\nu j N}$. Suppose (53) has k_j linearly independent solutions $\hat{Z}_{1j}, \ldots, \hat{Z}_{k_j j}$, which can be taken as a basis $\hat{N}_{\mathcal{F}}^{(j)}$. Then equation (54) also has k_j linearly independent solutions $\hat{Z}_{1j*}, \ldots, \hat{Z}_{k_j j*}$, which can be taken as a basis $\hat{N}_{\mathcal{F}}^{(j)*}$. Let us expand the component $\hat{Q}_{\nu j N}$ of sum (55) with respect to the basis $\hat{N}_{\mathcal{F}}^{(j)}$:

$$\hat{Q}_{\nu j N} = \sum_{i=1}^{k_j} \alpha_{\nu j i} \hat{Z}_{ij}.$$

The difference $\hat{Q}_{\nu j T} = \hat{Q}_{\nu, j} - \hat{Q}_{\nu j N}$ belongs to the image $\hat{T}_{j}^{(j)}$ and, therefore, is orthogonal to the subspace $\hat{N}_{\mathcal{F}}^{(j)*}$. As a result, we get a system of linear algebraic equations for finding the coefficients $\alpha_{\nu j i}$

$$\sum_{i=1}^{k_j} \alpha_{\nu j i} \left\langle \hat{Z}_{ij}, \hat{Z}_{lj*} \right\rangle = \left\langle \hat{Q}_{\nu j}, \hat{Z}_{lj*} \right\rangle, \quad l = \overline{1, k_j}. \tag{56}$$

System (56) can be represented in the space $R^{(m_j, n)}$ as the matrix system

$$\sum_{i=1}^{k_j} \alpha_{\nu_j i} \operatorname{tr} \left\langle Z_{ij}, \bar{Z}_{lj*}^T \right\rangle = \operatorname{tr} \left\langle \hat{Q}_{\nu j}, \bar{Z}_{lj*}^T \right\rangle, \quad l = \overline{1, k_j}.$$

The operator S_ν is determined by the set of matrices $\Gamma_{\nu j}$, $j = 1, 2, \ldots$, which are solutions of the equations

$$G_{ij}^{(j)} \hat{\Gamma}_{\nu j} = \hat{Q}_{\nu j T}.$$

2.4. Models connected with group $SO(2)$. Let us consider Van der Pol equation (1) as a perturbed motion on $SO(2)$.

The system of zero approximation (1) yields the group (for details, see [2], Ch. 1).

$$x_1' = \exp(tU) x_1 = x_1 \cos t + x_2 \sin t,$$

$$x_2' = \exp(tU) x_2 = -x_1 \sin t + x_2 \cos t.$$

Pass to polar coordinates in the operator

$$x_1 = \rho \sin \varphi, \quad x_2 = \rho \cos \varphi. \tag{57}$$

The functions

$$\rho = \sqrt{x_1^2 + x_2^2}, \quad \varphi = \operatorname{arctg} \frac{x_1}{x_2}$$

are inverse to (57). Finally we obtain $U = \partial / \partial \varphi$.

The representation space $T_\otimes$ of the group $SO(2)$ generated by U is equal to the direct sum of the subspaces $T_{\otimes j}$

$$T_\otimes = T_{\otimes 1} + T_{\otimes 2} + \ldots .$$

Every subspace $T_{\otimes n}$ is defined by the basis

$$f^{(n)} = \| \rho^n \sin n\varphi, \rho^n \cos n\varphi \|, \quad n = 1, 2, \ldots .$$

The operator U has the matrix of representation $\mathcal{F}_{m_n}$ in the subspace $T_{\otimes n}$. This matrix can be calculated by the formulae

$$U f^{(n)} = f^{(n)} \mathcal{F}_{m_n}, \quad \mathcal{F}_{m_n} = \begin{Vmatrix} 0 & -n \\ n & 0 \end{Vmatrix}.$$

Illustrate application of the asymptotic decomposition method to system (1) in the representation space $T_\otimes$. Calculate only the first approximation. Let one term S_1 be in transformation (40) and the transformed operator be represented by the sum

$$U_0 = U + \varepsilon N_1.$$

According to the general theory, we should consider the operator equation

$$[U, S_1] = F_1, \quad F_1 \stackrel{\text{def}}{=} \tilde{U}. \tag{58}$$

After a change of variables (57), the operators $\partial / \partial x_1$, $\partial / \partial x_2$ turn into L_1, L_2 respectively

$$L_1 = \sin \varphi \frac{\partial}{\partial \rho} + \frac{\cos \varphi}{\rho} \frac{\partial}{\partial \varphi}, \quad L_2 = \cos \varphi \frac{\partial}{\partial \rho} - \frac{\sin \varphi}{\rho} \frac{\partial}{\partial \varphi}.$$

They possess the property

$$L_j x_i = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j, \ i, j = 1, 2. \end{cases}$$

Rewrite $\boldsymbol{U}$, $\tilde{\boldsymbol{U}}$ in new variables by using $\boldsymbol{L}_1$, $\boldsymbol{L}_2$

$$\boldsymbol{U} \equiv \frac{\partial}{\partial\varphi} \equiv \| \rho \sin\varphi, \rho\cos\varphi \| \mathcal{F} L,$$

$$\tilde{\boldsymbol{U}} = \| \rho \sin\varphi, \rho\cos\varphi \| Q_{11} L + \| \rho^3 \sin 3\varphi, \rho^3 \cos 3\varphi \| Q_{31} L,$$

where

$$\mathcal{F} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \quad Q_{11} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 - \frac{1}{4}\rho^2 \end{Vmatrix}, \quad Q_{31} = \begin{Vmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{Vmatrix},$$

$$L = \operatorname{col} \| \boldsymbol{L}_1, \boldsymbol{L}_2 \|.$$

Write the operator of the transformation $\boldsymbol{S}_1$ following the structure of the right-hand side of equation (58) in the form

$$\boldsymbol{S}_1 = \boldsymbol{S}_{11} + \boldsymbol{S}_{31},$$

where

$$\boldsymbol{S}_{11} = \| \rho \sin\varphi, \rho\cos\varphi \| \Gamma_{11} L,$$

$$\boldsymbol{S}_{31} = \| \rho^3 \sin 3\varphi, \rho^3 \cos 3\varphi \| \Gamma_{31} L,$$

Γ_{11}, Γ_{31} are the unknown second-order square matrices. In the general case, they depend on the variable ρ.

Substituting $\boldsymbol{U}$, $\tilde{\boldsymbol{U}}$ and $\boldsymbol{S}_1$ into equation (58), we obtain two independent subsystems of linear algebraic equations

$$\mathcal{F}_1 \Gamma_{11} - \Gamma_{11} \mathcal{F} = Q_{11}, \tag{59}$$

$$\mathcal{F}_3 \Gamma_{31} - \Gamma_{31} \mathcal{F} = Q_{31}. \tag{60}$$

Write the homogeneous equation corresponding to system (59) in the space $\hat{R}^{(2,2)}$ in the form

$$G_1 \hat{\Gamma}_{11} = 0, \tag{61}$$

where

$$G_{11} = \begin{Vmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{Vmatrix}.$$

Equation (61) has the independent solutions

$$Z_1 = \| 1, 0, 0, 1 \|, \quad Z_2 = \| 0, -1, -1, 0 \|.$$

The equation conjugate to (61) has the solutions

$$Z_1^* = \| 1, 0, 0, 1 \|, \quad Z_2^* = \| 0, -1, -1, 0 \|.$$

The projection $\operatorname{pr} \hat{Q}_{11}$ of the vector $\hat{Q}_{11}$ is obtained from the system of equations

$$\alpha_1 \langle Z_1, Z_1^* \rangle + \alpha_2 \langle Z_2, Z_1^* \rangle = \langle \hat{Q}_{11}, Z_1^* \rangle.$$

$$\alpha_1\langle Z_1, Z_2^*\rangle + \alpha_2\langle Z_2, Z_2^*\rangle = \langle \hat{Q}_{11}, Z_2^*\rangle.$$

After simple calculations, we obtain

$$\operatorname{pr}\hat{Q}_{11} = \left\| \frac{1}{2}-\frac{1}{8}\rho^2, \quad 0, \quad 0, \quad \frac{1}{2}-\frac{1}{8}\rho^2 \right\|,$$

$$\hat{Q}_{11} - \operatorname{pr}\hat{Q}_{11} = \left\| -\frac{1}{2}+\frac{1}{8}\rho^2, \quad 0, \quad 0, \quad \frac{1}{2}-\frac{1}{8}\rho^2 \right\|.$$

Since the vectors $\hat{Q}_{11}$, $\operatorname{pr}\hat{Q}_{11}$ are composed of the rows of the matrices Q_{11}, $\operatorname{pr}Q_{11}$, these matrices can be easily recovered

$$\operatorname{pr}Q_{11} = \begin{Vmatrix} \frac{1}{2}-\frac{1}{8}\rho^2 & 0 \\ 0 & \frac{1}{2}-\frac{1}{8}\rho^2 \end{Vmatrix}, \quad Q_{11} - \operatorname{pr}Q_{11} = \begin{Vmatrix} -\frac{1}{2}+\frac{1}{8}\rho^2 & 0 \\ 0 & \frac{1}{2}-\frac{1}{8}\rho^2 \end{Vmatrix}.$$

The vector $\hat{\Gamma}_{11}$ is obtained from the equation $Q_1\hat{\Gamma}_1 = \hat{Q}_{11} - \operatorname{pr}\hat{Q}_{11}$. By using the vector $\hat{\Gamma}_{11}$, the matrix Γ_{11} is recovered

$$\Gamma_{11} = \begin{Vmatrix} 0 & -\frac{1}{2}+\frac{1}{8}\rho^2 \\ 0 & 0 \end{Vmatrix}.$$

By using the matrix Γ_{11}, the operator S_{11} can be writtten as

$$S_{11} = \|\rho\sin\varphi, \rho\cos\varphi\|\,\Gamma_{11}\,L = \left(-\frac{1}{2}+\frac{1}{8}\rho^2\right)\rho\sin\varphi\,\boldsymbol{L}_2.$$

The solution of equation (60) is analogous to that of equation (59). Give now only the result

$$\operatorname{pr}Q_{31} = 0,$$

$$\boldsymbol{S}_{31} = \|\rho^3\sin 3\varphi, \rho^3\cos 3\varphi\|\,\Gamma_{31}\,L, \quad \Gamma_{31} = \begin{Vmatrix} 0 & \frac{1}{32} \\ -\frac{3}{32} & 0 \end{Vmatrix}.$$

Thus, the operator $\boldsymbol{N}_1$ defined by the matrix can be written in the final form

$$\boldsymbol{N}_1 = \|\rho\sin\varphi, \rho\cos\varphi\|\,\operatorname{pr}Q_{11}\,L \equiv$$

$$\equiv \left(\frac{1}{2}-\frac{1}{8}\rho^2\right)\rho\sin\varphi\,\boldsymbol{L}_1 + \left(\frac{1}{2}-\frac{1}{8}\rho^2\right)\rho\cos\varphi\,\boldsymbol{L}_2 \equiv \rho\left(\frac{1}{2}-\frac{1}{8}\rho^2\right)\frac{\partial}{\partial\rho}.$$

By using the operator $\boldsymbol{U}_0 = \boldsymbol{U} + \varepsilon\boldsymbol{N}_1$, we restore the centralized system in the first approximation

$$\dot{\rho} = \frac{\varepsilon}{2}\left(1-\frac{1}{4}\rho^2\right)\rho,$$
$$\dot{\varphi} = 1. \tag{62}$$

Compare the asymptotic decomposition algorithm in the representation space $T_{\otimes}$ of trigonometrical functions described in this section with the analogous algorithm in the space of polynomials $T(V_{\otimes})$ (see [4] and section 1).

We should mention now three very important items.

First is a substantial decrease of calculating efforts. This fact takes place due to lowering the order of the matrices of representations $\mathcal{F}_j$ of the operator $\boldsymbol{U}$ in the subspaces $T_{\otimes j}$ in comparison with the subspace $V_{\otimes j}$. Really, in the first case, the order of the matrices $\mathcal{F}_j$ is unchangeable and is equal to 2. In the second case, it grows proportionally to the index j according to the theorem 2 [4].

The second item relates to a better geometrical and physical interpretation of the new variables ρ and φ. For example, there are stationary points $\rho_0 = \pm 2$ in the first equation of system (62). The positive point makes geometrical sense of circle radius. This circle corresponds to a stable limiting cycle in variables of the initial system.

The third item concerns the definite optimal number of terms in expansions while using the space of representation of trigonometrical functions.

Finally, let us compare the alghorithm of asymptotic decomposition with the existing methods. The results of this subsection may be obtained by application of Krylov – Bogolyubov asymptotic method. The results in section 1 may be obtained by using the normal form method.

Both these results may be obtained by the asymptotic decomposition method depending on the choice of the representation space U in the zero approximation system. If the representation space $T_{\otimes}$ of the group $SO(2)$ is chosen, then we obtain the results of Krylov – Bogolyubov asymptotic method. If the representation space $V_{\otimes}$ of the general linear group $GL(2)$ (the space of homogeneous polynomials) is chosen, then we obtained the results of the method of normal forms.

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