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SOME RESULTS IN QUASITOPOLOGICAL HOMOTOPY GROUPS
ДЕЯКІ РЕЗУЛЬТАТИ З КВАЗІТОПОЛОГІЧНИХ ГОМОТОПІЧНИХ ГРУП

We show that the nth quasitopological homotopy group of a topological space is isomorphic to $(n-1)$th quasitopological homotopy group of its loop space and by this fact we obtain some results about quasitopological homotopy groups. Finally, using the long exact sequence of a based pair and a fibration in qTop introduced by Brazas in 2013, we obtain some results in this field.

Доведено, що n-та квазітопологічна гомотопічна група топологічного простору є ізоморфною $(n-1)$-й квазітопологічній гомотопічній групі його простору петель, та отримано деякі результати, що відносяться до квазітопологічних гомотопічних груп. Насамкінець за допомогою довгої точної послідовності для базової пари та розшарування у qTop, що запропонував Бразас у 2013 р., отримано деякі результати у цій області.

1. Introduction. Endowed with the quotient topology induced by the natural surjective map $q\colon \Omega^n(X,x) \to \pi_n(X,x)$, where $\Omega^n(X,x)$ is the nth loop space of (X,x) with the compact-open topology, the familiar homotopy group $\pi_n(X,x)$ becomes a quasitopological group which is called the quasitopological nth homotopy group of the pointed space (X,x), denoted by $\pi_n^{qtop}(X,x)$ (see [3–5, 10]).

It was claimed by Biss [3] that $\pi_1^{qtop}(X,x)$ is a topological group. However, Calcut and McCarthy [7] and Fabel [8] showed that there is a gap in the proof of [3] (Proposition 3.1). The misstep in the proof is repeated by Ghane et al. [10] to prove that $\pi_n^{qtop}(X,x)$ is a topological group [10] (Theorem 2.1) (see also [7]).

Calcut and McCarthy [7] showed that $\pi_1^{qtop}(X,x)$ is a homogeneous space and more precisely, Brazas [5] mentioned that $\pi_1^{qtop}(X,x)$ is a quasitopological group in the sense of [1].

Calcut and McCarthy [7] proved that for a path connected and locally path connected space X, $\pi_1^{qtop}(X)$ is a discrete topological group if and only if X is semilocally 1-connected (see also [5]). Pakdaman et al. [12] showed that for a locally $(n-1)$-connected space X, $\pi_n^{qtop}(X,x)$ is discrete if and only if X is semilocally n-connected at x (see also [10]). Also, they proved that the quasitopological fundamental group of every small loop space is an indiscrete topological group. We recall that a loop in X at x is called small if it is homotopic to a loop in every neighborhood U of x. Also the topological space X with non trivial fundamental group is called a small loop space if every loop of X is small.

In this paper, we obtain some results about quasitopological homotopy groups. One of the main results of Section 2 is Theorem 2.1.

By this fact we can show that some properties of a space can be transferred to its loop space. Also, we obtain several results in quasitopological homotopy groups. Moreover, we show that for a fibration $p\colon E \longrightarrow X$ with fiber F, the induced map $f_*\colon \pi_n^{qtop}(B,b_0) \longrightarrow \pi_{n-1}^{qtop}(F,\tilde{b_0})$ is continuous.

Brazas in his thesis [6] exhibited two long exact sequences of based pair (X,A) and fibration $p\colon E \longrightarrow X$ in qTop. In Section 3, we use these sequences and obtain some results in this filed.

2. Quasitopological homotopy groups. It is well-known that for a pointed topological space (X, x), for all $n \geq 1$ and $1 \leq k \leq n-1$, $\pi_n(X, x) \cong \pi_{n-k}(\Omega^k(X, x), e_x)$. In this section, we extend this result for quasitopological homotopy groups and we obtain some results about them. The following theorem is one of the main results of this paper.

Theorem 2.1. *Let (X, x) be a pointed topological space. Then, for all $n \geq 1$ and $1 \leq k \leq$ $\leq n-1$,*

$$\pi_n^{qtop}(X, x) \cong \pi_{n-k}^{qtop}(\Omega^k(X, x), e_x),$$

where e_x is the constant k-loop in X at x.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \Omega^n(X,x) & \xrightarrow{\ \phi\ } & \Omega^{n-k}(\Omega^k(X,x),e_x) \\ \Big\downarrow q & & q\Big\downarrow \\ \pi_n^{qtop}(X,x) & \xrightarrow{\ \phi_*\ } & \pi_{n-k}^{qtop}(\Omega^k(X,x),e_x), \end{array} \qquad (1)$$

where $\phi \colon \Omega^n(X, x) \longrightarrow \Omega^{n-k}(\Omega^k(X, x), e_x)$ given by $\phi(f) = f^\sharp$ is a homeomorphism with inverse $g \longmapsto g^\flat$ in the sense of [13]. Since the map q is a quotient map, the homomorphism ϕ_* is an isomorphism between quasitopological homotopy groups.

The following result is a consequence of Theorem 2.1.

***Corollary* 2.1.** *Let X be a locally $(n-1)$-connected. Then X is semilocally n-connected at x if and only if $\Omega^{n-1}(X, x)$ is semilocally simply connected at e_x, where e_x is the constant loop in X at x.*

Proof. Since X is a locally $(n-1)$-connected, by [12] (Theorem 6.7), X is semilocally n-connected at x if and only if $\pi_n^{qtop}(X, x)$ is discrete. By Theorem 2.1, $\pi_n^{qtop}(X, x) \cong$ $\cong \pi_1^{qtop}(\Omega^{n-1}(X, x), e_x)$. Also $\pi_1^{qtop}(\Omega^{n-1}(X, x), e_x)$ is discrete if and only if $\Omega^{n-1}(X, x)$ is semilocally simply connected at e_x by [12] (Theorem 6.7).

Note that the above result has been shown by Hidekazu Wada [17] (Remark) and Authors [11] (Lemma 3.1) with another methods.

***Corollary* 2.2.** *Let $(X, x) = \lim\limits_{\leftarrow}(X_i, x_i)$ be the inverse limit of an inverse system $\{(X_i, x_i), \varphi_{ij}\}_I$. Then, for all $n \geq 1$ and $1 \leq k \leq n-1$,*

$$\pi_n^{qtop}(X, x) \cong \pi_{n-k}^{qtop}(\lim_{\leftarrow} \Omega^k(X_i, x_i), e_x).$$

Virk [16] introduced the SG (small generated) subgroup of fundamental group $\pi_1(X, x)$, denoted by $\pi_1^{sg}(X, x)$, as the subgroup generated by the following elements

$$[\alpha * \beta * \alpha^{-1}],$$

where α is a path in X with initial point x and β is a small loop in X at $\alpha(1)$. Recall that a space X is said to be small generated if $\pi_1(X, x) = \pi_1^{sg}(X, x)$, also a space X is said to be semilocally small generated if for every $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U, x) \leq \pi_1^{sg}(X, x)$. Torabi et al. [15] proved that if X is small generated space, then $\pi_1^{qtop}(X, x)$ is an indiscrete topological group and the quasitopological fundamental group of a semilocally small generated space is a topological group. By Theorem 2.1, we obtain several results in quasitopological homotopy groups as follows:

***Corollary* 2.3.** *Let X be a topological space such that $\Omega^{n-1}(X,x)$ is small generated. Then $\pi_n^{qtop}(X,x)$ is an indiscrete topological group.*

Proof. Since $\Omega^{n-1}(X,x)$ is a small generated space, then $\pi_1^{qtop}(\Omega^{n-1}(X,x),e_x)$ is an indiscrete topological group, by [15] (Remark 2.11). Therefore $\pi_n^{qtop}(X,x)\cong\pi_1^{qtop}(\Omega^{n-1}(X,x),e_x)$ implies that $\pi_n^{qtop}(X,x)$ is an indiscrete topological group.

***Corollary* 2.4.** *Let X be a topological space such that $\Omega^{n-1}(X,x)$ is a semilocally small generated space. Then $\pi_n^{qtop}(X,x)$ is a topological group.*

Proof. Since $\Omega^{n-1}(X,x)$ is semilocally small generated, then $\pi_1^{qtop}(\Omega^{n-1}(X,x),e_x)$ is a topological group, by [15] (Theorem 4.1). Therefore $\pi_n^{qtop}(X,x)\cong\pi_1^{qtop}(\Omega^{n-1}(X,x),e_x)$ implies that $\pi_n^{qtop}(X,x)$ is a topological group.

Fabel [8] proved that $\pi_1^{qtop}(HE,x)$ is not topological group. By considering the proof of this result it seems that if $\pi_1(X,x)$ is an abelian group, then $\pi_1^{qtop}(X,x)$ is a topological group. He [9] also showed that for each $n\geq 2$ there exists a compact, path connected, metric space X such that $\pi_n^{qtop}(X,x)$ is not a topological group. In the following example we show that there is a metric space Y with Abelian fundamental group such that $\pi_1^{qtop}(Y,y)$ is not a topological group.

***Example* 2.1.** Let $n\geq 2$, X be the compact, path connected, metric space introduced in [9] such that $\pi_n^{qtop}(X,x)$ is not a topological group. By Theorem 2.1 $\pi_1^{qtop}(\Omega^{n-1}(X,x),e_x)$ is not a topological group. Since for every $n\geq 2$, $\pi_n(X,x)$ is an Abelian group, hence there is a metric space $Y=\Omega^{n-1}(X,x)$ with Abelian fundamental group such that $\pi_1^{qtop}(Y,y)$ is not a topological group.

In [4] (Proposition 3.25), it is proved that the quasitopological fundamental groups of shape injective spaces are Hausdorff. By Theorem 2.1 we have the following result.

***Corollary* 2.5.** *Let X be a topological space such that $\Omega^{n-1}(X,x)$ is shape injective space. Then $\pi_n^{qtop}(X,x)$ is Hausdorff.*

Proposition 2.1 [15]. *For a pointed topological space (X,x), if $\{[e_x]\}$ is closed (or equivalently the topology of $\pi_1^{qtop}(X,x)$ is T_0), then X is homotopically Hausdorff.*

We generalized the above proposition as follows:

Proposition 2.2. *For a pointed topological space (X,x), if $\{[e_x]\}$ is closed (or equivalently the topology of $\pi_n^{qtop}(X,x)$ is T_0), then X is n-homotopically Hausdorff.*

Proof. By Theorem 2.1 since $\pi_n^{qtop}(X,x)$ is T_0, hence $\pi_1^{qtop}(\Omega^{n-1}(X,x),e_x)$ is T_0. Therefore by previous proposition $\Omega^{n-1}(X,x)$ is homotopically Hausdorff which implies that X is n-homotopically Hausdorff by [11] (Lemma 3.5).

***Corollary* 2.6.** *Let X be a topological space such that $\Omega^{n-1}(X,x)$ is shape injective space. Then X is n-homotopically Hausdorff.*

Proof. It follows from Corollary 2.5 and Proposition 2.2.

Let (B,b_0) be a pointed space and $p\colon E\longrightarrow B$ be a fibration with fiber F. Consider its mapping fiber, $Mp=\{(e,\omega)\in E\times B^I : \omega(0)=b_0 \text{ and } \omega(1)=p(e)\}$. If $\tilde{b_0}\in p^{-1}(b_0)$, then the injection map $k\colon\Omega(B,b_0)\longrightarrow Mp$ given by $k(\omega)=(\tilde{b_0},\omega)$ induces a homomorphism $f_*\colon\pi_n(B,b_0)\longrightarrow\pi_{n-1}(F,\tilde{b_0})$ [13].

Theorem 2.2. *Let (B,b_0) be a pointed space and $p\colon E\longrightarrow B$ be a fibration. If $\tilde{b_0}\in p^{-1}(b_0)$, then $f_*\colon\pi_n^{qtop}(B,b_0)\longrightarrow\pi_{n-1}^{qtop}(F,\tilde{b_0})$ is continuous for all $n\geq 1$.*

Proof. We consider the following commutative diagram:

$$\begin{array}{ccc} \Omega^{n-1}(\Omega(B,b_0),e_{b_0}) & \xrightarrow{k_\sharp} & \Omega^{n-1}(Mp,*) \\ \Big\downarrow q & & q\Big\downarrow \\ \pi_{n-1}^{qtop}(\Omega(B,b_0),e_{b_0}) & \xrightarrow{k_*} & \pi_{n-1}^{qtop}(Mp,*), \end{array} \tag{2}$$

where q is the quotient map and $k_\sharp$ is the induced map of $k\colon \Omega(B,b_0) \longrightarrow Mp$ by the functor Ω^{n-1}. Since $k_\sharp$ is continuous and q is a quotient map, $k_*\colon \pi_{n-1}^{qtop}(\Omega(B,b_0),e_{b_0}) \longrightarrow \pi_{n-1}^{qtop}(F,\tilde{b_0})$ is continuous. By Theorem 2.1, $\pi_{n-1}^{qtop}(\Omega(B,b_0),e_{b_0})$ is isomorphic to $\pi_n^{qtop}(B,b_0)$. Therefore, $f_*\colon \pi_n^{qtop}(B,b_0) \longrightarrow \pi_{n-1}^{qtop}(F,\tilde{b_0})$ is continuous.

3. Long exact sequence of $\pi_n^{qtop}(X)$. Brazas [6] (Theorem 2.49) proved that for every based pair (X, A) with inclusion $i\colon A \longrightarrow X$, there is a long exact sequence in the category of quasitopological groups as follows:

$$\ldots \longrightarrow \pi_{n+1}^{qtop}(A) \to \pi_{n+1}^{qtop}(X) \to \pi_{n+1}^{qtop}(X,A) \to \pi_n^{qtop}(A) \longrightarrow \ldots$$

$$\cdots \longrightarrow \pi_1^{qtop}(X) \longrightarrow \pi_1^{qtop}(X,A) \longrightarrow \pi_0^{qtop}(A) \longrightarrow \pi_0^{qtop}(X).$$

He [6] (Proposition 2.20) also showed that for every fibration $p\colon E \longrightarrow B$ of path connected spaces with fiber F, there is a long exact sequence in the category of quasitopological groups as follows:

$$\cdots \longrightarrow \pi_n^{qtop}(E) \to \pi_n^{qtop}(B) \to \pi_{n-1}^{qtop}(F) \to \pi_{n-1}^{qtop}(E) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1^{qtop}(B) \longrightarrow \pi_0^{qtop}(F) \longrightarrow \pi_0^{qtop}(E) \longrightarrow \pi_0^{qtop}(B). \tag{3}$$

In follow, we obtain some results and examples by these exact sequences.

***Example* 3.1.** Consider the pointed pair (HA, HE), where HA is the harmonic archipelago and HE is the hawaiian earring. Then by [6] (Theorem 2.49), there is a long exact sequence in $qTop$:

$$\cdots \longrightarrow \pi_{n+1}^{qtop}(HE) \to \pi_{n+1}^{qtop}(HA) \to \pi_{n+1}^{qtop}(HA,HE) \to \pi_n^{qtop}(HE) \longrightarrow \ldots$$

$$\cdots \longrightarrow \pi_1^{qtop}(HA) \longrightarrow \pi_1^{qtop}(HA,HE) \longrightarrow \pi_0^{qtop}(HE) \longrightarrow \pi_0^{qtop}(HA).$$

Recall that a short exact sequence $E\colon 0 \longrightarrow H \overset{i}{\to} X \overset{\pi}{\to} G \longrightarrow 0$ of topological Abelian groups will be called an extension of topological groups if both i and π are continuous and open homomorphisms when considered as maps onto their images. Also, the extension E is called split if and only if it is equivalent to the trivial extension $E_0 : 0 \longrightarrow H \overset{i_H}{\to} H \times G \overset{\pi_G}{\to} G \longrightarrow 0$ [2].

Theorem 3.1 ([2], Theorem 1.2)**.** *Let $E\colon 0 \longrightarrow H \overset{i}{\to} X \overset{\pi}{\to} G \longrightarrow 0$ be an extension of topological Abelian groups. The following are equivalent*:

(1) *E splits.*

(2) *There exists a right inverse for π.*

(3) *There exists a left inverse for i.*

The above results hold for quasitopological groups, too.

Proposition 3.1. *If $r\colon X \longrightarrow A$ is a retraction, then there are isomorphisms in quasitopological groups, for all $n \geq 2$,*

$$\pi_n^{qtop}(X) \cong \pi_n^{qtop}(A) \times \pi_n^{qtop}(X,A).$$

Proof. Consider the pointed pair (X, A). By [6] (Theorem 2.49), there is a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}^{qtop}(X) \longrightarrow \pi_{n+1}^{qtop}(X, A) \longrightarrow \pi_n^{qtop}(A) \xrightarrow{i_*} \pi_n^{qtop}(X) \longrightarrow \pi_n^{qtop}(X, A) \longrightarrow \ldots .$$

Since r is a retraction and i_* is an injection, there is a short exact sequence

$$0 \longrightarrow \pi_n^{qtop}(A) \xrightarrow{i_*} \pi_n^{qtop}(X) \rightarrow \pi_n^{qtop}(X, A) \longrightarrow 0.$$

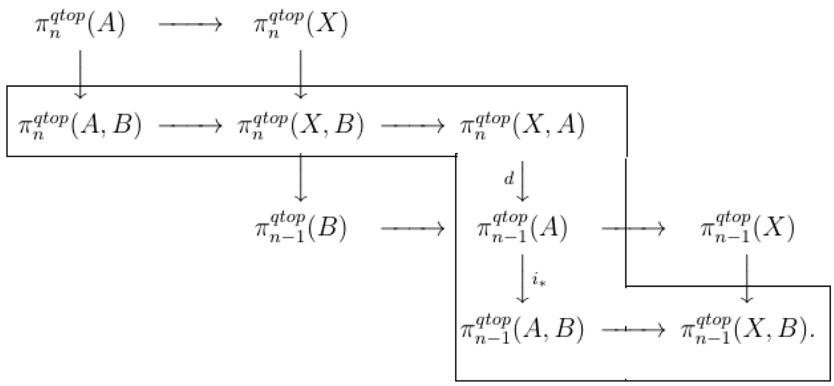
Moreover, this sequence is an extension. Indeed, the map i_* and π_* are continuous and open homomorphisms when considered as maps onto their images. Therefore,

$$\pi_n^{qtop}(X) \cong \pi_n^{qtop}(A) \times \pi_n^{qtop}(X, A).$$

Proposition 3.2. *Let $B \subseteq A \subseteq X$ be pointed spaces. Then there is a long exact sequence of the triple (X, A, B) in qTop:*

$$\cdots \longrightarrow \pi_{n+1}^{qtop}(X, A) \longrightarrow \pi_n^{qtop}(A, B) \longrightarrow \pi_n^{qtop}(X, B) \longrightarrow \pi_n^{qtop}(X, A) \longrightarrow \pi_{n-1}^{qtop}(A, B) \longrightarrow \ldots .$$

Proof. Consider the following commutative diagram and chase a long diagram as follows:



The following results are immediate consequences of sequence (3).

***Corollary* 3.1.** *If $p\colon E \longrightarrow B$ is a fibration with E contractible, then $f_*: \pi_n^{qtop}(B, b_0) \longrightarrow \longrightarrow \pi_{n-1}^{qtop}(F, \tilde{b_0})$ is an isomorphism in quasitopological groups for all $n \geq 2$ and $f_*: \pi_1^{qtop}(B, b_0) \longrightarrow \longrightarrow \pi_0^{qtop}(F)$ is an isomorphism in Set.*

***Corollary* 3.2.** *Let (X, x) be a pointed topological space. Then $\pi_n^{qtop}(X, x) \cong \pi_{n-1}^{qtop}(\Omega(X, x), e_x)$ in quasitopological groups for all $n \geq 2$, where e_x is the constant loop in X at x and $\pi_1^{qtop}(X, x) \cong \cong \pi_0^{qtop}(\Omega(X, x))$ in Set.*

Proof. By [14] (Proposition 4.3), the map $p\colon PX \longrightarrow X$ is a fibration with fiber $\Omega(X, x)$, where $PX = (X, x)^{(I,0)}$. By [6] (Proposition 2.20), the sequence

$$\cdots \longrightarrow \pi_n^{qtop}(PX, e_x) \longrightarrow \pi_n^{qtop}(X, x) \longrightarrow \pi_{n-1}^{qtop}(\Omega(X, x), e_x) \longrightarrow \pi_{n-1}^{qtop}(PX, e_x) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1^{qtop}(X, x) \longrightarrow \pi_0^{qtop}(\Omega(X, x)) \longrightarrow \pi_0^{qtop}(PX) \longrightarrow \pi_0^{qtop}(X)$$

is exact in qTop. By [14] (Proposition 4.4), (PX, e_x) is contractible and therefore the result holds by Corollary 3.1.

***Corollary* 3.3.** *If* $p\colon \tilde{X} \longrightarrow X$ *is a covering projection, then for all* $n \geq 2$, $\pi_n^{qtop}(\tilde{X}) \cong \pi_n^{qtop}(X)$ *in quasitopological groups and* $\pi_1^{qtop}(\tilde{X})$ *can be embedded in* $\pi_1^{qtop}(X)$.

Proof. This result follows by sequence (3) and this fact that the fiber F of the covering projection p is discrete and therefore $\pi_n^{qtop}(F)$ is trivial for all $n \geq 1$.

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