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Z. Kamont, prof. (Inst. Math. Univ., Gdańsk)

FINITE-DIFFERENCE APPROXIMATIONS FOR FIRST-ORDER PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS

СКІНЧЕННОРІЗНИЦЕВА АПРОКСИМАЦІЯ ДИФЕРЕНЦІАЛЬНО-ФУНКЦІОНАЛЬНИХ РІВНЯНЬ З ЧАСТИННИМИ ПОХІДНИМИ ПЕРШОГО ПОРЯДКУ

We consider initial-boundary-value problems of the Dirichlet type for nonlinear equations. We give sufficient conditions of convergence of a general class of one-step difference methods. We assume that the right-hand side of the equation satisfies an estimate of Perron type with respect to the functional argument.

Для нелінійних рівнянь розглядається аналог задачі Діріхле з крайовою початковою умовою. Наведені достатні умови для збіжності загального класу однокрокових різницевих методів. Припускається, що права частина рівняння задовольняє аналог оцінки Перрона відносно функції, що є аргументом.

1. Introduction. Denote by $C(X, Y)$ the class of all continuous functions from X into Y, where X and Y are metric spaces. Define $E = [0, a] \times [-b, b]$, where $a \in R$, $b = (b_1, \dots, b_n) \in R^n$, $a > 0$, and $b_i > 0$ for $i = 1, \dots, n$. Let $\tau_0 \in R_+$, $\tau = (\tau_1, \dots, \tau_n) \in R_+^n$, $R_+ = [0, +\infty)$, and $D = [-\tau_0, 0] \times [-\tau, \tau]$. Define also $c = (c_1, \dots, c_n) = b + \tau$ and $E^{(0)} = ([-\tau_0, a] \times [-c, c]) \setminus ([0, a] \times [-b, b])$. If $z: E^{(0)} \cup E \to R$ is a function of the variables $(x, y) = (x, y_1, \dots, y_n)$ and there exist derivatives $D_{y_i} z$, $i = 1, \dots, n$, then we write $D_y z = (D_{y_1}, \dots, D_{y_n})$. For $(x, y) \in E$ and $z: E^{(0)} \cup E \to R$, we define a function $z_{(x,y)}: D \to R$ by $z_{(x,y)}(t, s) = z(x + t, y + s)$, where $(t, s) = (t, s_1, \dots, s_n) \in D$. The function $z_{(x,y)}$ is the restriction of z to the set $[x - \tau_0, x] \times [y - \tau, y + \tau]$, and this restriction is shifted to the set D. Let $\Omega = E \times C(D, R) \times R^n$ and assume that $\varphi: E^{(0)} \to R$ and $f: \Omega \to R$ are given functions. The paper deals with the initial-boundary-value problem

$$\begin{aligned} D_x z(x, y) &= f(x, y, z_{(x,y)}, D_y z(x, y)), \\ z(x, y) &= \varphi(x, y) \quad \text{for } (x, y) \in E^{(0)}. \end{aligned} \tag{1}$$

Below, we give examples of equations which can be derived from (1) by specifying the operator f.

***Example* 1.** Assume that $F: E \times R \times R^n \to R$ and $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n): E \to R^{1+n}$. We define $f(x, y, w, q) = F(x, y, w(\alpha_0(x, y) - x, \alpha'(x, y) - y), q)$, $(x, y, w, q) \in \Omega$, $\alpha' = (\alpha_1, \dots, \alpha_n)$. Then we have an equation with a deviated argument

$$D_x z(x, y) = F(x, y, z(\alpha(x, y)), D_y z(x, y)).$$

***Example* 2.** For F introduced above, we define

$$f(x, y, w, q) = F\left(x, y, \int_D w(t, s)\, dt\, ds, q\right), \qquad (x, y, w, q) \in \Omega.$$

Then (1) is the initial-boundary-value problem for the differential-integral equation

$$D_x z(x, y) = F\left(x, y, \int_D z(x + t, y + s)\, dt\, ds, D_y z(x, y)\right).$$

Our formulation of the differential-functional problem is also motivated by a general model of the functional dependence in ordinary differential-functional equations [1].

We consider classical solutions of (1). Finite-difference approximations relative to initial-value problems or initial-boundary-value problems for first-order partial differential equations have been investigated in [2–7]. Generalizations of these results to a differential-functional case can be found in [8, 9]. The main problem in these investigations is to find a suitable difference equation or a difference-functional equation which would satisfy a consistency condition with respect to the original problem and be stable. The method of difference inequalities or simple theorems on recurrence inequalities are used in the investigation of the stability. The authors have assumed that given functions have partial derivatives with respect to all arguments with the exception of (x, y). In [8, 9], it is assumed that the right-hand sides of differential-functional equations satisfy the Lipschitz condition with respect to the functional argument.

In the present paper, we introduce nonlinear estimations with respect to the functional argument. More precisely, we assume that the function f of the variables (x, y, w, q) has partial derivatives with respect to $q = (q_1, \dots, q_n)$ and satisfies a nonlinear estimation of the Perron type with respect to the functional argument. Note that the conditions indicated above are identical with the assumptions that guarantee the uniqueness of solutions of initial or initial-boundary-value problems for differential or differential-functional equations [1, 10, 11].

In this paper, we use general ideas for finite difference approximations relative to partial differential equations which were introduced in [1–6, 8, 10, 12–18]. For further bibliography concerning approximate solutions of first-order partial differential equations, see the references in the papers cited above and in [8, 12–14, 17].

An error estimate implying the convergence of difference schemes is obtained in [14] by the method of difference inequalities. Therefore, the authors have assumed in [14] that the right-hand sides of equations are nondecreasing with respect to the functional argument. In this paper, we omit the assumption on this monotonicity.

We prove a general theorem on the error estimate of approximate solutions of difference-functional equations of the Volterra type.

For $x \in [-\tau_0, a]$, we define $E_x = \{(t, y) \in E: t \le x\}$ and $E_x^{(0)} = \{(t, y) \in E^{(0)}: t \le x\}$ and denote by $\|z\|_x$ the supremum norm of $z \in C(E_x^{(0)} \cup E_x, R)$. We shall use vector inequalities with understanding that the same inequalities hold for their components. For $N = (N_1, \dots, N_n) \in N^n$, where N is the set of natural numbers, we put $N + 1 = (N_1 + 1, \dots, N_n + 1)$. For $y = (y_1, \dots, y_n)$ and $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$, $y, \bar{y} \in R^n$, we define $y * \bar{y} = (y_1 \bar{y}_1, \dots, y_n \bar{y}_n)$. We introduce a mesh in $E^{(0)} \cup E$. Let $d = (d_0, d_1, \dots, d_n) \in R^{1+n}$, $d_i > 0$ for $i = 0, 1, \dots, n$, and let $I_d \subset (0, d]$. Suppose that, for $h = (h_0, h') \in I_d$, $h' = (h_1, \dots, h_n)$, there exists $N_0 \in N$, $N = (N_1, \dots, N_n) \in N^n$, such that $N_0 h_0 = \tau_0$, $N * h' = c$. In what follows, we introduce additional assumptions concerning h and I_d. Assume that $\tilde{N}_0 \in N$ and $\tilde{N} = (\tilde{N}_1, \dots, \tilde{N}_n) \in N^n$ are defined by

$$\tilde{N}_0 h_0 \le a < (\tilde{N}_0 + 1) h_0 \quad \text{and} \quad N * h' < b \le (\tilde{N} + 1) * h'.$$

For $m = (m_0, m_1, \dots, m_n)$, where m_i are integers, we denote $m' = (m_1, \dots, m_n)$ and $x^{(m_0)} = m_0 h_0$, $y^{(m')} = \left(y_1^{(m_1)}, \dots, y_n^{(m_n)}\right)$, where $y^{(m')} = m' * h'$. Let $M_h = \{m': -N \le m' \le N\}$ and $\tilde{M}_h = \{m': -\tilde{N} \le m' \le \tilde{N}\}$. We define

$$E_h = \{(x^{(m_0)}, y^{(m')}): m_0 = 0, 1, \dots, \tilde{N}_0, \quad m' \in \tilde{M}_h\},$$

$$E_h^{(0)} = \{(x^{(m_0)}, y^{(m')}): m_0 = -N_0, -N_0 + 1, \dots, 0, \quad m' \in M_h\} \cup$$

$$\cup \{(x^{(m_0)}, y^{(m')}) : m_0 = 1, \dots, \tilde{N}_0, \quad m' \in M_h \setminus \tilde{M}_h\}.$$

Then E_h and $E_h^{(0)}$ are sets of mesh points in E and $E^{(0)}$, respectively. For a function $z : E_h^{(0)} \cup E_h \to R$, we write $z^{(m)} = z(x^{(m_0)}, y^{(m')})$.

Now we define discrete analogs of E_x, $\|z\|_x$, D, and $z_{(x,y)}$. We define

$$E_{h,m_0} = \{(x^{(i)}, y^{(m')}) \in E_h^{(0)} \cup E_h : i \le m_0\}$$

and

$$\|z\|_{h,m_0} = \max\{|z^{(i,m')}| : (x^{(i)}, y^{(m')}) \in E_{h,m_0}\},$$

where $-N_0 \le m_0 \le \tilde{N}_0$. Let $M_h^* = \{m : m_0 = 0, 1, \dots, \tilde{N}_0 - 1, \ m' \in \tilde{M}_h\}$.

Now we define a mesh in the set D. There exists $K = (K_1, \dots, K_n) \in N^n$ such that $K * h' \le \tau \le (K+1) * h'$. Let

$$D_h = \{(x^{(m)}, y^{(m')}) : m_0 = -N_0, -N_0 + 1, \dots, 0, \quad -K \le m' \le K\}.$$

If $z : E_h^{(0)} \cup E_h \to R$ and $0 \le m_0 \le N_0$, $m' \in \tilde{M}_h$, then $z_{(m)} : D_h \to R$ is a function defined by

$$z_{(m)}(x^{(s_0)}, y^{(s')}) = z(x^{(m_0+s_0)}, y^{(m'+s')}),$$

$$(s_0, s') = (s_0, s_1, \dots, s_n), \quad -N_0 \le s_0 \le 0, \quad -K \le s' \le K.$$

Let $S = \{s' = (s_1, \dots, s_n) : s_i \in \{-1, 0, 1\}$ for $i = 1, \dots, n\}$ and $z : E_h^{(0)} \cup E_h \to \to R$. We define operators A, Δ_0, and $\Delta = (\Delta_1, \dots, \Delta_n)$ in the following way:

$$Az^{(m)} = \sum_{s' \in S} a_{s',m} z^{(m_0, m'+s')},$$

$$\Delta_0 z^{(m)} = h_0^{-1}[z^{(m_0+1,m')} - Az^{(m)}], \tag{2}$$

$$\Delta_i z^{(m)} = h_i^{-1} \sum_{s' \in S} c_{s',m}^{(i)} z^{(m_0, m'+s')}, \quad i = 1, \dots, n,$$

where $m_0 = 0, 1, \dots, \tilde{N}_0$, $m' \in \tilde{M}_h$, $a_{s',m}, c_{s',m}^{(i)} \in R$, and $\Delta z^{(m)} = (\Delta_1 z^{(m)}, \dots$ $\dots, \Delta_n z^{(m)})$. We will approximate $D_x z(x, y)$, $D_y z(x, y)$, and $z(x, y)$ by $\Delta_0 z^{(m)}$, $\Delta z^{(m)}$, and $Az^{(m)}$, respectively.

For any two sets X and Y, we denote by $F(X, Y)$ the class of all functions defined on X and taking values in Y. We introduce now an operator $T_h : F(E_h^{(0)} \cup$ $\cup E_h, R) \to F(E^{(0)} \cup E, R)$ as follows: Put $S^* = \{s = (s_0, s_1, \dots, s_n) : s_i \in \{0,$ $1\}$ for $i = 0, 1, \dots, n\}$. Let $z \in F(E_h^{(0)} \cup E_h, R)$ and $(x, y) \in E^{(0)} \cup E$. Then there exists m such that $(x^{(m_0)}, y^{(m')})$, $(x^{(m_0+1)}, y^{(m'+1)}) \in E_h^{(0)} \cup E_h$, $m' + 1 =$ $= (m_1 + 1, \dots, m_n + 1)$, $x^{(m_0)} \le x \le x^{(m_0+1)}$, and $y^{(m')} \le y \le x^{(m'+1)}$. We define

$$(T_h z)(x, y) =$$

$$= \sum_{s \in S^*} z^{(m+s)} \left(\frac{x - x^{(m_0)}}{h_0}\right)^{s_0} \left(1 - \frac{x - x^{(m_0)}}{h_0}\right)^{1-s_0} \left(\frac{y - y^{(m')}}{h'}\right)^{s'} \left(1 - \frac{y - y^{(m')}}{h'}\right)^{1-s'} \tag{3}$$

where

$$\left(\frac{y-y^{(m')}}{h'}\right)^{s'} = \prod_{i=1}^{n}\left(\frac{y_i - y_i^{(m_i)}}{h_i}\right)^{s_i},$$
$$\left(1-\frac{y-y^{(m')}}{h'}\right)^{1-s'} = \prod_{i=1}^{n}\left(1-\frac{y_i - y_i^{(m_i)}}{h_i}\right)^{1-s_i} \tag{4}$$

and we take $0^0 = 1$ in (4). Thus, we have $T_h z\colon E^{(0)} \cup E \to R$. We will interpolate functions $z\colon E_h^{(0)} \cup E_h \to R$ by means of $T_h z$.

***Remark* 1** [8]. (i) If $z\colon E_h^{(0)} \cup E_h \to R$, then $T_h z \in C(E^{(0)} \cup E, R)$.

(ii) Suppose that the function $z : E^{(0)} \cup E \to R$ belongs to the class C^1 and $|D_x z(x,y)| \le C$ and $|D_{y_i} z(x,y)| \le C$, $i = 1, \dots, n$, for $(x,y) \in E$. Let $z_h = z|_{E_h^{(0)} \cup E_h}$. Then $\|T_h z - z\|_x \le C|h|$ for $x \in [-\tau_0, a]$, where $|h| = h_0 + h_1 + \dots + h_n$.

We consider the following difference method for problem (1)

$$\Delta_0 z^{(m)} = f(x^{(m_0)}, y^{(m')}, (T_h z)_{(m)}, \Delta z^{(m)}), \quad m \in M_h^*,$$
$$z^{(m)} = \varphi_h^{(m)} \quad \text{on } E_h^{(0)}, \tag{5}$$

where $\varphi_h\colon E_h^{(0)} \to R$ is a given function and $(T_h z)_{(m)} = (T_h z)_{(x^{(m_0)}, y^{(m')})}$.

2. Error estimations of approximate solutions of difference-functional equations. Suppose that $F_h\colon E_h \times F(E_h^{(0)} \cup E_h, R) \to R$, $h \in I_d$. For $(x^{(m_0)}, y^{(m')}, z) \in E_h \times F(E_h^{(0)} \cup E_h, R)$ we denote $F_h[m,z] = F_h(x^{(m_0)}, y^{(m')}, z)$. We say that F_h satisfies the Volterra condition if, for each $(x^{(m_0)}, y^{(m')}) \in E_h$ and for $z, \bar{z} \in F(E_h^{(0)} \cup E_h, R)$ such that $z|_{E_{h,m_0}} = \bar{z}|_{E_{h,m_0}}$, we have $F_h[m,z] = F_h[m,\bar{z}]$. Let $X_h = \{x^{(-N_0)}, x^{(-N_0+1)}, \dots, x^{(0)}, x^{(1)}, \dots, x^{(\tilde{N}_0)}\}$, let $X_h^* = \{x^{(0)}, \dots, x^{(\tilde{N}_0)}\}$, and let $V_h\colon F(E_h^{(0)} \cup E_h, R) \to F(X_h, R_+)$ be given by $(V_h z)(x^{(m_0)}) = \max\{|z^{(m)}|\colon m' \in M_h\}$. In the sequel, we will write $(V_h z)(m_0)$ instead of $(V_h z)(x^{(m_0)})$. Suppose that $\sigma_h\colon X_h^* \times F(X_h, R_+) \to R_+$. If $(x^{(m_0)}, \eta) \in X_h^* \times F(X_h, R_+)$, then we denote $\sigma_h[m_0, \eta] = \sigma_h(x^{(m_0)}, \eta)$. We say that the function σ_h satisfies the Volterra condition if, for each $x^{(m_0)} \in X_h^*$ and for $\eta, \bar{\eta} \in F(X_h, R_+)$ such that $\eta(x^{(i)}) = \bar{\eta}(x^{(i)})$ for $-N_0 \le i \le m_0$, we have $\sigma_h[m_0, \eta] = \sigma_h(m_0, \bar{\eta})$.

For a given $\varphi_h\colon E_h^{(0)} \to R$, we consider the problem

$$z^{(m_0+1, m')} = F_h[m,z], \quad m \in M_h^*,$$
$$z^{(m)} = \varphi_h^{(m)} \quad \text{on } E_h^{(0)}. \tag{6}$$

If we assume that F_h satisfies the Volterra condition, then there exists exactly one solution of (6). Now we prove a theorem on the estimation of the difference between the exact and approximate solutions of (6).

Theorem 1. *Suppose that*

1°) *the function* $F_h\colon E_h \times F(E_h^{(0)} \cup E_h, R) \to R$, $h \in I_d$, *satisfies the Volterra*

condition and $\varphi_h : E_h^{(0)} \to R$;

2°) *there exists a function* $\sigma_h : X^* \times F(X_h, R_+) \to R_+$ *such that*

i) σ_h *is nondecreasing with respect to the functional argument and satisfies the Volterra condition;*

ii) *for* $m \in M_h^*$ *and* $z, \bar{z} \in F(E_h^{(0)} \cup E_h, R)$, *we have*

$$|F_h[m, z] - F_h[m, \bar{z}]| \le \sigma_h[m_0, V_h(z - \bar{z})];$$

3°) $\bar{u}_h : E_h^{(0)} \cup E_h \to R$ *is a solution of* (6), $\bar{v}_h : E_h^{(0)} \cup E_h \to R$, *and there exist* $\beta_h : X_h \to R_+$, $\gamma_h : X_h^* \to R_+$, $h \in I_d$, *such that*

$$\left|\bar{v}_h^{(m)} - \varphi_h^{(m)}\right| \le \beta_h^{(m_0)} \quad \text{on } E_h^{(0)}, \tag{7}$$

$$\left|\bar{v}_h^{(m_0+1,m')} - F_h[m, \bar{v}_h]\right| \le \gamma_h^{(m_0)} \quad \text{for } m \in M_h^*, \tag{8}$$

and

$$\beta_h^{(m_0+1)} \ge \sigma_h[m_0, \beta_h] + \gamma_h^{(m_0)}, \quad m_0 = 0, 1, \ldots, \tilde{N}_0 - 1.$$

Under these assumptions, for $m_0 = -N_0, -N_0 + 1, \ldots, 0, 1, \ldots, \tilde{N}_0$ *and* $m' \in M_h$, *we have*

$$\left|\bar{v}_h^{(m)} - \bar{u}_h^{(m)}\right| \le \beta_h^{(m_0)}. \tag{9}$$

Proof. It follows from (7) that estimation (9) holds for $-N_0 \le m_0 \le 0$, $m' \in M_h$. Assume that (9) is satisfied on E_{h,m_0}. Then we have from assumption 2° and (8)

$$\left|\bar{v}_h^{(m_0+1,m')} - \bar{u}_h^{(m_0+1,m')}\right| \le \left|\bar{v}_h^{(m_0+1,m')} - F_h[m, \bar{v}_h]\right| +$$

$$+ |F_h[m, \bar{v}_h] - F_h[m, \bar{u}_h]| \le \gamma_h^{(m_0)} + \sigma_h[m_0, V_h(\bar{v}_h - \bar{u}_h)] \le$$

$$\le \gamma_h^{(m_0)} + \sigma_h[m_0, \beta_h] \le \beta_h^{(m_0+1)}, \quad m' \in \tilde{M}_h.$$

This estimation and (7) imply

$$\left|\bar{v}_h^{(m_0+1,m')} - \bar{u}_h^{(m_0+1,m')}\right| \le \beta_h^{(m_0+1)}, \quad m' \in M_h.$$

This completes the proof of (9).

***Remark* 2.** If the assumptions of Theorem 1 are satisfied and β_h is nondecreasing on X_h, then $\|\bar{v}_h - \bar{u}_h\|_{h,m_0} \le \beta_h^{(m_0)}$ for $-N_0 \le m_0 \le \tilde{N}_0$.

3. Convergence of the difference method. For $w \in C(D, R)$ we denote by $\|w\|_{C(D)}$ the supremum norm of w. For the above w, we define $Vw : [-\tau_0, 0] \to R_+$ by

$$(Vw)(t) = \max\{|w(t, s)| : s \in [-\tau, \tau]\}, \quad t \in [-\tau_0, 0].$$

If $\eta : [-\tau_0, a_0) \to R_+$ and $a_0 > 0$, $x \in [0, a_0]$, then $\eta_{(x)} : [-\tau_0, 0) \to R$ is a function given by $\eta_{(x)}(t) = \eta(x + t)$, $t \in [-\tau_0, 0]$. Suppose that $N^* \in N$ is defined by $N^* h_0 < a_0 \le (N^* + 1)h_0$. If $\eta : [-\tau_0, a_0) \to R$, then η_{h_0} is the restriction of η to the set $J^* = \{x^{(-N_0)}, x^{(-N_0+1)}, \ldots, x^{(N^*)}\}$. Suppose now that $\beta : J^* \to R$. Denote by $L[h_0, \beta]$ a function given by

$$L[h_0, \beta] : [-\tau_0, a^*] \to R, \quad a^* = N^* h_0,$$

$$L[h_0, \beta](x) = \beta^{(i+1)} h_0^{-1}(x - x^{(i)}) + \beta^{(i)}[1 - h_0^{-1}(x - x^{(i)})], \quad x \in [x^{(i)}, x^{(i+1)}].$$

In the sequel, we will need the following assumption:

Assumption H_0. Suppose that the function $\sigma : [0, a_0) \times C([-\tau_0, 0], R_+) \to R_+$, $a_0 > a$, satisfies the conditions

1°) σ is continuous on $[0, a_0) \times C([-\tau_0, 0], R_+)$ and $\sigma(x, \theta) = 0$ for $x \in [0, a_0)$, where $\theta(t) = 0$ for $t \in [-\tau_0, 0]$;

2°) if (x, w), $(\bar{x}, \bar{w}) \in [0, a_0) \times C([-\tau_0, 0], R_+)$ and $x \le \bar{x}$, $w \le \bar{w}$, then $\sigma(x, w) \le \sigma(\bar{x}, \bar{w})$;

3°) the function $\eta(x) = 0$, for $x \in [-\tau_0, a_0)$, is a unique solution of the problem

$$\eta'(x) = \sigma(x, \eta_{(x)}), \quad \eta(x) = 0 \quad \text{for} \quad x \in [-\tau_0, 0]. \tag{10}$$

Assumption H_1. Suppose that

1°) the function $\varphi : E^{(0)} \to R$ is of class C^1, $f \in C(\Omega, R)$, and

$$|f(x, y, w, q) - f(x, y, \bar{w}, q)| \le \sigma(x, V(w - \bar{w})) \quad \text{on} \quad \Omega, \tag{11}$$

where σ is given by Assumption H_0;

2°) for each $P = (x, y, w, q) \in \Omega$ there exist partial derivatives $(D_{q_1} f(P), \dots, D_{q_n} f(P)) = D_q f(P)$ and $D_q f(x, y, w, \cdot) \in C(R^n, R^n)$, where $(x, y, w) \in E \times C(D, R)$;

3°) for each $Q^{(m)} = (x^{(m_0)}, y^{(m')}, w, q) \in E_h \times C(D, R) \times R^n$, we have

$$a_{s',m} + h_0 \sum_{i=1}^{n} h_i^{-1} c_{s',m}^{(i)} D_{q_i} f(Q^{(m)}) \ge 0, \quad m \in M_h^*; \tag{12}$$

4°) there exists $c_0 > 0$ such that $h_i \le c_0 h_0$, $i = 1, \dots, n$, and $h_i h_j^{-1} \le c_0$, $i, j = 1, \dots, n$.

Assumption H_2. Suppose that the operators A and Δ satisfy the conditions

1°) for $m_0 = 0, 1, \dots, \tilde{N}_0$, $m' \in \tilde{M}_h$, we have

$$\sum_{s' \in S} a_{s',m} = 1, \quad \sum_{s' \in S} c_{s',m}^{(i)} = 0, \quad i = 1, \dots, n,$$

$$\sum_{s' \in S} s_j a_{s',m} = 0, \quad \sum_{s' \in S} s_j c_{s',m}^{(i)} = \delta_{ij}, \quad i, j = 1, \dots, n,$$

where δ_{ij} is the Kronecker symbol;

2°) there exists $\bar{c} > 0$ such that $m_0 = 0, 1, \dots, \tilde{N}_0$, $m' \in \tilde{M}_h$ we have

$$\sum_{s' \in S} |a_{s,m}| \le \bar{c}, \quad \sum_{s' \in S} |c_{s',m}^{(i)}| \le \bar{c}, \quad i = 1, \dots, n.$$

The following theorem enables us to get an estimation between exact and approximate solutions of (1).

Theorem 2. *Suppose that*

1°) *assumption* $H_0 - H_2$ *are satisfied and* v_h *is a solution of* (5);

2°) $u : E^{(0)} \cup E \to R$ *is a solution of* (1) *and the function* $u|_E$ *is of class* C^2;

3°) *there exists* $\beta : I_d \to R_+$ *such that*

$$|\varphi^{(m)} - \varphi_h^{(m)}| \le \beta(h) \quad \text{on} \quad E_h^{(0)},$$

and $\lim_{h\to 0} \beta(h) = 0$.

Under these assumptions, there exist $\gamma: I_d \to R_+$ *and* $\varepsilon_0 > 0$ *such that*

$$\| u_h - v_h \|_{h, m_0} \le \beta(h) \quad \text{for} \quad m_0 = 0, 1, \dots, \tilde{N}_0, \quad |h| \le \varepsilon_0, \tag{13}$$

where $u_h = u|_{E_h^{(0)} \cup E_h}$ *and*

$$\lim_{h\to 0} \gamma(h) = 0, \tag{14}$$

Proof. We apply Theorem 1 to prove (13), (14). Suppose that δ_h is defined by

$$\Delta_0 u_h^{(m)} = f(x^{(m_0)}, y^{(m')}, (T_h u_h)_{(m)}, \Delta u_h^{(m)}) + \delta_h^{(m)}, \quad m \in M_h^*.$$

It follows from condition 2° in Assumption H_2 that there exists $\alpha: I_d \to R_+$ such that $|\delta_h^{(m)}| \le \alpha(h)$ for $m \in M_h^*$ and $\lim_{h\to 0} \alpha(h) = 0$. We define $F_h: E_h \times F(E_h^{(0)} \cup E_h, R) \to R$ by

$$F_h[m, z] = A z^{(m)} + h_0 f(x^{(m_0)}, y^{(m')}, (T_h z)_{(m)}, \Delta z^{(m)}).$$

Then we have

$$v_h^{(m_0+1, m')} = F_h[m, v_h], \quad m \in M_h^*,$$

$$v_h^{(m)} = \varphi_h^{(m)} \quad \text{on} \quad E_h^{(0)}$$

and

$$|F_h[m, u_h] - u_h^{(m_0+1, m')}| \le h_0 \alpha(h), \quad m \in M_h^*,$$

$$|u_h^{(m)} - v_h^{(m)}| \le \beta(h) \quad \text{on} \quad E_h^{(0)}.$$

Suppose that $z, \bar{z} \in F(E_h^{(0)} \cup E_h, R)$. Then we have for $m \in M_h^*$,

$$F_h[m, z] - F_h[m, \bar{z}] = A z^{(m)} - A \bar{z}^{(m)} + h_0 B_h[m, z, \bar{z}] + h_0 C_h[m, z, \bar{z}], \tag{15}$$

where

$$B_h[m, z, \bar{z}] = f(x^{(m_0)}, y^{(m')}, (T_h z)_{(m)}, \Delta z^{(m)}) - f(x^{(m_0)}, y^{(m')}, (T_h \bar{z})_{(m)}, \Delta z^{(m)}),$$

$$C_h[m, z, \bar{z}] = f(x^{(m_0)}, y^{(m')}, (T_h \bar{z})_{(m)}, \Delta z^{(m)}) - f(x^{(m_0)}, y^{(m')}, (T_h \bar{z})_{(m)}, \Delta \bar{z}^{(m)}).$$

It follows from Assumption H_1 and from (2) that

$$|A z^{(m)} - A \bar{z}^{(m)} + h_0 C_h[m, z, \bar{z}]| =$$

$$= \left| \sum_{s' \in S} (z^{(m_0, m'+s')} - \bar{z}^{(m_0, m'+s')}) \left[a_{s', m} + h_0 \sum_{i=1} h_i^{-1} c_{s', m}^{(i)} D_{q_i} f(Q) \right] \right|,$$

where Q is an intermediate point. Inequality (12) and Assumption H_2 imply

$$|A z^{(m)} - A \bar{z}^{(m)} + h_0 C_h[m, z, \bar{z}]| \le V_h(z - \bar{z})(m_0). \tag{16}$$

It follows from (11) that

$$|B_h[m, z, \bar{z}]| \le \sigma_h(x^{(m_0)}, V[(T_h z)_{(m)} - (T_h \bar{z})_{(m)}]), \quad m \in M_h^*. \tag{17}$$

Put $S_0^* = \{s' = (s_1, \dots, s_n) : s_j \in \{0, 1\}, i = 1, \dots, n\}$. It is easy to see that, for $y^{(m')} \le y \le y^{(m'+1)}$, we have

$$\sum_{s' \in S_0^*} \left(\frac{y - y^{(m')}}{h'}\right)^{s'} \left(1 - \frac{y - y^{(m')}}{h'}\right)^{1-s'} = 1.$$

Then

$$|(T_h z)(x, y) - (T_h \bar{z})(x, y)| \le \sum_{s' \in S_0^*} \left| z^{(m_0, m'+s')} - \bar{z}^{(m_0, m'+s')} \right| \times$$

$$\times \left(1 - \frac{x - x^{(m_0)}}{h_0}\right)\left(\frac{y - y^{(m')}}{h'}\right)^{s'} \left(1 - \frac{y - y^{(m')}}{h'}\right)^{1-s'} +$$

$$+ \sum_{s' \in S_0^*} \left| z^{(m_0+1, m'+s')} - \bar{z}^{(m_0+1, m'+s')} \right| \times$$

$$\times \left(\frac{x - x^{(m_0)}}{h_0}\right)\left(\frac{y - y^{(m')}}{h'}\right)^{s'} \left(1 - \frac{y - y^{(m')}}{h'}\right)^{1-s'} \le$$

$$\le \left(1 - \frac{x - x^{(m_0)}}{h_0}\right) \max\{|z^{(m_0, m')} - \bar{z}^{(m_0, m')}| : m' \in M_h\} +$$

$$+ \frac{x - x^{(m_0)}}{h_0} \max\{|z^{(m_0, m')} - \bar{z}^{(m_0, m')}| : m' \in M_h\}.$$

The above estimations and (17) imply

$$|B_h[m, z, \bar{z}]| \le \sigma_h(x^{(m_0)}, (L[h_0, V_h(z - \bar{z})])_{(x^{(m_0)})}). \tag{18}$$

It follows from (15) – (18) that

$$|F_h[m, z] - F_h[m, \bar{z}]| \le \sigma_h[m_0, V_h(z - \bar{z})], \quad m \in M_h^*,$$

where

$$\sigma_h[m_0, \beta] = \beta^{(m_0)} + h_0 \sigma(x^{(m_0)}, (L[h_0, \beta])_{(x^{(m_0)})}), \tag{19}$$

$$m_0 = 0, 1, \dots, \tilde{N}_0, \quad \beta \in F(X_h, R_+).$$

Consider the initial-value problem

$$\eta'(x) = \sigma(x, (L[h_0, \eta_{h_0}])_{(x)}) + \alpha(h),$$

$$\eta(x) = \beta(x) \quad \text{for} \quad x \in [-\tau_0, 0]. \tag{20}$$

It follows from Assumption H_0 and the theorem on continuous dependence of solutions on initial values and on the right-hand sides of equations that there exists $\varepsilon_0 > 0$ such that, for $|h| < \varepsilon_0$, there exists a solution $\omega(\cdot, h)$ of (20). This solution is defined on $[-\tau_0, a]$ and $\lim_{h \to 0} \omega(x, h) = 0$ uniformly with respect to $x \in [-\tau_0, a]$. Since $\omega(\cdot, h)$ is a convex function on $[0, a]$,

$$\omega(x^{(m_0+1)}, h) \ge \omega(x^{(m_0)}, h) + h_0 \sigma(x^{(m_0)}, (L[h_0, \omega_{h_0}(\cdot, h)])_{(x^{(m_0)})}) +$$

$$+ h_0 \alpha(h), \quad m_0 = 0, 1, \dots, \tilde{N}_0 - 1,$$

where $\omega_{h_0}(\cdot, h)$ is the restriction of $\omega(\cdot, h)$ to the set X_h.

Thus, we see that all the assumptions of Theorem 1 are satisfied and, consequently,

$$| u_h^{(m)} - v_h^{(m)} | \leq \omega(x^{(m_0)}, h), \quad m_0 = 0, 1, \ldots, \tilde{N}_0,$$

where $|h| < \varepsilon_0$. Now we obtain (13), (14) with $\gamma(h) = \omega(a, h)$, which completes the proof.

***Remark* 3.** If the assumptions of Theorem 2 hold with

$$\sigma(x, \eta) = L \max \{|\eta(t)| : t \in [-\tau_0, 0]\}, \quad L \in R_+,$$

$$(x, \eta) \in [0, a_0] \times C([-\tau_0, 0], R_+),$$

i.e., the Lipschitz condition is satisfied,

$$|f(x, y, w, q) - f(x, y, \overline{w}, q)| \leq L \| w - \overline{w} \|_{C(D)} \text{ on } \Omega,$$

then we have the estimations

$$\| u_h - v_h \|_{h, m_0} \leq \beta(h) \exp [L x^{(m_0)}] \alpha(h) L^{-1} [\exp [L x^{(m_0)}] - 1] \text{ if } L > 0,$$

$$\| u_h - v_h \|_{h, m_0} \leq \beta(h) + x^{(m_0)} \alpha(h) \text{ if } L = 0,$$

where $m_0 = 0, 1, \ldots, \tilde{N}_0$.

***Remark* 4.** Suppose that $\sigma_0 \in C([0, a_0] \times R_+, R_+)$ and

$$\sigma(x, \eta) = \sigma_0(x, \max \{\eta(t) : t \in [-t_0, 0]\}) \text{ for } (x, \eta) \in [0, a_0) \times C([-\tau_0, 0], R_+)$$

then estimation (11) has the form

$$|f(x, y, w, q) - f(x, y, \overline{w}, q)| \leq \sigma_0(x, \| w - \overline{w} \|_{C(D)}) \text{ on } \Omega.$$

The comparison problem (10) is equivalent to

$$\eta'(x) = \sigma_0(x, \eta(x)), \quad \eta(0) = 0,$$

in this case.

4. Examples. Inequality (12) is the main assumption in the theorem on the convergence of the difference method (5). We give examples of operators Δ_0 and Δ and formulate the main assumption (12) for these Δ_0 and Δ.

If $1 \leq j \leq n$, $m = (m_1, \ldots, m_n)$, then we define

$$j(m) = (m_0, \ldots, m_{j-1} m_j + 1, m_{j+1}, \ldots, m_n),$$

$$-j(m) = (m_0, \ldots, m_{j-1} m_j - 1, m_{j+1}, \ldots, m_n).$$

***Example* 3.** Suppose that $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_n)$ is a fixed point in $(-b, b)$. Suppose that $h_i < \min \{b_i - \overline{y}_i, \overline{y}_i + b_i\}$, $i = 1, \ldots, n$. Consider method (5) with

$$\Delta_0 z^{(m)} = h_0^{-1} [z^{(m_0 + 1, m')} - z^{(m)}],$$

$$\Delta_i z^{(m)} = h_i^{-1} [z^{(i(m))} - z^{(m)}] \text{ if } y_i^{(m_i)} \geq \overline{y}_i,$$

$$\Delta_i z^{(m)} = h_i^{-1} [z^{(m)} - z^{(-i(m))}] \text{ if } y_i^{(m_i)} < \overline{y}_i, \quad i = 1, \ldots, n.$$

Under these assumptions, condition (12) is equivalent to

$$D_{q_i} f(P) \operatorname{sign} (y_i - \overline{y}_i) \geq 0, \quad i = 1, \ldots, n, \quad P = (x, y, w, q) \in \Omega, \tag{21}$$

$$1 - h_0 \sum_{i=1}^{n} h_i^{-1} |D_{q_i} f(P)| \geq 0, \quad P \in \Omega. \tag{22}$$

***Remark* 5.** Suppose that Assumption H_0 and estimation (21) hold true. If (11) is satisfied then the solution of (1) is unique and depends continuously on the initial-boundary function $\varphi \in C(E^{(0)}, R)$. This property of (1) can be proved by the method of differential inequalities. We omit details.

***Example* 4.** Consider method (5) with

$$\Delta_0 z^{(m)} = z^{(m_0+1, m')} - (2n)^{-1} \sum_{i=1}^{n} [z^{(i(m))} + z^{(-i(m))}],$$

$$\Delta_i z^{(m)} = (2h_i)^{-1} [z^{(i(m))} - z^{(-i(m))}], \quad i = 1, \ldots, n.$$

Then condition (12) is equivalent to

$$1 - n h_0 h_i^{-1} |D_{q_i} f(P)| \geq 0, \quad i = 1, \ldots, n, \quad P \in \Omega.$$

Now we consider a numerical example. We will use the following assertion:

Lemma 1. *Suppose that* $z : E_h^{(0)} \cup E_h \to R$ *and denote* $x = y_0$, $x^{(m_0)} = y_0^{(m_0)}$ *in* (3). *If* $y^{(m)} = (y_0^{(m_0)}, y_1^{(m_1)}, \ldots, y_n^{(m_n)})$, $y^{(m+1)} = (y_0^{(m_0+1)}, y_1^{(m_1+1)}, \ldots, y_n^{(m_n+1)})$, $y^{(m)}, y^{(m+1)} \in E_h^{(0)} \cup E_h$, *then*

$$\int_{y^{(m)}}^{y^{(m+1)}} (T_h z)(x, y)\, dx\, dy = \frac{1}{2^{n+1}} \prod_{i=0}^{n} h_i \sum_{s' \in S^*} z^{(m+s)}. \tag{23}$$

We omit a simple proof of Lemma 1.

***Example* 5.** Let $E = [0, 1] \times [-2, 2] \times [-2, 2]$ and

$$E_0 = ([-1, 1] \times [-2.5, 2.5] \times [-2.5, 2.5]) \setminus ((0, 1] \times (-2, 2) \times (-2, 2)).$$

We define $y = (y_1, y_2)$, $s = (s_1, s_2)$ and

$$f_0(x, y) = \sin(x-1)\left[\frac{1}{6} y_2(y_2^2 - 1) - 2\right] - \frac{1}{6} y_2(y_2^2 - 1) -$$

$$- \frac{1}{3} x y_1(y_1^2 - 1), \quad (x, y) \in E,$$

$$g_1(x, y) = \frac{x}{3} y_1(y_1^2 - 1),$$

$$g_2(x, y) = \frac{1}{6} y_2(y_2^2 - 1)[1 - \sin(x-1)],$$

$$\varphi(x, y) = 12 + y_1 + y_2 + \sin x + \cos x, \quad (x, y) \in E_0,$$

$$D = \left\{(t, s) : t \in [-1, 0],\ s_j \in \left[-\frac{1}{2}, \frac{1}{2}\right],\ i = 1, 2\right\}.$$

Consider the problem

$$D_x z(x, y) = -\int_D z_{(x,y)}(t, s)\, dt\, ds + z(x-1, y) + g_1(x, y) D_{y_1} z(x, y) +$$

$$+ g_2(x, y) D_y z(x, y) + f_0(x, y), \quad (x, y) \in E,$$

$$z(x, y) = \varphi(x, y) \text{ for } (x, y) \in E_0, \tag{24}$$

and the difference method

$$z^{(i+1,j,k)} = \frac{1}{4}\left[z^{(i,j+1,k)} + z^{(i,j-1,k)} + z^{(i,j,k+1)} + z^{(i,j,k-1)}\right] -$$

$$- h_0 \left\{ \int_D (T_h z)_{(i,j,k)}(t,s)\,dt\,ds + (T_h z)(x^{(i)} - 1,\ y_1^{(j)},\ y_2^{(k)}) + \right.$$

$$+ g_1(x^{(i)},\ y_1^{(j)},\ y_2^{(k)})(2h_1)^{-1}\left[z^{(i,j+1,k)} - z^{(i,j-1,k)}\right] +$$

$$\left. + g_2(x^{(i)},\ y_1^{(j)},\ y_2^{(k)})(2h_2)^{-1}\left[z^{(i,j,k+1)} - z^{(i,j,k-1)}\right] + f_0(x^{(i)},\ y_1^{(j)},\ y_2^{(k)}) \right\}, \quad (25)$$

where

$$z^{(i,j,k)} = \varphi^{(i,j,k)} \quad \text{on} \quad E_0.$$

It is easy to see that, for $h_1 = h_2 \geq 5h_0$, method (25) satisfies condition (12). We calculate the integral $\int_D (T_h z)_{(i,j,k)}(t,s)\,dt\,ds$ using Lemma 1. We take $h_0 = 10^{-2}$, $h_1 = h_2 = 10^{-1}$, $N_0 = \tilde{N}_0 = 100$, and $N_1 = N_2 = 25$. Denote by u and v_h solutions of (24) and (25) respectively. Let $\delta : \{0, 1, \ldots, N_0\} \to R_+$ be given by

$$\delta_i = \max\left\{\left|u(x^{(i)},\ y_1^{(j)},\ y_2^{(k)}) - v_h(x^{(i)},\ y_1^{(j)},\ y_2^{(k)})\right| : j, k = 0, \pm 1, \ldots, \pm 20\right\},$$

$$i = 0, 1, \ldots, 100.$$

The values of $x^{(i)}$, $v_h(x^{(i)}, 0, 0)$, δ_i are listed in the following table.

$x^{(i)}$	$v_h(x^i, 0, 0)$	δ_i
$x^{(10)} = 0.10$	13.094199	0.000638
$x^{(20)} = 0.20$	13.177354	0.001382
$x^{(30)} = 0.30$	13.250857	0.002216
$x^{(40)} = 0.40$	13.307355	0.003125
$x^{(50)} = 0.50$	13.352920	0.004088
$x^{(60)} = 0.60$	13.384892	0.005086
$x^{(70)} = 0.70$	13.402962	0.006098
$x^{(80)} = 0.80$	13.406964	0.007099
$x^{(90)} = 0.90$	13.396870	0.008067
$x^{(100)} = 1.0$	13.372787	0.008976

***Remark* 6.** The results of the paper can be easily extended to the initial-boundary-value problems for weakly coupled systems of nonlinear equations.

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