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GELFAND PAIR ASSOCIATED WITH A HOPH ALGEBRA AND A COIDEAL*

ПАРА ГЕЛЬФАНДА, ЯКА ПОВ'ЯЗАНА З АЛГЕБРОЮ ХОПФА ТА КОІДЕАЛОМ

A pair of a compact quantum group and a coideal in its dual Hopf $*$-algebra is considered. A notion of a Gelfand pair and a strict Gelfand pair is introduced. For a strict Gelfand pair, two hypercomplex systems dual to each other are constructed. As an example, the quantum analog of the pair $(U(n), SO(n))$ is considered.

Розглянуто пару компактної квантової групи та коідеалу у її дуальній $*$-алгебрі Хопфа. Вводиться поняття пари Гельфанда та строгої пари Гельфанда. Для строгої пари Гельфанда будується пара гіперкомплексних систем, які є дуальними одна до одної. Як приклад розглянуто квантовий аналог пари $(U(n), SO(n))$.

Considering a pair of a locally compact group G and its compact subgroup K, with Haar measures ν_G and ν_K, respectively, one can endow the algebra of functions on G that are biinvariant with respect to the subgroup K with a natural hypergroup structure [1] or a closely related structure of a hypercomplex system (see [2] and survey [3]). A similar construction has been carried out for a pair of compact quantum groups H_1, H_2, where the subgroup K was replaced by a Hopf algebra epimorphism $\pi: H_1 \to H_2$, see [4, 5], and in the case of noncompact quantum groups, see [6] (for related topics, see also [7, 8]). However, existence of such an epimorphism π seems to be a rather restrictive condition. It is not evident whether such an epimorphism exists, for example, in the case where $H_1 = U_q(n)$, $K = SO_q(n)$ [9]. In this paper, we construct a pair of hypercomplex systems dual to each other for Hoph algebras, which are in duality, and a coideal. Similarly to the classical case and to the case of a pair of quantum groups with an epimorphism, we define a notion of a Gelfand pair and a strict Gelfand pair. Some elements of this construction can be found in [10, 11]. As an example, we consider the compact quantum group $U_q(n)$ and a coideal corresponding to the quantum group $SO_q(n)$. In this case, the characters of the constructed hypercomplex system, which are zonal spherical functions, correspond to the Macdonald's polynomials. This example was studied in depth in [11].

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Let $\boldsymbol{H} = (H, d, 1, \Delta, \varepsilon, S)$ and $\hat{\boldsymbol{H}} = (\hat{H}, \hat{d}, \hat{1}, \hat{\Delta}, \hat{\varepsilon}, \hat{S})$ be Hopf algebras [12] with the corresponding multiplication, unity, comultiplication, counit, and antipode.

Definition 1. *We will say that the Hopf algebras* $\boldsymbol{H}$ *and* $\hat{\boldsymbol{H}}$ *are in duality if there exists a nondegenerate pairing* $\langle \cdot, \cdot \rangle : \hat{H} \times H \to C$ *such that*:

$$\begin{aligned} \langle \xi, d(f, g) \rangle &= \langle \hat{\Delta}(\xi), f \otimes g \rangle, \\ \langle \hat{d}(\xi, \eta), f \rangle &= \langle \xi \otimes \eta, \Delta(f) \rangle, \\ \langle \hat{1}, f \rangle &= \varepsilon(f), \\ \langle \xi, 1 \rangle &= \hat{\varepsilon}(\xi), \\ \langle \hat{S}(\xi), f \rangle &= \langle \xi, S(f) \rangle \end{aligned} \tag{1}$$

for all ξ, η *in* $\hat{H}$ *and all* f, g *in* H.

In the sequel, instead of writing $\hat{d}(\xi, \eta)$, $d(f, g)$, and $\langle \xi, f \rangle$, we will write

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$\xi \cdot \eta$, $f \cdot g$, and $\xi(f)$.

For every $\xi \in \hat{H}$, we will define two linear mappings ξ^l and ξ^r in $\mathrm{End}(H)$ as follows:

$$\xi^l = (\xi \otimes \mathrm{id}) \circ \Delta,$$
$$\xi^r = (\mathrm{id} \otimes \xi) \circ \Delta.$$

Lemma 1. *The mappings $\xi \to \xi^r$ and $\xi \to \xi^l$ are faithful representation and antirepresentation, respectively, of the unital algebra $\hat{H}$ in* $\mathrm{End}(H)$.

Proof. Indeed, let $\xi_1, \xi_2 \in \hat{H}$ and $f \in H$. Then

$$\xi_1^l\left(\xi_2^l(f)\right) = \xi_1^l((\xi_2 \otimes \mathrm{id}) \circ \Delta(f)) = (\xi_1 \otimes \mathrm{id}) \circ \Delta((\xi_2 \otimes \mathrm{id}) \circ \Delta(f)) =$$
$$= (\xi_1 \otimes \mathrm{id}) \circ (\xi_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \Delta(f) =$$
$$= (\xi_1 \otimes \xi_2 \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta(f) = ((\xi_2 \cdot \xi_1) \otimes \mathrm{id}) \circ \Delta(f) =$$
$$= (\xi_2 \cdot \xi_1)^l(f).$$

Moreover,

$$\hat{1}^l(f) = \left(\hat{1} \otimes \mathrm{id}\right) \circ \Delta(f) = (\varepsilon \otimes \mathrm{id}) \circ \Delta(f) = f.$$

Since, $\varepsilon(\xi^l(f)) = \xi(f)$, we see that $\xi^l(f) = 0$ for all f implies that $\xi(f) = 0$ for all f and, because the pairing is nondegenerate, $\xi = 0$. Similarly, one can show that ξ^r is a faithful representation.

Let now $\hat{M} \subset \hat{H}$ be some subset of $\hat{H}$.

Definition 2. *An element $f \in H$ will be called left-invariant (right-invariant) with respect to $\hat{M}$ if for all $\xi \in \hat{M}$, $\xi^l(f) = \hat{\varepsilon}(\xi) f$ ($\xi^r(f) = \hat{\varepsilon}(\xi) f$). An element $f \in H$ will be called biinvariant with respect to $\hat{M}$ if it is left- and right-invariant with respect to $\hat{M}$.*

We will denote the set of all left-invariant (right-invariant) elements of H with respect to $\hat{M}$ by $\hat{M} \backslash H$ ($H / \hat{M}$) and the set of all bi-invariant elements with respect to $\hat{H}$ by $\hat{M} \backslash H / \hat{M}$. It is clear that $\hat{M} \backslash H / \hat{M} = \hat{M} \backslash H \cap H / \hat{M}$.

Definition 3. *Let $\hat{M}_0 \subset \hat{H}$ and let $\hat{M}_0 \backslash H$ be the subset of all left-invariant elements of H. We will say that the set $\hat{M} \supseteq \hat{M}_0$ is maximal with respect to $\hat{M}_0 \backslash H$ if for every $\xi \in \hat{H}$ such that $\xi^l(f) = \hat{\varepsilon}(\xi) f$ for all $f \in \hat{M}_0 \backslash H$, we have $\xi \in \hat{M}$. The notion of a maximal set with respect to $H / \hat{M}_0$ and $\hat{M}_0 \backslash H / \hat{M}_0$ is defined similarly.*

Remark. It is clear that if $\hat{M}$ is maximal with respect to $\hat{M}_0 \backslash \hat{H}$, then $\hat{M} \backslash H = \hat{M}_0 \backslash H$. A similar statement holds if $\hat{M}$ is maximal with respect to $H / \hat{M}_0$ or $\hat{M}_0 \backslash H / \hat{M}_0$.

Lemma 2. *If the set $\hat{M} \subset \hat{H}$ is maximal with respect to $\hat{M}_0 \backslash H$ ($H / \hat{M}_0$, $\hat{M}_0 \backslash H / \hat{M}_0$), then $\hat{M}$ is a subalgebra of $\hat{H}$ and $\hat{1} \in \hat{M}$.*

Proof. Let $\hat{M}$ be maximal with respect to $\hat{M}_0 \backslash H$, $f \in \hat{M}_0 \backslash H$, and ξ_1, $\xi_2 \in \hat{M}$. Then, by using Lemma 1, we have

$$(\xi_1 \cdot \xi_2)^l(f) = \xi_2^l\left(\xi_1^l(f)\right) = \hat{\varepsilon}(\xi_1)\, \xi_2^l(f) = \hat{\varepsilon}(\xi_1)\, \hat{\varepsilon}(\xi_2) f = \hat{\varepsilon}(\xi_1 \cdot \xi_2) f.$$

Hence, all $f \in \hat{M}_0 \backslash H$ are left-invariant with respect to the element $\xi_1 \cdot \xi_2$, and since $\hat{M}$ is maximal with respect to $\hat{M}_0 \backslash H$, $\xi_1 \cdot \xi_2 \in \hat{M}$. Because $\hat{1}^l(f) = f = \hat{\varepsilon}\left(\hat{1}\right) f$, it follows that $\hat{1} \in \hat{M}$. We prove the statement similarly if $\hat{M}$ is maximal with respect to $\hat{M}_0\backslash H$ or $\hat{M}_0\backslash H/\hat{M}_0$.

Lemma 3. *Let $\hat{M}\subset\hat{H}$ be an arbitrary linear subset of $\hat{H}$, which contains $\hat{1}$, and let $\hat{M}_0$ be a set of all $\xi\in\hat{M}$ such that $\hat{\varepsilon}(\xi)=0$. Then a) $\hat{M}_0\backslash H = \hat{M}\backslash H$; b) $H/\hat{M}_0 = H/\hat{M}$; c) $\hat{M}_0\backslash H/\hat{M}_0 = \hat{M}\backslash H/\hat{M}$.*

Proof. We will prove only (a), since the other statements can be proved similarly. Because $\hat{M}_0\subset\hat{M}$, $\hat{M}_0\backslash H\supset\hat{M}\backslash H$. Let now $f\in\hat{M}_0\backslash H$ and $\xi\in\hat{M}$. Consider the element $\xi-\hat{\varepsilon}(\xi)\hat{1}\in\hat{M}_0$. Because $f\in\hat{M}_0\backslash H$, it follows that $(\xi-\hat{\varepsilon}(\xi)\hat{1})^l(f)=0$. But this means that $\xi^l(f)=\hat{\varepsilon}(\xi)f$. Since $\xi\in\hat{M}$ is arbitrary, $f\in\hat{M}\backslash H$, and so $\hat{M}_0\backslash H=\hat{M}\backslash H$.

In what follows, we will assume that $\hat{M}_0$ is a coideal.

Definition 4. *Let $\hat{H}$ be a Hopf algebra and $\hat{M}_0\subset\hat{H}$ be a linear subspace of $\hat{H}$ such that*

$$\hat{\Delta}(\hat{M}_0)\subset\hat{M}_0\otimes\hat{H}+\hat{H}\otimes\hat{M}_0, \tag{2}$$

$$\hat{\varepsilon}(\hat{M}_0)=0, \tag{3}$$

then we say that $\hat{M}_0$ is a coideal in $\hat{H}$.

Remark. For a pair of Hopf algebras H_1 and H_2 and a Hopf algebra epimorphism $\pi: H_1\to H_2$, there is a Hopf algebra imbedding $\hat{\pi}:\hat{H}_2\to\hat{H}_1$ in the dual Hopf algebras. The subset of $\hat{\pi}(\hat{H}_2)$, on which $\hat{\varepsilon}_1$ equals to zero, is a coideal in $\hat{H}_1$ and the definition of invariant elements in H_1 with respect to H_2 given in [4] is the same as in Definition 2 if $\hat{M}_0$ is assumed to be a coideal.

Lemma 4. *Let $\hat{M}_0\subset\hat{H}$ be a coideal. Then $\hat{M}_0\backslash H$, $H/\hat{M}_0$, and $\hat{M}_0\backslash H/\hat{M}_0$ are unital subalgebras of H.*

Proof. Let $f, g\in\hat{M}_0\backslash H$ and let $\Delta(f)=\sum_k f_k^1\otimes f_k^2$, $\Delta(g)=\sum_l g_l^1\otimes g_l^2$ for some $f_k^1, f_k^2, g_l^1, g_l^2\in H$. Then for all $\xi\in\hat{M}_0$, $\hat{\Delta}(\xi)=\sum_m \xi_m^1\otimes\xi_m^2$, we have

$$\xi^l(f\cdot g)=(\xi\otimes\mathrm{id})\circ\Delta(f\cdot g)=(\xi\times\mathrm{id})(\Delta(f)\cdot\Delta(g))=$$

$$=(\xi\otimes\mathrm{id})\left(\sum_{k,l}\left(f_k^1\otimes f_k^2\right)\cdot\left(g_l^1\otimes g_l^2\right)\right)=(\xi\otimes\mathrm{id})\left(\sum_{k,l} f_k^1\cdot g_l^1\otimes f_k^2\cdot g_l^2\right)=$$

$$=\sum_{k,l}\xi\left(f_k^1\cdot g_l^1\right)f_k^2\cdot g_l^2=\sum_{k,l}\hat{\Delta}(\xi)\left(f_k^1\otimes g_l^1\right)f_k^2\cdot g_l^2=$$

$$=\sum_m\sum_{k,l}\xi_m^1\left(f_k^1\right)\xi_m^2\left(g_l^1\right)f_k^2\cdot g_l^2=\sum_m\left(\sum_k\xi_m^1\left(f_k^1\right)f_k^2\right)\cdot\left(\sum_l\xi_m^2\left(g_l^1\right)g_l^2\right)=$$

$$=\sum_m\xi_m^{1^l}(f)\cdot\xi_m^{2^l}(g).$$

Since $\hat{M}_0$ is a coideal, it follows from (2) that, for all m, either $\xi_m^1\in\hat{M}_0$ or $\xi_m^2\in\hat{M}_0$, which means that, for all m, either $\xi_m^{1^l}(f)=0$ or $\xi_m^{2^l}(g)=0$ because $f\in\hat{M}_0\backslash H$. Thus, $\xi^l(f\cdot g)=0$ and $f\cdot g\in\hat{M}_0\backslash H$. In the same way, one can prove that $H/\hat{M}_0$ and $\hat{M}_0\backslash H/\hat{M}_0$ are subalgebras of H.

It follows from (1) that 1 is invariant with respect to any $\xi\in\hat{H}$ and so $1\in$

$\in \hat{M}_0 \backslash H\,(H / \hat{M}_0,\ \hat{M}_0 \backslash H / \hat{M}_0)$. The rest of the lemma is proved similarly.

Let now $\hat{M}_0 \backslash H\,(H / \hat{M}_0,\ \hat{M}_0 \backslash H / \hat{M}_0)$ be a subalgebra of left-invariant (right-invariant, biinvariant) elements of H with respect to a coideal $\hat{M}_0$. Let $\hat{M} \supset \hat{M}_0$ be maximal with respect to $\hat{M}_0 \backslash H\,(H / \hat{M}_0,\ \hat{M}_0 \backslash H / \hat{M}_0)$.

Definition 5. *An element $\eta \in \hat{H}$ is called left-invariant with respect to $\hat{M}$ if for all $\xi \in \hat{M}$, $\eta \cdot \xi = \hat{\varepsilon}(\xi)\eta$. An element η is right-invariant with respect to $\hat{M}$ if $\xi \cdot \eta = \hat{\varepsilon}(\xi)\eta \quad \forall \xi \in \hat{M}$, and we call it biinvariant if it is left- and right-invariant with respect to $\hat{M}$.*

Denote the set of all left-invariant (right-invariant, biinvariant) elements of $\hat{H}$ with respect to $\hat{M}$ by $\hat{M} \backslash \hat{H}\ (\hat{H} / \hat{M},\ \hat{M} \backslash \hat{H} / \hat{M})$.

Lemma 5. *The set $\hat{M} \backslash \hat{H}\ (\hat{H} / \hat{M})$ is a left (right) ideal in $\hat{H}$. The set $\hat{M} \backslash \hat{H} / \hat{M}$ is a subalgebra of $\hat{H}$. Moreover, if $\hat{M}$ contains elements other than $\hat{0}$ and $\hat{1}$, then $\hat{1} \notin \hat{M} \backslash \hat{H} / \hat{M}$.*

Proof. Let us prove, for example, that $\hat{M} \backslash \hat{H}$ is a left ideal in $\hat{H}$. Indeed, let $\zeta \in \hat{H}$, $\eta \in \hat{M} \backslash \hat{H}$, and $\xi \in \hat{M}$. Then

$$(\zeta \cdot \eta) \cdot \xi = \zeta \cdot (\eta \cdot \xi) = \hat{\varepsilon}(\xi)\zeta \cdot \eta.$$

It is also clear that $\hat{1} \cdot \xi = \xi \neq \hat{\varepsilon}(\xi)\hat{1}$ for any ξ different from $\hat{0}$ and $\hat{1}$.

Lemma 6. *An element $\eta \in \hat{H}$ is left (right)-invariant with respect to $\hat{M}$ if* $\operatorname{Im}(\eta^l) \subset \hat{M} \backslash \hat{H}\,(\operatorname{Im}(\eta^r) \subset H \backslash \hat{M})$.

Proof. Let η be left-invariant. Then for all $\xi \in \hat{M}$ and $f \in H$,

$$\xi^l(\eta^l(f)) = (\eta \cdot \xi)^l = \hat{\varepsilon}(\xi)\,\eta^l(f),$$

hence, $\eta^l(f) \in \hat{M} \backslash H$.

Conversely, suppose that $\operatorname{Im}(\eta^l) \subset \hat{M} \backslash H$. Then for any $f \in H$ and $\xi \in \hat{M}$,

$$(\eta \cdot \xi)^l(f) = \xi^l(\eta^l(f)) = \hat{\varepsilon}(\xi)\,\eta^l(f).$$

Since $\eta \to \eta^l$ is a faithful anti-representation, $\eta \cdot \xi = \hat{\varepsilon}(\xi)\,\eta$, and so $\eta \in \hat{M} \backslash \hat{H}$. A similar argument can be used to prove the other case.

Definition 6. *An element $\nu_{\hat{M}}$ is called an invariant integral with respect to $\hat{M}$ if $\nu_{\hat{M}} \in \hat{M} \cap \hat{M} \backslash \hat{H} / \hat{M}$ and $\hat{\varepsilon}(\nu_{\hat{M}}) = 1$.*

Remark. a). By the definition, $\nu_{\hat{M}} \cdot \xi = \xi \cdot \nu_{\hat{M}} = \hat{\varepsilon}(\xi)\,\nu_{\hat{M}}$ for all $\xi \in \hat{M}$. b). If $\hat{M} = \hat{H}$, then in virtue of a), we get the invariant integral on the Hopf algebra H.

We assume that an invariant integral with respect to $\hat{M}$ exists and, in the sequel, we prove its existence under an additional condition on the subalgebra $\hat{M}$.

Lemma 7. *Let $\nu_{\hat{M}}$ be an invariant integral with respect to $\hat{M}$. Then a) $\nu_{\hat{M}}$ is an identity in the algebra $\hat{M} \backslash \hat{H} / \hat{M}$; b) $\nu_{\hat{M}}$ is unique.*

Proof. a). Since $\nu_{\hat{M}} \in \hat{M}$ and $\hat{\varepsilon}(\nu_{\hat{M}}) = 1$, it follows from the definition of $\hat{M} \backslash \hat{H} / \hat{M}$ that $\forall \eta \in \hat{M} \backslash \hat{H} / \hat{M}$,

$$\nu_{\hat{M}} \cdot \eta = \eta \cdot \nu_{\hat{M}} = \hat{\varepsilon}(\nu_{\hat{M}})\eta = \eta.$$

b). Since an identity in an algebra is unique, the statement is clear.

Remark. As an easy consequence of the definition of an invariant integral with respect to $\hat{M}$, Lemma 6, and Lemma 7, it follows that an element $\nu_{\hat{M}} \in \hat{H}$ such that $\hat{\varepsilon}(\nu_{\hat{M}}) = 1$ is an invariant integral if and only if $\nu^l_{\hat{M}}$ is a projection of H onto $\hat{M}\backslash H$, $\nu^r_{\hat{M}}$ is a projection of H onto $H/\hat{M}$. These projections commute and their product is a projection of H onto $\hat{M}\backslash H/\hat{M}$.

By using the invariant integral $\nu_{\hat{M}}$, we can obtain a description of the subalgebra $\hat{M}\backslash \hat{H}/\hat{M}$. In fact, we have the following lemma:

Lemma 8. *An element $\eta \in \hat{H}$ belongs to the subalgebra $\hat{M}\backslash \hat{H}/\hat{M}$ if and only if there exists an element $\eta_0 \in \hat{H}$ such that $\eta = \nu_{\hat{M}} \cdot \eta_0 \cdot \nu_{\hat{M}}$.*

Proof. Indeed, if $\eta \in \hat{M}\backslash \hat{H}/\hat{M}$, then we recall that $\nu_{\hat{M}}$ is an identity in the algebra $\hat{M}\backslash \hat{H}/\hat{M}$ and set $\eta_0 = \eta$.

Conversely, if $\eta = \nu_{\hat{M}} \cdot \eta_0 \cdot \nu_{\hat{M}}$ for some $\eta_0 \in \hat{H}$, then it follows from the definition of $\nu_{\hat{M}}$ that, for all $\xi \in \hat{M}$, we have

$$\eta \cdot \xi = \nu_{\hat{M}} \cdot \eta_0 \cdot \nu_{\hat{M}} \cdot \xi = \hat{\varepsilon}(\xi)\nu_{\hat{M}} \cdot \eta_0 \cdot \nu_{\hat{M}} = \hat{\varepsilon}(\xi)\eta,$$

hence, $\eta \in \hat{M}\backslash \hat{H}$. By using the same type of argument, we can show that $\eta \in \in \hat{H}/\hat{M}$, and so $\eta \in \hat{M}\backslash \hat{H}/\hat{M}$.

Let $(\hat{M}\backslash H/\hat{M})^*$ denote the set of all linear functionals on $\hat{M}\backslash H/\hat{M}$. The description of the algebra $\hat{M}\backslash \hat{H}/\hat{M}$ is given by the following theorem:

Theorem 1. *The linear spaces $(\hat{M}\backslash H/\hat{M})^* \cap \hat{H}$ and $\hat{M}\backslash \hat{H}/\hat{M}$ are isomorphic.*

Proof. If we define a linear mapping $\Gamma : (\hat{M}\backslash H/\hat{M})^* \cap \hat{H} \to \hat{M}\backslash \hat{H}/\hat{M}$ by $\Gamma(\eta) = \nu_{\hat{M}} \cdot \eta_0 \cdot \nu_{\hat{M}}$, it would follow from Lemma 8 that Γ is an isomorphism.

Corollary. *Suppose that $f \in \hat{M}\backslash H/\hat{M}$ and $\eta_1, \eta_2 \in (\hat{M}\backslash H/\hat{M})^* \cap \hat{H}$. Then*

$$\eta_1 \cdot \eta_2(f) = (\eta_1 \otimes \nu_{\hat{M}} \otimes \eta_2) \circ (\Delta \otimes \mathrm{id}) \circ \Delta(f).$$

Proof. Let $f \in \hat{M}\backslash H/\hat{M}$. By using Lemma 8 and the remark to Lemma 7, we have $f = \nu^r_{\hat{M}} \circ \nu^l_{\hat{M}}(f)$, and so

$$\eta_1 \cdot \eta_2(f) = \nu_{\hat{M}} \cdot \eta_1 \cdot \nu_{\hat{M}} \cdot \eta_2 \cdot \nu_{\hat{M}}(f) = \eta_1 \cdot \nu_{\hat{M}} \cdot \eta_2(\nu^l_{\hat{M}} \circ \nu^r_{\hat{M}}(f)) =$$
$$= (\eta_1 \otimes \nu_{\hat{M}} \otimes \eta_2) \circ (\Delta \otimes \mathrm{id}) \circ \Delta(f).$$

By using the isomorphism between $\hat{M}\backslash \hat{H}/\hat{M}$ and $(\hat{M}\backslash H/\hat{M})^* \cap \hat{H}$, we can endow the latter with an algebra structure. This is equivalent to defining a new comultiplication $\tilde{\Delta}$ on the algebra $\hat{M}\backslash H/\hat{M}$. This comultiplication is given by

$$\tilde{\Delta} = (\mathrm{id} \otimes \nu_{\hat{M}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{4}$$

The properties of this comultiplication are summarized in the following theorem:

Theorem 2. *Let a mapping $\tilde{\Delta}$ be defined by* (4). *Then*

a) $\tilde{\Delta}$ maps $\hat{M}\backslash H/\hat{M}$ into $\hat{M}\backslash H/\hat{M} \otimes \hat{M}\backslash H/\hat{M}$;

b) $\tilde{\Delta}$ is coassociative, i.e.,

$$(\mathrm{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes \mathrm{id}) \circ \tilde{\Delta};$$

c) $\nu_{\hat{M}}$ is a counit on $\hat{M}\backslash H/\hat{M}$ with respect to $\tilde{\Delta}$,

$$(\nu_{\hat{M}} \otimes \mathrm{id}) \circ \tilde{\Delta} = (\mathrm{id} \otimes \nu_{\hat{M}}) \circ \tilde{\Delta} = \mathrm{id};$$

d) if ν is an invariant integral on H, then ν is also an invariant integral with respect to $\tilde{\Delta}$,

$$(\nu \otimes \mathrm{id}) \circ \tilde{\Delta}(h) = (\mathrm{id} \otimes \nu) \circ \tilde{\Delta}(h) = \nu(h) \cdot 1.$$

Now, we introduce a notion of a Gelfand pair for a Hopf algebra and a coideal.

Definition 7. *Let H and $\hat{H}$ be two Hoph algebras in duality, let $\hat{M}_0$ be a coideal in $\hat{H}$, and let $\hat{M}$ be a maximal subalgebra of $\hat{H}$ with respect to the algebra of biinvariant elements $\hat{M}_0 \backslash H / \hat{M}_0$. Then we say that the pair $(\hat{H}, \hat{M})$ is a Gelfand pair if the algebra $\hat{M} \backslash \hat{H} / \hat{M}$ is commutative. We will say that $(\hat{H}, \hat{M})$ is a strict Gelfand pair if it is a Gelfand pair and the algebra $\hat{M} \backslash H / \hat{M}$ is commutative.*

Now, we assume that H and $\hat{H}$ are $*$-Hopf algebras. We will say that H and $\hat{H}$ are in duality as $*$-Hopf algebras if for all $\xi \in \hat{H}$ and $f \in H$,

$$\xi^*(f) = \overline{\xi\left(S(f)^*\right)}, \tag{5}$$

where the same symbol $*$ denotes the involution in H and in $\hat{H}$.

Lemma 9. *Let $\hat{M}_0$ be a coideal in $\hat{H}$ and $\hat{M}$ be maximal with respect to $\hat{M}_0 \backslash H / \hat{M}_0$. Then $\hat{M}_0 \backslash H / \hat{M}_0$ is a $*$-subalgebra of H if and only if $S^{-1}(\hat{M}^*) \subset \hat{M}$.*

Proof. Let $f \in \hat{M}_0 \backslash H / \hat{M}_0$ and $\xi \in \hat{M}$. We have

$$\xi^l(f^*) = (\xi \otimes \mathrm{id}) \circ \Delta(f^*) = \left((\xi^*)^* \otimes \mathrm{id}\right) \circ (* \otimes *) \circ \Delta(f) =$$

$$= \left(\overline{\xi}^* \otimes \mathrm{id}\right) \circ (* \otimes \mathrm{id}) \circ (S \otimes \mathrm{id}) \circ (* \otimes *) \circ \Delta(f) =$$

$$= \left(\overline{\xi}^* \otimes \mathrm{id}\right) \circ (S^{-1} \otimes *) \circ \Delta(f) = \left(S^{-1}(\xi^*)^l(f)\right)^* .$$

Similarly, $\xi^r(f^*) = \left(S^{-1}(\xi^*)^r(f)\right)^*$ and so the statement is clear.

Now, we consider the case when H is a compact quantum group and $\hat{H}$ is the $*$-Hopf algebra of all continuous linear functionals on H. We know [13] that H can be represented as

$$H = \sum_{\alpha \in P} \sum_{i,j=1}^{d_\alpha} C u_{i,j}^{\alpha}, \tag{6}$$

where P is a discrete set, $u_{i,j}^{\alpha}$ are matrix elements of d_α-dimensional unitary corepresentation of H ($d_\alpha < \infty$ for all α), and there exists an invariant integral ν on H, which is a state and such that (6) defines an orthogonal decomposition in the sense of the inner product given by

$$\langle f, g \rangle = \nu(f \cdot g^*). \tag{7}$$

Lemma 10. *Let H be a compact quantum group and let $\hat{M}_0$ be a coideal in $\hat{H}$. Let $\hat{M}$ be maximal with respect to $\hat{M}_0 \backslash H / \hat{M}_0$. Set $\hat{M}_1 = \hat{M} \cap \hat{M}^*$ and assume that $\hat{M}_0 \backslash H / \hat{M}_0 = \hat{M}_1 \backslash H / \hat{M}_1$. Then for each unitary corepresentation $V_\alpha \to V_\alpha \otimes H$, there exists a basis in V_α such that*

$$\hat{M}\backslash H/\hat{M} = \sum_{\alpha}\sum_{i,j=1}^{d'_\alpha} C u^{\alpha}_{i,j}, \quad \hat{M}\backslash H = \sum_{\alpha}\sum_{i=1}^{d'_\alpha}\sum_{j=1}^{d_\alpha} C u^{\alpha}_{i,j},$$
$$H/\hat{M} = \sum_{\alpha}\sum_{i=1}^{d_\alpha}\sum_{j=1}^{d'_\alpha} C u^{\alpha}_{i,j}. \tag{8}$$

Proof. Consider a unitary representation $V_\alpha \to V_\alpha \otimes H$ with the matrix elements $\{u^{\alpha}_{ij}\}^{d_\alpha}_{i,j=1}$. Let $\xi \in \hat{M}_1$ and denote by A_ξ the matrix $(A_\xi)_{ij} = \xi(u^{\alpha}_{ij})$. If $f = \sum_{i,j=1}^{d_\alpha} n_{ij}\, u^{\alpha}_{ij}$, then $\xi^l(f) = A_\xi N$ and $\xi^r(f) = N A_\xi$, where N is the matrix (n_{ij}). If f is left-invariant, then $A_\xi N = \hat{\varepsilon}(\xi)N$, i.e., the vectors $\{n^{\alpha}_{ij}\}^{d_\alpha}_{i=1}$ are eigenvectors of the matrix A_ξ with the eigenvalue $\hat{\varepsilon}(\xi)$ for all $j = 1, \ldots, d_\alpha$. Since $\hat{M}_1$ is closed with respect to the involution and $A_{\xi^*} = A^*_\xi$, there is an orthonormal basis in V_α such that the first d'_α vectors are the eigenvectors of the operators A_ξ corresponding to the eigenvalue $\hat{\varepsilon}(\xi)$ and $A_\xi = \hat{\varepsilon}(\xi) I_{d'_\alpha} \oplus A'_\xi$ for some $d_\alpha - d'_\alpha$-dimensional operators A'_ξ. It is clear that in such a basis, we have decompositions (8).

Lemma 11. *Let H be a compact quantum group, let $\hat{M}_0$ be an arbitrary coideal, and let $\hat{M}$ — an algebra maximal with respect to $\hat{M}_0\backslash H/\hat{M}_0$. Suppose that, for $\hat{M}_1 = \hat{M} \cap \hat{M}^*$, $\hat{M}\backslash H/\hat{M} = \hat{M}_1\backslash H/\hat{M}_1$. Then there exists an invariant integral with respect to $\hat{M}$. Moreover, it is a state.*

Proof. Let π be a projection with respect to (8) of H onto the linear subspace $\hat{M}\backslash H$ and let $\nu_{\hat{M}} = \varepsilon \circ \pi$. We will show that $(\varepsilon\circ\pi)^l$ is a projection onto $\hat{M}\backslash H$. Indeed, let $f = \sum_\alpha \sum_{i,j=1}^{d_\alpha} f^{i,j}_\alpha u^{\alpha}_{i,j}$. Then

$$(\varepsilon\circ\pi)^l(f) = (\varepsilon\circ\pi\otimes \mathrm{id})\left(\sum_\alpha\sum_{i,j=1}^{d_\alpha}\sum_{k=1}^{d_\alpha} f^{i,j}_\alpha u^{\alpha}_{i,k}\otimes u^{\alpha}_{k,j}\right) =$$

$$= \sum_\alpha\sum_{i,j=1}^{d_\alpha}\sum_{k=1}^{d'_\alpha} f^{i,j}_\alpha\, \varepsilon\left(u^{\alpha}_{i,k}\right) u^{\alpha}_{k,j} = \sum_\alpha\sum_{i=1}^{d'_\alpha}\sum_{j=1}^{d_\alpha} f^{i,j}_\alpha u^{\alpha}_{i,j} \in \hat{M}\backslash H.$$

In the same way, we show that $(\varepsilon\circ\pi)^r$ is a projection onto $H/\hat{M}$ and, hence, by the remark to Lemma 7, $\nu_{\hat{M}}$ is an invariant integral with respect to $\hat{M}$.

Since $\nu\left(u^{\alpha}_{ij}\, u^{\beta^*}_{ij}\right) = 0$ if $i \neq j$, π is an orthogonal projection and, hence, $\|\pi\| = 1$. Since ε is a homomorphism, $\|\varepsilon\| = 1$, and so $\|\nu_{\hat{M}}\| \le \|\pi\|\,\|\varepsilon\| = 1$. Consequently, $\|\nu_{\hat{M}}\| = \nu_{\hat{M}}(1) = 1$ and it is a state.

Assume now that $\hat{M}_0$ is a coideal such that the algebra $\hat{M}\backslash H/\hat{M}$ is commutative.

Let $\tilde{H}$ be a completion of the algebra $\hat{M}\backslash H/\hat{M}$ with respect to the C^*-norm $|\cdot| = \sup_\rho \|\rho(\cdot)\|$, where ρ runs over the set of all irreducible representations of H. We denote the spectrum of the commutative C^*-algebra $\tilde{H}$ by $\mathrm{Spec}(\tilde{H})$. We also use $\tilde{\Delta}$ and ν to denote the extensions of the corresponding mapping to $\tilde{H}$. Since any $p \in \mathrm{Spec}(\tilde{H})$ can be identified with a continuous homomorphism $p: \tilde{H} \to C$, there is an involution $\vee: \mathrm{Spec}(\tilde{H}) \to \mathrm{Spec}(\tilde{H})$ defined by

$$p^{\vee}(f) = \overline{p(S(f^*))} \qquad \forall f \in \tilde{H} \tag{9}$$

and, for any $p \in (\tilde{H})$, there is a generalized shift operator $L^p: \tilde{H} \to \tilde{H}$ given by

$$L^p(f) = (p \otimes \mathrm{id}) \circ \tilde{\Delta}(f) \qquad \forall f \in \tilde{H}. \tag{10}$$

We use $\tilde{H}$ and ν for GNS construction of a Hilbert space $L_2(\tilde{H})$ and consider L^p as operators on this space.

Lemma 12. *Let the involution* $\vee$ *and the generalized shift operators* L^p, $p \in \mathrm{Spec}(\tilde{H})$, *be given by (9) and (10). Then*

a) L^p *is a bounded operator for all* $p \in \mathrm{Spec}(\tilde{H})$ *and the mapping* $p \to L^p$ *is strongly continuous;*

b) $\varepsilon^{\vee} = \varepsilon$ *and* $\overline{L^{q^{\vee}}(S(f)^*)(p^{\vee})} = L^{p}(f)(q)$ *for all* $p \in L_2(\tilde{H})$;

c) $L^{\varepsilon} = \mathrm{id}$;

d) for any positive $f \in L_2(\tilde{H})$, $L^p(f)$ *is positive for all* $p \in \mathrm{Spec}(\tilde{H})$;

e) $L^p(1)(f) = 1$ *for all* $p, q \in \mathrm{Spec}(\tilde{H})$;

f) $(L^p)^* = L^{p^{\vee}}$, *where* $(L^p)^*$ *is the operator adjoint to* L^p in $L_2(\tilde{H})$.

Proof. a) It follows from the definition of L^p and the positivity property of $\tilde{\Delta}$ that L^p are bounded with $\|L^p\| \le 1$. Moreover, for any $f \in \tilde{H}$, $p \to L^p(f) = (p \otimes \mathrm{id}) \circ \tilde{\Delta}(f)$ is continuous.

b) Since $\overline{\varepsilon \circ S \circ *} = \varepsilon$, $\varepsilon^{\vee} = \varepsilon$. To prove the second part, let $f \in \tilde{H}$. Then

$$\overline{L^{q^{\vee}}(S(f)^*)(p^{\vee})} = (q \circ S \circ * \otimes p \circ S \circ *) \circ (\mathrm{id} \otimes \nu_2 \circ \pi \otimes \mathrm{id}) \circ (\tilde{\Delta}_1 \otimes \mathrm{id}) \circ \tilde{\Delta}_1(S(f)^*) =$$

$$= (p \circ S \circ * \circ S \circ * \otimes q \circ S \circ * \circ S \circ *) \circ (\mathrm{id} \otimes \nu_2 \circ \pi \otimes \mathrm{id}) \circ (\tilde{\Delta}_1 \otimes \mathrm{id}) \circ \tilde{\Delta}_1(f) =$$

$$= L^p(f)(q).$$

Now we use part a).

c) This is a direct consequence of part c) of Theorem 2.

d) Since $\tilde{\Delta}$ is positive, this follows from the fact that p is a homomorphism and property a).

e) Since $\tilde{\Delta}(1) = 1 \otimes 1$ and p, q are homomorphisms, $L^p(1)(q) = 1$.

f) It is sufficient to prove that $\langle L^p f, g \rangle = \langle f, L^{p^{\vee}} g \rangle$ on $\tilde{H}$ for all continuous functionals $p, q: \tilde{H} \to C$, $f, g \in \tilde{H}$, L^p considered as a linear operator $\tilde{H} \to \tilde{H}$, and the involution $\vee$ naturally extended over the space of continuous linear functionals on $\tilde{H}$. So let $f = u^{\alpha}_{ij}$ and $g = u^{\beta}_{mn}$ be matrix elements of unitary corepresentations of $\tilde{H}$. Then we have

$$\langle L^p u^{\alpha}_{ij}, u^{\beta}_{mn} \rangle = \left\langle \sum_k (p \otimes \mathrm{id})(u^{\alpha}_{ik} \otimes u^{\alpha}_{kj}), u^{\beta}_{mn} \right\rangle = \sum_k p(u^{\alpha}_{ik}) \langle u^{\alpha}_{kj}, u^{\beta}_{mn} \rangle =$$

$$= \delta_{\alpha\beta}\, p(u^{\alpha}_{im}) \langle u^{\alpha}_{mj}, u^{\alpha}_{mn} \rangle.$$

On the other hand,

$$\langle u^{\alpha}_{ij}, L^{q^{\vee}}(u^{\beta}_{mn}) \rangle = \left\langle u^{\alpha}_{ij}, \sum_k (p^{\vee} \otimes \mathrm{id})(u^{\beta}_{mk} \otimes u^{\beta}_{kn}) \right\rangle =$$

$$= \sum_k \overline{p^\vee\left(u^\beta_{mk}\right)} \left\langle u^\alpha_{ij}, u^\beta_{kn} \right\rangle = \delta_{\alpha\beta}\, \overline{p^\vee\left(u^\alpha_{mi}\right)} \left\langle u^\alpha_{ij}, u^\alpha_{in} \right\rangle.$$

Since [13] $\overline{p^\vee\left(u^\alpha_{mi}\right)} = p\left(u^\alpha_{im}\right)$ and $\left\langle u^\alpha_{ij}, u^\alpha_{in} \right\rangle = \left\langle u^\alpha_{mj}, u^\alpha_{mn} \right\rangle$, we see that the equality holds. By restricting to $\hat{M} \backslash H / \hat{M}$ and then extending to $\tilde{H}$, we prove (f).

It follows from Theorem 2.1 of [2] that if generalized shift operators satisfy the properties listed in Lemma 12, then they generate a hypercomplex system on Spec ($\tilde{H}$). So, we have following lemma:

Lemma 13. *Let $\tilde{H}$ be a commutative C^*-algebra defined above with the comultiplication $\tilde{\Delta}$, count ε, unit 1, antipode S, involution $*$, and the state ν. Then* Spec ($\tilde{H}$) *is a basis of a hypercomplex system.*

Now we will be considering a strict Gelfand pair $\left(\hat{H}, \hat{M}\right)$ with a complete set of characters $\Phi = \{\varphi_m\}_{m \in Q}$ of the above mentioned commutative hypercomplex system for some discrete set Q. Since the product of two characters is a positive definite function, we can apply the general construction of [2] to find a hypercomplex system dual to the above mentioned. This will be a hypercomplex system with a discrete basis Φ and, hence, it is a hypergroup [2].

For the basis $\Phi = \{\varphi_m\}_{m \in Q}$, we have

$$\varphi_l \varphi_m = \sum_{k \in Q} c^k_{lm} \varphi_k, \tag{11}$$

where c^k_{lm} are the Clebsch-Gordan coefficients in the decomposition into a sum of irreducible corepresentations of the tensor product of irreducible corepresentation of $\tilde{H}$.

Now by using the duality theorem from [2], we get the following theorem:

Theorem 3. *Suppose that $\left(\hat{H}, \hat{M}\right)$ is a strict Gelfand pair. Then there are two structures dual to each other: a commutative hypercomplex system with compact basis* Spec ($\tilde{H}$) *and the discrete commutative hypergroup with the basis Q. Here, φ_m are the characters of the hypercomplex system if considered as functions on* Spec ($\tilde{H}$) *and they are the characters of the hypergroup if considered as functions on Q.*

Remark. A direct construction of the dual hypergroup is given in an informal note by Koornwinder for a special case of a Gelfand pair, which is not necessarily strict.

Example. Consider the compact quantum group $U_q(n)$ and set $\boldsymbol{H} = U_q(n)$. It is known that [14]

$$U_q(n) = C\langle t_{ij}, t, 1\rangle / I_R.$$

Here, $C\langle t_{ij}, t, 1\rangle$ is a free algebra generated by the elements of the matrix $T = (t_{ij})$, $i, j = 1, \ldots, n$, the elements t, 1, and I_R is a two-sided ideal generated by the relations

$$RT_1T_2 = T_2T_1R,$$
$$tt_{ij} = t_{ij}t,$$
$$t \cdot \det_q(T) = \det_q(T) \cdot t = 1,$$

where $T_1 = T \otimes I$, $T_2 = I \otimes T$, I is the identity matrix in R^n, the matrix R is given by

$$R = \sum_{1 \le i, j \le n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + \left(q - q^{-1}\right) \sum_{1 \le i < j \le n} e_{ij} \otimes e_{ji},$$

$e_{ij} \in Mat(n \times n)$ are matrix units,

$$\det_q(T) = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1\sigma_1} \ldots t_{n\sigma_n},$$

S_n is the permutation group of the set $\{1, \ldots, n\}$, $l(\sigma)$-the length of the permutation σ, and $q \in C$.

For $q \in R$ and $|q| < 1$, the structure of a $*$-Hopf algebra is defined on the generators and is given by

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \quad \Delta(t) = t \otimes t,$$

$$\varepsilon(t_{ij}) = \Delta_{ij}, \quad \varepsilon(t) = 1,$$

$$S(t_{ij}) = (-q)^{i-j} \omega^{1,\ldots,\hat{j},\ldots,n}_{1,\ldots,\hat{i},\ldots,n} \cdot t, \quad S(\sigma) = \det_q(T),$$

where $\omega^{1,\ldots,\hat{j},\ldots,n}_{1,\ldots,\hat{i},\ldots,n}$ is a quantum minor defined to be the quantum determinant of the matrix (t_{rs}) with the indices r and s belonging to the sets $\{1, \ldots, i-1, i+1, \ldots, n\}$ and $\{1, \ldots, j-1, j+1, \ldots, n\}$, respectively, and the involution $*$ is defined by

$$t^*_{ij} = S(t_{ji}), \quad t^* = t.$$

The dual $*$-Hopf algebra, $u_q(n)$, is defined as the algebra [14]

$$u_q(n) = C\langle l^+_{ij}, l^-_{ij}, 1\rangle / I_{R^+},$$

where the free algebra $C\langle l^+_{ij}, l^-_{ij}, 1\rangle$ is generated by the elements $l^+_{ij}, l^-_{ij}, 1$, $i, j = 1, \ldots, n$, and the two-sided ideal I_{R^+} is generated by the relations

$$R^+ L^\pm_1 L^\pm_2 = L^\pm_2 L^\pm_1 R^+,$$

$$R^+ L^+_1 L^-_2 = L^-_2 L^+_1 R^+,$$

where $L^\pm_1 = L^\pm \otimes I$, $L^\pm_2 = I \otimes L^\pm$, $R^+ = PRP(P(l_1 \otimes l_2) = l_2 \otimes l_1)$. The coalgebra structure is given by

$$\Delta(l^\pm_{ij}) = \sum_{k=1}^{n} l^\pm_{ik} \otimes l^\pm_{kj},$$

and a nondegenerate pairing is defined to be

$$(L^\pm, T_1 \ldots T_k) = R^\pm_1 \ldots R^\pm_k,$$

where

$$T_i = I \otimes \ldots \underbrace{\otimes T \otimes}_{i} \ldots \otimes I,$$

$R^- = R^{-1}$, and $R^\pm_i$ acts as $R^\pm$ on the 0^{th} and i^{th} component of the tensor product $(R^n)^{\otimes(k+1)}$.

Let P be a free Z-module with the basis $\{\varepsilon_i\}_{i=1}^n$, i.e., $P = \sum_{i=1}^n Z\varepsilon_i$, and let $P^+ \subset P$ be such that

$$P^+ = \left\{ \Lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \right\}.$$

It is known [15] that

$$U_q = \underset{\Lambda \in P^+}{\otimes} W(\Lambda),$$

where $W(\Lambda)$ is an invariant finite-dimensional $u_q(n)$-submodule such that

$$l_{ii}^{+l}\left(u(\Lambda)\right) = q^{\lambda_i} u(\Lambda)$$

and the module $W(\Lambda)$ is generated by the element $u(\Lambda)$. We say that elements of $W(\Lambda)$ have the weight Λ.

Let $J = \sum_{i=1}^{n} q^{n-1} e_{ii}$ and let $\tilde{M}_0$ be a coideal generated by the matrix elements b_{ij} of

$$B = L^+ - JS(L^{-1})^t J^{-1}.$$

Theorem 4 [11]. *For every* $\Lambda \in P^+$, *the set of biinvariant elements,* $(\hat{M} \setminus U_q(n) / \hat{M})_\Lambda$ *equals to one if and only if*

$$\lambda_k - \lambda_{k+1} \in 2Z, \quad 1 \le k \le n-1. \tag{12}$$

Let $\Lambda_r = \sum_{i=1}^{r} \varepsilon_i$. Then every $\Lambda \in P^+$ satisfying (12) can be represented as

$$\Lambda = \sum_{i=1}^{n-1} 2m_r \Lambda_r + l\Lambda_n,$$

where m_r, $l \in Z_+$. Let also

$$\varphi(2\Lambda_r) = \sum_{i_1 < \ldots < i_r,\, j_1 < \ldots j_r} \left(\omega_{i_1 \ldots i_r}^{j_1 \ldots j_r}\right)^2 a_{i_1}^{-1} \ldots a_{i_r}^{-1} a_{j_1} \ldots a_{j_r}, \tag{13}$$

where $\omega_{i_1 \ldots i_r}^{j_1 \ldots j_r}$ is a quantum minor of the matrix T and $a_k = q^{n-k}$.

Theorem 5 [11]. *The subalgebra of biinvariant elements* $\hat{M} \setminus U_q(n) / \hat{M}$ *is commutative. It is generated by the elements* $\varphi(2\Lambda_r)$, $r = 1, \ldots, n$. *The involution on* $\varphi(2\Lambda_r)$ *is given by*

$$\varphi(2\Lambda_r)^* = \varphi(2\Lambda_{n-r} - 2\Lambda_n). \tag{14}$$

Moreover, one-dimensional subspaces $(\hat{M} \setminus U_q(n) / \hat{M})_\Lambda$ *are generated by the elements*

$$\tilde{\varphi}(\Lambda) = \varphi(2\Lambda_1)^{m_1} \ldots \varphi(2\Lambda_{n-1})^{m_{n-1}} \varphi(\Lambda_n)^l, \quad \Lambda \in P^+. \tag{15}$$

Corollary. *It follows from Theorem 4 and Theorem 5 that* $(u_q(n), \hat{M})$ *is a strict Gelfand pair.*

Let now $\tilde{H}$ denote the completion of the commutative algebra $\hat{M} \setminus U_q(n) / \hat{M}$ with respect to the C^*-algebra nor $\|\cdot\| = \sup_\rho \|\rho(\cdot)\|$, where ρ runs over the set of all irreducible representations of $U_q(n)$. Then the spectrum of $\tilde{H}$, Spec $(\tilde{H})$, can be identified with $T^n = \{(x_1, \ldots, x_n) \in C^n : |x_1| = \ldots = |x_n| = 1\}$ — the n-dimensional torus (see [11]). Hence, we can identify $\tilde{H}$ with the algebra of all continuous functions on the compact space T^n. The invariant integral ν on this algebra is given as follows [11]:

For a holomorphic function in the neighborhood of T^n, let $[F(x)]_0$ denote the constant term the Laurent series expansion of $F(x)$ and define the meromorphic function on T^n, $\omega(x; q, t)$, by

$$\omega(x; q, t) = \prod_{1\le i<j\le n} \frac{\left(x_i / x_j; q\right)_\infty \left(x_j / x_i; q\right)_\infty}{\left(t x_i / x_j; q\right)_\infty \left(t x_j / x_i; q\right)_\infty},$$

where $(a; q)_\infty = \prod_{k=0}^{\infty} \left(1 - a q^k\right)$. Then, for a function $\varphi \in \tilde{H}$, which can be identified with a symmetric polynomial on T^n, we set

$$\nu(\varphi) = \frac{[\varphi\omega(x; q^4, q^2)]_0}{\omega(x; q^4, q^2)}. \tag{16}$$

Since $\hat{M}_0^* = q^{-\rho} \hat{M}_0 q^{\rho}$, where $\rho = \sum_{k=1}^{n} (n-k)\varepsilon_k$, the conditions of Lemma 11 hold and Theorem 3 gives in this particular case the following:

Theorem 6. *For the strict Gelfand pair* $(u_q(n), \hat{M})$ *there is a commutative hypercomplex system with the compact basis* T^n, *the invariant integral given by* (16) *and the discrete commutative hypergroup with the basis* P^+. *They are dual to each other and their characters given by* (15) *and considered as functions on* T^n *are the Macdonald's symmetric polynomials.*

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