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GENERALIZED GREEN'S MATRIX FOR LINEAR PULSE BOUNDARY-VALUE PROBLEMS

УЗАГАЛЬНЕНА МАТРИЦЯ ГРІНА ДЛЯ ЛІНІЙНИХ ІМПУЛЬСНИХ КРАЙОВИХ ЗАДАЧ

An algebraic criterion of solvability and the structure of a general solution are found for a linear boundary-value problem for systems of differential equations with pulse effects. The generalized Green's matrix is constructed.

Одержано алгебраїчний критерій розв'язності та структуру загального розв'язку крайової задачі для систем диференціальних рівнянь з імпульсним ефектом. Побудовано узагальнену матрицю Гріна.

1. Introduction. We consider the linear system of differential equations

$$\dot{x} = A(t)x + \varphi(t), \quad t \in [a, b], \ t \neq \tau_i, \ i \neq \overline{1, p}, \tag{1}$$

with pulse effects in the boundary conditions

$$l(x) \equiv \sum_{i=1}^{p+1} l_i(x_i) = h. \tag{2}$$

Here, the $n \times n$-matrix function $A(t)$ is from $C[a, b]$, $\varphi(t)$ is a first-order discontinuous n-vector function for $t = \tau_i$, $i = \overline{1, p}$, i.e.,

$$\varphi(t) = \varphi_i(t), \quad t \in [\tau_{i-1}, \tau_i], \ i = \overline{1, p+1}, \ \tau_0 = a, \ \tau_{p+1} = b.$$

The linear functionals $l_i(x_i)$ are defined in the n-vector space of continuously differentiable functions $x_i(t)$ in $[\tau_{j-1}, \tau_j]$, $i = \overline{1, p+1}$; $l_i(x_i)$: $C^1[\tau_{i-1}, \tau_i] \to R^m$; h is an arbitrary m-vector from E^m.

Difference boundary-value problems are obtained in dependence on the concrete form of the linear functionals. For instance, multipoint boundary-value problems with p-points of pulse effects can be written by the Stieltjes integral

$$l_i(x_i) = \int_{\tau_{i-1}}^{\tau_i} [d\sigma_i(s)]\, C_i(s) x_i(s), \quad i = \overline{1, p+1}, \ \tau_0 = a, \ \tau_{p+1} = b,$$

where $\sigma_i(s) = \operatorname{diag}\left[\sigma_{11}^i(s) \ldots \sigma_{mm}^i(s)\right]$ are diagonal $m \times m$ matrices, elements of which are functions of bounded variation in $[\tau_{i-1}, \tau_i]$; $C_i(s)$ are $m \times n$ matrices, elements of which are first-order discontinuous function for $t = \tau_i$, $i = \overline{1, p}$.

We consider the problem of finding a first-order discontinuous function

$$x(t) = x_i(t), \quad t \in [\tau_{i-1}, \tau_i], \ i = \overline{1, p+1}, \ \tau_0 = a, \ \tau_{p+1} = b, \tag{3}$$

for $t = \tau_i$, $i = \overline{1, p+1}$, so that the vector differentiable functions $x_i(t)$ satisfy (1), (2).

The system (1), (2) is, in particular, considered in [1], where the boundary conditions are written in the form

$$C_i x(\tau_i - 0) + D_i x(\tau_i + 0) = d_i, \quad i = \overline{1, p},$$
$$\sum_{k=0}^{p+2} A_k x(t_k) = d_{p+1}, \quad t_k \neq \tau_i, \; i = \overline{1, p}, \tag{4}$$

where C_i, B_i, A_k are $n \times n$ matrices, the pulse points τ_k lie between the points t_k as follows:

$$t_k < \tau_k < t_{k+1}, \quad k \doteq \overline{1, p}. \tag{5}$$

The homogeneous multipoint boundary-value problem corresponding to (1), (4), (5) has a unique zero solution.

In this paper, this problem is solved by using the apparatus of the half-inverse matrices and the generalized Green's matrices. (For half-inverse matrices, see [2]; for generalized Green's matrix, see, e.g., [3 – 5].)

In [6], the same apparatus is applied to the system (1) with two-point boundary conditions and with one pulse effect in (5).

2. Principal results. We introduce the following notation: $\Phi(t)$ is a fundamental matrix of solutions of $\dot{x} = A(t)x$ with $\Phi(a) = I_n$ (I_n are unit $n \times n$ matrices); $l_i(\Phi)$, $i = \overline{1, p+1}$, are $m \times n$ matrices; $M = [l_1(\Phi) \ldots l_{p+1}(\Phi)]$ is an $m \times (p+1)n$ matrix; $M^{-} = [N_1 N_2 \ldots N_{p+1}]^T$ is an arbitrary half-inverse matrix of M, where N_i, $i = \overline{1, p+1}$, are $n \times m$ matrices.

1. The general solution of (1) has the form

$$x(t) = \Phi(t)c + \bar{x}(t), \quad c \in E^n, \tag{6}$$

with the coordinate representation $(\tau_0 = a, \; \tau_{p+1} = b)$

$$x_j(t) = \Phi(t)\Phi^{-1}(\tau_{j-1})x_j(\tau_{j-1}) + \bar{x}_j(t), \quad t \in [\tau_{j-1}, \tau_j], \; j = \overline{1, p+1}. \tag{7}$$

The function $\bar{x}(t)$ is first-order discontinuous and it is a particular solution of (1), which we choose in the form

$$\bar{x}(t) = \int_a^b K(t, s)\varphi(s)ds; \quad s, t \in [a, b]\,; \; s, t \neq \tau_i, \; i = \overline{1, p}, \tag{8}$$

where

$$K(t, s) = \frac{1}{2}\Phi(t)\Phi^{-1}(s)\,\mathrm{sign}\,(t - s),$$
$$t, s \in [a, b]\,; \quad t, s \neq \tau_i, \quad i = \overline{1, p}.$$

The domain of definition, $a \le t, \; s \le b$, of the functions $K(t, s)$ is decomposed into subdomains so that $(j = \overline{1, p+1})$

$$K(t, s) = \begin{cases} K_i^j(t, s) = \dfrac{1}{2}\Phi(t)\Phi^{-1}(s), & i = 1, 2, \ldots, j-1, j1 \quad \text{in } i \le j, \\ K_i^j(t, s) = -\dfrac{1}{2}\Phi(t)\Phi^{-1}(s), & i = j2, j+1, \ldots, p+1 \quad \text{in } i \le j. \end{cases}$$

If $i = j$, then j takes the values $j1$ and $j2$ corresponding to the parts under and over the diagonal of the square $\tau_{j-1} \le t \le \tau_j$, $\tau_{j-1} \le s \le \tau_j$. The form of $K(t, s)$ in the subdomains determined by the pulse effect implies the coordinate representation of (8)

$$\bar{x}_i(t) = \int_a^b K_i(t,s)\varphi(s)ds = \int_a^{\tau_1} K_i^1(t,s)\varphi_1(s)ds + \int_{\tau_1}^{\tau_2} K_i^2(t,s)\varphi_2(s)ds +$$

$$+ \ldots + \int_{\tau_{i-1}}^{t} K_i^{j1}(t,s)\varphi_i(s)ds + \int_t^{\tau_i} K_i^{j2}(t,s)\varphi_i(s)ds + \ldots +$$

$$+ \int_{\tau_p}^{b} K_i^{p+1}(t,s)\varphi_{p+1}(s)ds, \quad t \in [\tau_{i-1}, \tau_i], \; i = \overline{1, p+1}. \tag{9}$$

We substitute (6) into the boundary conditions (2) and obtain the algebraic system with the solutions $x_i(\tau_{i-1})$, $i = \overline{1, p+1}$

$$M[\Phi^{-1}(a)x_1(a)\Phi^{-1}(\tau_1)x_2(\tau_1)\ldots\Phi^{-1}(\tau_p)x_{p+1}(\tau_p)]^T =$$

$$= h - \sum_{i=1}^{p+1} l_i(\bar{x}_i), \tag{10}$$

where M is the $m \times (p+1)n$ matrix introduced above.

The necessary and sufficient condition of existence of a solution of (10) is

$$(I_m - MM^-)\left(h - \sum_{i=1}^{p+1} l_i(\bar{x}_i)\right) = 0. \tag{11}$$

We denote by

$$P_M^* = I - M^- M(p+1)n \times (p+1)n$$

the matrix projector of $R^{(p+1)n}$ on the space $N(M)$ of M, i.e., $P_M^* : R^{(p+1)n} \to \to N(M)$. Then the general solution of (10) has the form

$$[\Phi^{-1}(a)x_1(a)\Phi^{-1}(\tau_1)x_2(\tau_1)\ldots\Phi^{-1}(\tau_p)x_{p+1}(\tau_p)]^T =$$

$$= M^-\left(h - \sum_{i=1}^{p+1} l_i(\bar{x}_i)\right) + P_M^* u, \tag{12}$$

where u is an arbitrary $(p+1)n \times 1$ vector, the elements of which are numbers.

We represent the matrix P_M^* in a hypermatrix form

$$P_M^* = \left[P_M^1 P_M^2 \ldots P_M^{p+1}\right]^T,$$

where P_M^i, $i = \overline{1, p+1}$, are $n \times (p+1)n$ matrices. Then, by (12), we obtain $x_{j+1}(\tau_j)$,

$$x_{j+1}(\tau_j) = \Phi(\tau_j)N_j\left(h - \sum_{i=1}^{p+1} l_i(\bar{x}_i)\right) + \Phi(\tau_j)P_M^{j+1}u, \quad j = \overline{0, p}, \; \tau_0 = a. \tag{13}$$

Now we substitute (13) and (9) in (7) and, after some transformations, we obtain

$$x_j(t) = \Phi(t)P_M^j u + \Phi(t)N_j h + \int_a^b G_{0j}(t,s)\varphi(s)ds, \; t \in [\tau_{j-1}, \tau_j], \; j = \overline{1, p+1}, \tag{14}$$

where

$$G_{0j}(t,s) = K_j(t,s) - \Phi(t)N_j \sum_{i=1}^{p+1} l_i K_i(\cdot,s), \quad j = \overline{1, p+1}. \tag{15}$$

In this case, it is necessary to keep in mind that

$$\int_a^b G_{0q}(t,s)\varphi(s)ds = \int_a^{\tau_1} G_{0q}^1(t,s)\varphi_1(s)ds + \ldots + \int_{\tau_{q-1}}^t G_{0q}^{q1}(t,s)\varphi_q(s)ds +$$

$$+ \int_t^{\tau_q} G_{0q}^{q2}(t,s)\varphi_q(s)ds + \ldots + \int_{\tau_p}^b G_{0q}^{p+1}(t,s)\varphi_{p+1}(s)ds, \quad q = \overline{1, p+1}, \tag{16}$$

and

$$G_{0q}^l(t,s) = K_q^l(t,s) - \Phi(t)N_q \sum_{i=1}^{p+1} l_i K(\cdot,s), \quad t \in [\tau_{q-1}, \tau_q],$$

$$q = \overline{1, p+1}, \quad l = 1, 2, \ldots, q-1, q1, q2, q+1, \ldots, p+1. \tag{17}$$

Lemma 1. *The matrix function*

$$G_0(t,s) = K(t,s) - \Phi(t)NlK(\cdot,s), \tag{18}$$

where

$$G_0(t,s) = G_{0q}(t,s), \quad t \in [\tau_{q-1}, \tau_q], \; s \in [a,b], \; q = \overline{1, p+1}, \tag{19}$$

and

$$G_{0q}(t,s) = \begin{cases} G_{0q}^l(t,s), \; s < t, \; l = 1, 2, \ldots, q-1, \; q1; & t \in [\tau_{q-1}, \tau_q], \\ G_{0q}^l(t,s), \; s > t, \; l = q2, \; q+1, \ldots, p+1; & t \in [\tau_{q-1}, \tau_q], \end{cases} \tag{20}$$

is $n \times n$-dimensional generalized Green's matrix for the pulse boundary-value problem (1), (2).

Proof. We prove Lemma 1 by direct verification of the conditions (see, e.g., [3]):

$$\frac{d}{dt} G_0(t,s) = A(t)G_0(t,s), \quad s \in [a,b], \; t \neq s \neq \tau_i, \; i = \overline{1, p},$$

$$lG_0(\cdot,s) = (I_m - MM^-)lK(\cdot,s),$$

$$t = a, \; t = b, \; t = \tau_i, \; s \in [a,b], \; s \neq \tau_i, \tag{21}$$

$$G_0(s+0,s) - G_0(s-0,s) = I_n, \quad t = s \neq \tau_i.$$

2. Let D be a square $(p+1)n \times (p+1)n$ square matrix

$$D = \int_a^b x_0^T(t)x_0(t)dt = \sum_{i=1}^{p+1} \int_{\tau_{i-1}}^{\tau_i} x_{0i}^T(t)x_{0i}(t)dt, \quad \tau_0 = a, \; \tau_{p+1} = b, \tag{22}$$

where $x_0(t)u = \Phi(t)P_m u$ is the general solution of the boundary-value system $\dot{x} = A(t)x, \; l(x) = 0$.

Lemma 2. *Let* $\det D \neq 0$. *Then the matrix-function*

$$G(t,s) = G_0(t,s) - \Phi(t)P_m D^{-1} \int_a^b x_0^T(t) G_0(t,s)\,dt, \tag{23}$$

where

$$\begin{gathered} G(t,s) = G_j(t,s) = G_{0j}(t,s) - \Phi(t)P_m^j D^{-1} \int_a^b x_0^T(t) G_0(t,s)\,dt, \\ t \in [\tau_{j-1}, \tau_j],\quad j = \overline{1, p+1},\quad \tau_0 = a,\quad \tau_{p+1} = b,\quad s \in [a,b], \end{gathered} \tag{24}$$

and

$$G_j(t,s) = \begin{cases} G_j^l(t,s), & \text{for } s < t,\ l = 1, 2, \ldots, j-1,\ j1;\quad t \in [\tau_{j-1}, \tau_j], \\ G_j^l(t,s), & \text{for } s > t,\ l = j2,\ j+1, \ldots, p+1;\ t \in [\tau_{j-1}, \tau_j], \end{cases} \tag{25}$$

$$G_j^l(t,s) = G_{0j}^l(t,s) - \Phi(t)P_M^j D^{-1} \int_a^b x_0^T(t) G_0(t,s)\,dt,$$

is the unique Green's matrix-function for pulse system (1), (2) *orthogonal to an arbitrary solution of the correspondent homogeneous system.*

Proof. Since $G(t,s)$ (from (23)–(25)) is expressed in terms of $G_0(t,s)$ (from (18)–(20)) and, for $G_0(t,s)$, Lemma 1 is true, we directly verity conditions (21). It remains to verify the additional orthogonal condition

$$\int_a^b x_0^T(t) G(t,s)\,dt = 0 \quad \text{at} \quad s \in [a,b],\ s \neq \tau_j,\ j = \overline{1, p}.$$

Indeed, (I is a unit $(p+1)n \times (p+1)n$ matrix)

$$\int_a^b x_0^T(t) G(t,s)\,dt = \int_a^b x_0^T(t) \left[G_0(t,s) - \Phi(t) P_M D^{-1} \int_a^b x_0^T(\tau) G_0(\tau,s)\,d\tau \right] dt =$$

$$= \int_a^b x_0^T(t) G_0(t,s)\,dt - \int_a^b x_0^T(t)\Phi(t) P_M\,dt\,D^{-1} \int_a^b x_0^T(\tau) G_0(\tau,s)\,d\tau =$$

$$= \left[I - \int_a^b x_0^T(t) x_0(t)\,dt\,D^{-1} \right] \int_a^b x_0^T(\tau) G_0(t,s)\,d\tau =$$

$$= [I - DD^{-1}] \int_a^b x_0^T(\tau) G_0(\tau,s)\,d\tau = 0.$$

Theorem. *Let condition* (11) *be satisfied. If* $\det D \neq 0$, *then the general solution of* (1), (2) $x(t,u)$ *from the set of first-order discontinuous functions defined on* $[a,b]$ *has the form*

$$x(t,u) = x_0(t)u + x^*(t), \quad u \in E_{(p+1)n}. \tag{26}$$

Here, $x_0(t)u = \Phi(t)P_M u$ *is a general solution of the associated homogeneous system* $\dot{x} = A(t)x$, $l(x) = 0$; $\bar{x}^*(t)$ *is the unique particular solution of* (1), (2), *satisfying the orthogonal condition*

$$\int_a^b x_0^T(t)\,x(t,u)\,dt = 0 \tag{27}$$

and having the form

$$\bar{x}^*(t) = \int_a^b G(t,s)\varphi(s)\,ds + H(t), \quad t,s \in [a,b],$$
$$H(t) = \Phi(t)\left[N - P_M D^{-1}\int_a^b x_0^T(t)\Phi(t)N\,dt\right]h. \tag{28}$$

Proof. By using the coordinate representation (14), we obtain the general solution of (1), (2) in the form

$$x(t,u) = x_0(t)u + \bar{x}_0(t), \quad u \in E_{(p+1)n}, \tag{29}$$

where

$$x_0(t)u = \Phi(t)P_M u \quad \text{and} \quad \bar{x}_0(t) = \Phi(t)Nh + \int_a^b G_0(t,s)\varphi(s)\,ds$$

are a general solution of the homogeneous system $\dot{x} = A(t)x,\ l(x) = 0$ and a particular solution of (1), (2), respectively. We substitute $x(t,u)$ from (29) into (27) and find a $(p+1)n \times 1$ vector u from the algebraic system

$$Du = B, \tag{30}$$

where D is the matrix from (22), and the $(p+1)n \times 1$ matrix B has the form

$$B = -\int_a^b x_0^T(t)\,\bar{x}_0(t)\,dt = -\int_a^b x_0^T(t)\Phi(t)Nh\,dt - \int_a^b\int_a^b x_0^T(t)G_0(t,s)\varphi(s)\,ds\,dt.$$

Since $\det D \neq 0$, relation (30) has the unique solution $u = D^{-1}B$, which we substitute in (29). By means of Lemmas 1 and 2, the general solution of (1), (2) is obtained in form (26), where the unique particular solution is $\bar{x}^*(t)$ from (28) with the coordinate representation at $t \in [\tau_{j-1}, \tau_j]$

$$\bar{x}_j^*(t) = \int_a^b G_j(t,s)\varphi(s)\,ds + H_j(t) = \int_a^{\tau_1} G_j^1(t,s)\varphi_1(s)\,ds + \ldots +$$
$$+ \int_{\tau_{j-1}}^{t} G_j^{j1}(t,s)\varphi_j(s)\,ds + \int_t^{\tau_j} G_j^{j2}(t,s)\varphi_j(s)\,ds + \ldots +$$
$$+ \int_{\tau_p}^{b} G_j^{p+1}(t,s)\varphi_{p+1}(s)\,ds + H_j(t), \tag{31}$$
$$H_j(t) = \Phi(t)\left[N_j - P_M^j D^{-1}\int_a^b x_0^T(t)\Phi(t)N\,dt\right]h.$$

3. Example. We consider the two-point boundary-value problem with one pulse effect

$$\dot{x} = \varphi(t), \quad \varphi(t) = [\varphi_1(t)\ \varphi_2(t)]^T, \quad x = [x_1\ x_2]^T, \tag{32}$$

$$[1\ \ -1]x(a) + [1\ \ -1]x(\tau_1 - 0) + [1/2\ \ 1/2]x(\tau_1 + 0) + \\ + [-1/2\ \ 1/2]x(b) = h. \tag{33}$$

Here,

$$\Phi(t) = I_2,\ \Phi^{-1}(t) = I_2,\ l_1(\Phi) = [2\ \ -2],\ l_2(\Phi) = [0\ \ 0],$$

$$M = [l_1(\Phi)\ \ l_2(\Phi)] = [2\ \ -2\ \ 0\ \ 0].$$

For half-inverse matrix of M, we choose the matrix $M^- = [1\ \ -1/2\ \ 0\ \ 0]^T$. By computing, we find $I_1 - MM^- = 0$,

$$P_M^* = I_4 - M^-M = \begin{bmatrix} -1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$P_M^1 = \begin{bmatrix} -1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},\quad P_M^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad N_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix},\quad N_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, the function $K(t, s)$ and the expression $l_1(\bar{x}_1) + l_2(\bar{x}_2)$ have the form

$$K(t, s) = \begin{cases} \frac{1}{2} I_2, & t > s, \\ -\frac{1}{2} I_2, & t < s, \end{cases}$$

$$l_1(\bar{x}_1) + l_2(\bar{x}_2) = \begin{bmatrix} -\frac{3}{2}\int\limits_{\tau_1}^{b} \varphi_{21}(s)\,ds & \frac{1}{2}\int\limits_{\tau_1}^{b} \varphi_{22}(s)\,ds \end{bmatrix}^T.$$

Since $I_1 - MM^- = 0$, condition (11) is always satisfied. By means of $x_{01}(t) = P_M^1$, $x_{02}(t) = P_M^2$ for D, we have

$$D = \begin{bmatrix} 2(\tau_1 - a) & 2(\tau_1 - a) & 0 & 0 \\ 2(\tau_1 - a) & 4(\tau_1 - a) & 0 & 0 \\ 0 & 0 & b - \tau_1 & 0 \\ 0 & 0 & 0 & b - \tau_1 \end{bmatrix}.$$

As a consequence of $\det D = 4(b - \tau_1)^2(\tau_1 - a)^2 \neq 0$, we get

$$D^{-1} = \begin{bmatrix} (\tau_1 - a)^{-1} & -\frac{1}{2}(\tau_1 - a)^{-1} & 0 & 0 \\ -\frac{1}{2}(\tau_1 - a)^{-1} & \frac{1}{2}(\tau_1 - a)^{-1} & 0 & 0 \\ 0 & 0 & (b - \tau_1)^{-1} & 0 \\ 0 & 0 & 0 & (b - \tau_1)^{-1} \end{bmatrix}.$$

In this case, for (33), by relation (17), we obtain

$$G_{01}^{11}(t,s) = \frac{1}{2}I_2, \quad G_{01}^{12}(t,s) = -\frac{1}{2}I_2, \quad G_{01}^{2}(t,s) = \frac{1}{4}\begin{bmatrix} 4 & -2 \\ -3 & -1 \end{bmatrix},$$

$$G_{02}^{1}(t,s) = \frac{1}{2}I_2, \quad G_{02}^{21}(t,s) = \frac{1}{2}I_2, \quad G_{02}^{22}(t,s) = -\frac{1}{2}I_2.$$

By applying relation (31), we get

$$H_1(t) = [2h \quad h]^T, \quad H_2(t) = [0 \quad 0]^T,$$

$$\bar{x}_1^*(t) = \left(1 - \frac{1}{\tau_1 - a}\right) \times$$

$$\times \left\{ \int_a^t \begin{bmatrix} \varphi_{11}(s) \\ -\varphi_{12}(s) \end{bmatrix} ds - \int_t^{\tau_1} \begin{bmatrix} \varphi_{11}(s) \\ \varphi_{12}(s) \end{bmatrix} ds + \frac{1}{4}\begin{bmatrix} 1 & -3 \\ -8 & 4 \end{bmatrix} \int_{\tau_1}^b \begin{bmatrix} \varphi_{21}(s) \\ \varphi_{22}(s) \end{bmatrix} ds \right\} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} h,$$

$$\bar{x}_2^*(t) = \frac{1}{2}\left(1 - \frac{1}{b - \tau_1}\right) \left\{ \int_a^{\tau_1} \begin{bmatrix} \varphi_{11}(s) \\ \varphi_{12}(s) \end{bmatrix} ds + \int_{\tau_1}^t \begin{bmatrix} \varphi_{21}(s) \\ \varphi_{22}(s) \end{bmatrix} ds - \int_t^b \begin{bmatrix} \varphi_{21}(s) \\ \varphi_{22}(s) \end{bmatrix} ds \right\}.$$

Therefore, the general solution of (32), (33) is

$$x_1(t) = \begin{bmatrix} -1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T + \bar{x}_1^*(t),$$

$$x_2(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T + \bar{x}_2^*(t).$$

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