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THE CHARACTERISTICS OF THE SUM OF A MULTIDIMENSIONAL SERIES

We study the relationship between the asymptotic behavior of coefficients of a multidimensional series of exponents and the asymptotic behavior of its sum near a point on the boundary of the domain of convergence. Growth characteristics, an order $\rho_Q(a)$, and a type $\sigma_{Q\beta}(a)$ in an octant $Q(a)$ are determined. The dependence of growth characteristics on the coordinates of points of the boundary of the domain of convergence is established.

Вивчається зв'язок асимптотичної поведінки коефіцієнтів багатовимірного ряду експонент з асимптотичною поведінкою його суми поблизу точки на межі збіжності. Визначені характеристики зростання, порядок $\rho_Q(a)$ та тип $\sigma_{Q\beta}(a)$ в октанті $Q(a)$. Встановлена залежність характеристик зростання від координат точок межі збіжності.

Introduction. In this article, the growth of the n-dimensional sum of the series

$$G(z_1, \dots, z_n) = \sum_{p=1}^{\infty} A_p \exp(\langle \lambda_p, z \rangle) \tag{1}$$

is investigated, where $A_p \in \mathbb{C}$, $\lambda_p = (\lambda_p^{(1)}, \dots, \lambda_p^{(n)})$ with positive numbers $\{\lambda_p^{(k)}\} \subset \mathbb{R}$, $k = 1, \dots, n$, and $\langle z, \lambda_p \rangle$ is a scalar product. It is known [1] that series (1) is convergent in a tubular domain $T = B + i\mathbb{R}^n \subset \mathbb{C}^n$, $B \subset \mathbb{R}^n$, where B is convex and octant-like.

Assume that $\lim\limits_{p \to +\infty} |\lambda_p|^{-1} \ln p = 0$, where $|\lambda_p| = \sum_{k=1}^{n} \lambda_p^{(k)}$. Then the domains of convergence and absolute convergence of series (1) coincide. Series (1) converges uniformly on every compact set of the domain T and the function is holomorphic in T. The boundary of the basis B is of special interest in the multidimensional case. The relationship between the asymptotic behavior of the sum of the series and the asymptotic behavior of the coefficients is studied near the boundary of the octants.

1. The order of growth in an octant. Let us take a point a on the boundary ∂B and an open octant $Q(a)$ with the vertex at a point Q so that $Q(a) = \{x \in B \mid x_k < a_k,\ k = 1, \dots, n\}$. Then $Q(a) \subset B$, and let us take an arbitrary point $x \in B \cap Q(a)$ inside $Q(a)$. The Euclidean distance between x and a is denoted by $d = d(x, a)$. We shall reach the point a in all possible ways inside $Q(a)$ if the distance $d \to 0+$. Let

$$M(G, x_1, \dots, x_n) = \sup_{y \in \mathbb{R}^n} \{|G(z_1, \dots, z_n)|\}, \quad u_k = a_k - x_k > 0,\ k = 1, \dots, n.$$

Definition 1. *The value*

$$\rho_Q(a) = \overline{\lim_{d \to 0+}} \frac{\ln \ln M(G, x_1, \dots, x_n)}{\ln \sum_{k=1}^{n} (1/u_k)}.$$

is called the order of growth in the octant $Q(a)$.

Theorem 1. *Let* $\lim\limits_{p \to +\infty} \ln^{-1} |\lambda_p| \ln \ln p = 0$ *and let* $E(a, p) = \ln |A_p| + \langle a, \lambda_p \rangle$. *Then the order of growth* $\rho_Q(a_1, \dots, a_n)$ *is given by*

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$$\overline{\lim_{p\to+\infty}} \frac{\ln^+ E(a,p)}{\ln|\lambda_p|} = \frac{\rho_Q(a)}{\rho_Q(a)+1}.$$

Proof. Let $G(z_1, \ldots, z_n)$, while achieving the point a in the octant $Q(a)$, have the order $\rho_Q(a) \in \mathbb{R}^+$. Then, for all sufficiently small $\varepsilon > 0$, $\exists\, \beta \in \mathbb{R}$, $\beta > \rho = \rho_Q(a) + \varepsilon$. Note that the functions

$$S(u_1, \ldots, u_n) = \ln \sum_{k=1}^{n} u_k^{-1} \quad \text{and} \quad t(u_1, \ldots, u_n) = \ln \left[\sum_{k=1}^{n} \frac{1}{u_k^{\beta}} \right]^{1/\beta}$$

are equivalent for $d \to 0+$. Then, according to the definition of $\rho_Q(a)$ for $\varepsilon > 0$, $\exists\, U_\varepsilon(a)$: $\forall\, x \in U_\varepsilon(a) \cap Q(a) \subset B$ and $\forall\, 0 < d < d_1(\varepsilon)$, the following inequality holds:

$$\ln M(G, x_1, \ldots, x_n) < \left[\sum_{k=1}^{n} u_k^{-\beta} \right]^{\rho/\beta}. \tag{2}$$

In addition, for all $x \in B$, the inequality

$$E(a,p) < \left[\sum_{k=1}^{n} u_k^{-\beta} \right]^{\rho/\beta} + \sum_{k=1}^{n} u_k \lambda_p^{(k)} \tag{3}$$

holds $\forall\, (x_1, \ldots, x_n) \in U_\varepsilon(a) \cap Q(a)$ and $\forall\, 0 < d < d_1$.

Consider a point $u^*(u_1^*, \ldots, u_n^*)$ with the coordinates

$$u_k^* < \rho^{1/(\rho+1)} \left[\lambda_p^{(k)}\right]^{-1/(\rho+1)} \left\{ \sum_{k=1}^{n} \left[\lambda_p^{(k)}\right]^{\beta/(\beta+1)} \right\}^{(\rho-\beta)/\beta(\rho+1)}, \quad k = 1, \ldots, n,$$

where $\beta - \rho > 0$. It is easy to see that, for $p \to +\infty$, $u_k^* \to 0+$, $k = 1, \ldots, n$. Therefore, the coordinates of the point u^* satisfy (3)

$$E(a,p) = \rho^{1/(\rho+1)} (\rho^{-1} + 1) \left\{ \sum_{k=1}^{n} \left[\lambda_p^{(k)}\right]^{\beta/(\beta+1)} \right\}^{\rho(1+\beta)/\beta(1+\rho)}.$$

Hence,

$$\overline{\lim_{p\to+\infty}} \frac{\ln^+ E(a,p)}{\ln\left\{\sum_{k=1}^{n} \left[\lambda_p^{(k)}\right]^{\beta/(\beta+1)}\right\}^{(\beta+1)/\beta}} \le \frac{\rho}{\rho+1}. \tag{4}$$

It is easy to see that the values

$$\ln \left\{ \sum_{k=1}^{n} \left[\lambda_p^{(k)}\right]^{\beta/(\beta+1)} \right\}^{(\beta+1)/\beta}$$

and $\ln|\lambda_p|$ are equivalent for $p \to +\infty$; thus, (4) turns into

$$\frac{\gamma}{\gamma+1} = \overline{\lim_{p\to+\infty}} \frac{\ln^+ E(a,p)}{\ln|\lambda_p|} \le \frac{\rho_Q(a)+\varepsilon}{\rho_Q(a)+\varepsilon+1}, \quad \varepsilon \to 0+ \Rightarrow \gamma \le \rho_Q(a).$$

Let us prove that $\gamma \ge \rho_Q(a)$. Assume that $(\gamma + \varepsilon + 1)^{-1}(\gamma + \varepsilon) = t > 0$ and $\forall\, \varepsilon > 0$, $\exists\, p_1 = p_1(\varepsilon) \in \mathbb{N}$: $\forall\, p \ge p_1$. We use the equivalence between $\ln|\lambda_p|$ and $\ln T^{(\beta+1)/\beta}$, where $T = \sum_{k=1}^{n} \left[\lambda_p^{(k)}\right]^{\beta/(1+\beta)}$. Then $|A_p| < \exp\{T^{t(\beta+1)/\beta}\} \times e^{-\langle a, \lambda_p\rangle}$ or $\forall\, p$

and $k = 1, \dots, n$, $j = 1, \dots, n$, $k \neq j$. Then

$$\Delta_n = \frac{(2t)^n}{(\beta+1)^n} [t(\beta+1) - \beta]^n \Delta \prod_{k=1}^{n} r_k^{-2/(\beta+1)} T_1^{nt(\beta+1)/2-2n},$$

where

$$\Delta = \begin{vmatrix} \left[1 - T_1/\omega r_1^{\beta/(\beta+1)}\right] & \dots & \dots & 1 \\ 1 & \left[1 - T_1/\omega r_2^{\beta/(\beta+1)}\right] & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & \left[1 - T_1/\omega r_h^{\beta/(\beta+1)}\right] \end{vmatrix},$$

$\omega = t(\beta+1) - \beta$, consists of the units off the main diagonal. Finally, we get

$$\Delta_n = (-1)^n (2t)^n (\beta+1)^{-n+1} T_1^{n[t(\beta+1)/2-2]} (1-t) \times$$
$$\times \prod_{k=1}^{n} \left(r_k^{-2/(\beta+1)} u_k^{\beta} \right) \left[\sum_{k=1}^{n} u_k^{-\beta} \right]^n.$$

Thus, the determinant Δ_n is alternating with respect to n. It is easy to see that, at the stationary point $r^{(0)}$, we have $R(h) < 0$. We must show that the value $\xi(r^{(0)})$ is the largest in the closed octant $\overline{R}(0) = \{ (r_1, \dots, r_n) | \ r_k \geq 0, \ k = 1, \dots, n \}$. Let $r_n = 0$. Since $\xi(r)$ has a local maximum at the point $r^{*(0)}$, the function

$$\xi_1(r_1, \dots, r_{n-1}) = 2 \left[\sum_{k=1}^{n-1} r_k^{\beta/(\beta+1)} \right]^{t(\beta+1)/\beta} - \sum_{k=1}^{n-1} r_k u_k$$

has a local maximum at the point $\left(r_1^{*(0)}, \dots, r_{n-1}^{*(0)} \right)$:

$$\xi_1\left(r_1^{*(0)}, \dots, r_{n-1}^{*(0)} \right) = \left[\sum_{k=1}^{n-1} u_k^{-\beta} \right]^{(\gamma+\varepsilon)/\beta} \left(\frac{2\gamma + 2\varepsilon}{\gamma + \varepsilon + 1} \right)^{\gamma+\varepsilon+1} \frac{1}{\gamma+\varepsilon}$$

and, since $\xi(\underbrace{0, \dots, 0}_{n}) = 0$, the value of $\xi_1(r_1^{*(0)}, \dots, r_{n-1}^{*(0)})$ is the largest at the boundary $\partial R(0)$ in view of the fact that

$$u_1^{-\beta} \leq u_1^{-\beta} + u_2^{-\beta} \leq \dots \leq \sum_{k=1}^{n-1} u_k^{-\beta}$$

under the condition $u_1 \leq u_2 \leq \dots \leq u_n$, which does not spoil the unity of proof.

Thus,

$$\sum_{k=1}^{n-1} u_k^{-\beta} \leq \sum_{k=1}^{n} u_k^{-\beta},$$

and we conclude that the inequality $\xi_1(r_1^{*(0)}, \dots, r_{n-1}^{*(0)}) < \xi(r_1^{(0)}, \dots, r_n^{(0)})$ implies that, at the point $r^{(0)}(r_1^{(0)}, \dots, r_n^{(0)})$, the function $\xi(r_1, \dots, r_n)$ reaches the absolute maximum in $\overline{R}(0)$, and, by taking (7) into account, we get

$$M(G, x_1, \dots, x_n) < B(\varepsilon) \exp\left\{ \max_{\overline{R}(0)} \xi(r_1, \dots, r_n) \right\} \sum_{p}^{+\infty} \exp[-T^{t(\beta+1)/\beta}]$$

or

$$M(G, x_1, \dots, x_n) <$$
$$< \frac{B(\varepsilon)}{\gamma+\varepsilon}\left[\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}\right]^{\gamma+\varepsilon+1}\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{(\gamma+\varepsilon)/\beta}\sum_{p}^{\infty}\exp\left[-T^{t(\beta+1)/\beta}\right]. \tag{8}$$

The series on the right-hand side of (8) converges if $\lim_{p\to+\infty} \ln^{-1}|\lambda_p|\ln\ln p = 0$. Indeed, since $\ln|\lambda_p|$ and $\ln T$ are equivalent for $p \to +\infty$, this implies that $\forall\, \varepsilon > 0,\ \forall\, p \ge p_1(\varepsilon)\ [\ln p]^{t(\beta+1)/\beta} < T^{t(\beta+1)/\beta}$.

If the number $\varepsilon > 0$ is taken so that $t(\beta+1)/\beta > 2\varepsilon$, then $\forall\, p \ge p_2(\varepsilon) > > p_1$,

$$\sum_{p \ge p_2}^{+\infty} \exp\left(-T^{t(\beta+1)/\beta}\right) < \sum_{p \ge p_2}^{+\infty} p^{-\ln p} < +\infty, \quad \sum_{p}^{+\infty} \exp\left(-T^{t(\beta+1)/\beta}\right) < K \in \mathbb{R}$$

and (8) transforms into

$$M(G, x_1, \dots, x_n) < A(\varepsilon)\exp\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{(\gamma+2\varepsilon)/\beta}$$

$\forall\, 0 < d < d_1(\varepsilon)$, where $A(\varepsilon) = KB(\varepsilon)$.

For $\forall\, 0 < d < d' < d_1(\varepsilon)$, the following inequality holds:

$$M(G, x_1, \dots, x_n) < \exp\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{(\gamma+3\varepsilon)/\beta}$$

or

$$\varlimsup_{d\to 0+} \frac{\ln\ln M(G, x_1, \dots, x_n)}{\ln\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{1/\beta}} \le \gamma + 3\varepsilon.$$

Since the functions $s(u_1, \dots, u_n)$ and $t(u_1, \dots, u_n)$ are equivalent for $d \to 0+$, we have

$$\rho_Q(a_1, \dots, a_n) = \varlimsup_{d\to 0+} \frac{\ln\ln M(G, x_1, \dots, x_n)}{\ln\sum_{k=1}^{n} 1/u_k} \le$$
$$\le \gamma + 3\varepsilon, \quad \varepsilon \to 0+ \Rightarrow \rho_Q(a) \le \gamma$$

and the theorem is proved for $\rho_Q(a) \in [0, +\infty)$.

2. The type in the octant $Q(a)$.

Definition 2. *The value*

$$\sigma_{Q\beta}(a_1, \dots, a_n) = \varlimsup_{d\to 0+} \frac{\ln M(G, x_1, \dots, x_n)}{\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{\rho_Q(a)/\beta}} \tag{9}$$

is called the type in the octant $Q(a)$ for $\rho_Q(a) \in (0, +\infty)$, where $\rho_Q(a) < \beta < < +\infty$.

Theorem 2. *Let* $\lim_{p\to+\infty} \ln^{-1}|\lambda_p|\ln\ln p = 0$. *Then the type of growth $\sigma_{Q\beta}(a_1, \dots, a_n)$ can be calculated as follows:*

$$\sigma_{Q\beta}(a_1, \dots, a_n) =$$

$$= \frac{[\rho_Q(a)]^{\rho_Q(a)}}{[\rho_Q(a)+1]^{\rho_Q(a)+1}} \varlimsup_{p\to+\infty} \frac{[E^+(a,p)]^{\rho_Q(a)+1}}{\left\{\sum_{k=1}^n [\lambda_p^{(k)}]^{\beta/(\beta+1)}\right\}^{\rho_Q(a)(\beta+1)/\beta}}.$$

Proof. Let the function $G(z_1, \ldots, z_n)$ have the type $\sigma_{Q\beta}(a) \in \mathbb{R}^+$ for $\rho_Q(a) \in (0, +\infty)$. Then, from Definition 2 and the Cauchy inequality, we find that $\forall\, \varepsilon > 0,\ \forall\, d < d_1(\varepsilon)$

$$E(a,p) < (\sigma_{Q\beta}(a) + \varepsilon)\left[\sum_{k=1}^n u_k^{-\beta}\right]^{\rho_Q(a)/\beta} + \langle u, \lambda_p\rangle, \tag{10}$$

by setting in the inequality (10) the point coordinates $u^{(0)}\left(u_1^{(0)}, \ldots, u_n^{(0)}\right)$,

$$u_k^{(0)} < \frac{[\sigma\rho_Q(a)]^{1/\rho_Q(a)+1}}{[\lambda_p^{(k)}]^{1/(\beta+1)}}\, T_2^{(\rho_Q(a)-\beta)/(\beta(\rho_Q(a)+1))} > 0, \quad k = 1, \ldots, n,$$

where

$$T_2 = [\rho_Q(a)\sigma]^{\beta/(\beta+1)}\left[\sum_{k=1}^n u_k^{-\beta}\right]^{(\rho_Q(a)+1)/(\beta+1)}, \quad \sigma = \sigma_{Q\beta}(a) + \varepsilon.$$

Note that $\forall\, p \geq p_2(\varepsilon)$ $u^{(0)} \in U_\varepsilon(a) \cap Q(a)$. We have

$$\frac{[E^+(a,p)]^{\rho_Q(a)+1}}{T_2^{\rho_Q(a)(\beta+1)/\beta}} < \sigma\,\rho_Q(a)\,[\rho_Q^{-1}(a) + 1]^{\rho_Q(a)+1}.$$

By passing to the limit as $p \to +\infty$ and then as $\varepsilon \to 0+$, we see that

$$\varlimsup_{p\to+\infty} \frac{[E^+(a,p)]^{\rho_Q(a)+1}}{\left\{\sum_{k=1}^n [\lambda_p^{(k)}]^{\beta/(\beta+1)}\right\}^{\rho_Q(a)(\beta+1)/\beta}} \leq$$

$$\leq \sigma_{Q\beta}(a)\,\rho_Q(a)\,[\rho_Q^{-1}(a) + 1]^{\rho_Q(a)+1},$$

$$\gamma_0\,\rho_Q(a)\left[1 + \frac{1}{\rho_Q(a)}\right]^{\rho_Q(a)+1} = \varlimsup_{p\to+\infty} \frac{[E^+(a,p)]^{\rho_Q(a)+1}}{\left\{\sum_{k=1}^n [\lambda_p^{(k)}]^{\beta/(\beta+1)}\right\}^{\rho_Q(a)(\beta+1)/\beta}} \leq$$

$$\leq \sigma_{Q\beta}(a)\,\rho_Q(a)\left[1 + \frac{1}{\rho_Q(a)}\right]^{\rho_Q(a)+1}.$$

Therefore, $\gamma_0 \leq \sigma_{Q\beta}(a)$. Let us prove that $\gamma_0 \geq \sigma_{Q\beta}(a)$. Assume that

$$\gamma_0 + \varepsilon = m_0 > 0, \quad 1 > \frac{\rho_Q(a)}{\rho_Q(a)+1} = t > 0, \quad t^{-1}\, m_0^{1/(\rho_Q(a)+1)} = A.$$

Hence, for any $\varepsilon > 0$ and any p, the following inequality holds:

$$|A_p| < A(\varepsilon)\exp\left\{A\left[\sum_{k=1}^n [\lambda_p^{(k)}]^{\beta/(\beta+1)}\right]^{t(\beta+1)/\beta}\right\} - \langle a, \lambda_p\rangle,$$

where $A(\varepsilon)$ is a constant for fixed $\varepsilon > 0$. Hence,

$$|G(z_1, \ldots, z_n)| < A(\varepsilon) \sum_{p}^{+\infty} \exp\{2A\,T^{t(\beta+1)/\beta} - \langle u, \lambda_p \rangle\}\, e^{-AT^{t(\beta+1)/\beta}}. \quad (11)$$

Similarly, we can show (as in the case of Theorem 1) that the function

$$\eta(v_1, \ldots, v_n) = 2A\left[\sum_{k=1}^{n} v_k^{\beta/(\beta+1)}\right]^{t(\beta+1)/\beta} - \sum_{k=1}^{n} u_k v_k,$$

where $v_k \geq 0$, $k = 1, \ldots, n$, has an absolute maximum at the unique stationary point $v^{(0)}$,

$$\eta\left(v_1^{(0)}, \ldots, v_n^{(0)}\right) < \left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{\rho_Q(a)/\beta} [\sigma_{Q\beta}(a) + \varepsilon]\, 2^{\rho_Q(a)+1}.$$

Thus, (11) can be written as

$$M(G, x_1, \ldots, x_n) < A(\varepsilon)\exp\left\{\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{\rho_Q(a)/\beta} \sigma\, 2^{\rho_Q(a)+1}\right\} \times$$

$$\times \sum_{p}^{\infty} \exp\left\{-A\left[\sum_{k=1}^{n} [\lambda_p^{(k)}]^{\beta/(\beta+1)}\right]^{t(\beta+1)/\beta}\right\}. \quad (12)$$

Under the condition that $\lim\limits_{p \to +\infty} \ln\ln p \ln^{-1}|\lambda_p| = 0$, the series on the right-hand side of (12) converges and $\forall\ 0 < d^{(1)} < d''$

$$M(G, x_1, \ldots, x_n) < \exp\left\{(\sigma + \varepsilon)\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{\rho_Q(a)/\beta}\right\},$$

$$\frac{\ln M(G, x_1, \ldots, x_n)}{\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{\rho_Q(a)/\beta}} < \gamma_0 + 2\varepsilon,$$

or

$$\sigma_{Q\beta}(a_1, \ldots, a_n) = \overline{\lim_{d \to 0+}} \frac{\ln M(G, x_1, \ldots, x_n)}{\left[\sum_{k=1}^{n} u_k^{-\beta}\right]^{\rho_Q(a)/\beta}} \leq$$

$$\leq \gamma_0 + 2\varepsilon, \quad \varepsilon \to 0+ \Rightarrow \sigma_{Q\beta}(a) \leq \gamma_0.$$

The theorem is proved for $0 \leq \sigma_{Q\beta}(a) < +\infty$. Obviously, Theorem 2 is also true for $\sigma_{Q\beta}(a) = +\infty$.

1. *Gromov V. P.* About the theory of repeating Dirichlet series. I // Scien. papers of Acad. of Sciences of Armenian. Mathematics. – 1970. – № 5. – P. 449–458.

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