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STORY OF RATIONAL APPROXIMATION FOR THE CLASS OF STIELTJES FUNCTIONS: FROM STIELTJES TO RECENT OPTIMAL ESTIMATIONS OF ERRORS

ПРО РАЦІОНАЛЬНІ АПРОКСИМАЦІЇ ДЛЯ КЛАСІВ СТІЛТЬЄСІВСЬКИХ ФУНКЦІЙ: ВІД СТІЛТЬЄСА ДО НОВИХ ОПТИМАЛЬНИХ ОЦІНОК ПОХИБОК

The history of obtaining inequalities for Padé approximant error in the Stieltjes case is given. Inequalities optimal with respect to the order are obtained for these approximants by using results of J. Vinuesa and A. P. Magnus.

Наведено історію встановлення нерівностей для похибок апроксимацій Паде у стілтьєсівському випадку. Доведено оптимальні за порядком нерівності для цих апроксимацій з використанням результатів Вінуеса та Магнуса.

We follow the story of bounds for Padé approximant errors in the Stieltjes case. Classical inequalities are due to Stieltjes himself. In 1979, the author remarked the existence of an optimal form (with respect to the order) of inequalities between the Padé approximant errors. The contributions of J. Vinuesa and A. P. Magnus allowed us to prove completely this result in 1992.

In his 1894's papers, Stieltjes proved that the following continued fraction

$$\frac{a_1}{1} - \frac{a_2 z}{1} - \frac{a_3 z}{1} - \dots, \tag{1}$$

where the coefficients satisfy some conditions, converges to the analytic function

$$f(z) = \int_0^{1/R} \frac{d\mu(x)}{1 - xz} \tag{2}$$

in the whole complex plane, except the cut on the real positive axis. The term "continued fraction" must be called "Padé approximants". Stieltjes died in 1894 in Toulouse. Fractions (1) and functions (2) became "Stieltjes Fractions" and "Stieltjes functions", respectively. One of the partial results of Stieltjes work concerns the inequalities between the approximants of (1). These inequalities will be later expressed in terms of "Padé" terminology. But it is necessary to remark that this type of inequality was probably considered in Stieltjes time as a secondary result. However, they presently play a crucial role in numerical analysis and error estimations. This was our motivation to take again the analysis of these inequalities and to ask for their optimal form.

Historical comment. Henri Padé's work, even in the time of Stieltjes, was related to the structure of a set (table) of approximants of continued fractions. In the 1930's, Van Vleck first gave the names "Padé approximant" and "Padé table" to these objects, but, probably, Padé himself never heard of it (Padé died in 1953). It is curious to state that Padé did not appreciate Stieltjes' work. In fact, he thought that the convergence of the continued fractions was the consequence of his incoherent theory of holoid fractions, and reproached Stieltjes the fact that he did not use his "theory". Probably because of Hermite's disapproval (Padé began as a student of Hermite), Padé definitely put an end to his research work. His last paper appeared in 1907. He died fifty years later.

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The Padé approximants reappear in 1948 in Wall's book [1]. In the 1950's, theoretical physicists rediscovered Padé approximants once and for all and now this domain is greatly developed.

Let us recall [2] that a rational function P_m/Q_n denoted briefly by $[m/n]$, where P_m and Q_n are the polynomials of respective degrees not greater than m and n, is called the Padé approximant to the function f if it satisfies the following relation:

$$f(z) - [m/n](z) = O(z^{m+n+1}). \tag{3}$$

This means that the error is of the order $(m+n+1)$. On any antidiagonal $(m+n = = \text{const})$ of the Padé table, all errors are of the same order. Conversely, for instance, following a diagonal, the order of errors changes. In this case and the case of non-rational Stieltjes functions, the classical inequality for the Padé approximant errors

$$0 < f(x) - [n/n+1](x) < f(x) - [n-1/n](x) \quad \forall x \in \left]0, R\right[\tag{4}$$

is no longer balanced with respect to the order of two sides. It follows that if x tends to zero, the left-hand side tends to zero more rapidly than the right-hand side, and, consequently, this inequality is, in this sense, trivial. Indeed, our idea was to balance all inequalities with respect to the series order.

Let me give now the final complete result obtained last year [3].

Theorem 1. *Let* $[m/n] = P_{m,n}/Q_{m,n}$ *be the Padé approximant to the non-rational Stieltjes function* f *defined by* (2). *Then the following inequalities occur:*

$$\forall m, n \geq 0, \quad \forall x \in \left]0, R\right[:$$

$$0 < f(x) - [m+1/n](x) < \alpha_{m,n}(x) \frac{x}{R} \{f(x) - [m/n](x)\}, \tag{5}$$

where

$$0 < \alpha_{m,n}(x) = \frac{Q_{m,n}(x)\, Q_{m+1,n}(R)}{Q_{m,n}(R)\, Q_{m+1,n}(x)} \leq 1 \tag{6}$$

and

$$0 < f(x) - [m/n+1](x) < \beta_{m,n}(x) \frac{x}{R} \{f(x) - [m/n](x)\}, \tag{7}$$

where

$$0 < \beta_{m,n}(x) = \frac{Q_{m,n}(x)\, Q_{m,n+1}(R)}{Q_{m,n}(R)\, Q_{m,n+1}(x)} \leq 1. \tag{8}$$

The introduction of our paper [4] for the Boulder Conference in 1988 explains very well the beginning of the problem of the order balanced inequalities:

"For the first time, ten years ago, the inequalities in question were quietly used by the authors to prove the existence of valleys in the c-table ([5], Formulas (29) and (31)). It was so natural for us to consider that everything about the Padé approximants to the Stieltjes functions coming from Stieltjes' work was well known. However, in the literature [6, 7] we have not been able to find this.

Having discovered the above accident we proved the valley property in another way (not published) and two conjectured inequalities became "open problem". Today we can give a complete proof of these inequalities."

It was wrong. In fact, in this time, we proved the above inequalities without the α's and β's coefficients and only for the lower half of the Padé table. For this proof, we used two well-known lemmas.

Lemma 1 [8]. *Let* f *be the Stieltjes function* f *defined by* (2) *and let* $S_{n,k}(t, x)$ *be an orthogonal polynomial of degree* n *in* t *defined by*

$$S_{n.k}(t, x) = \overline{Q}_{n+k,\,n}(t) / \overline{Q}_{n+k,\,n}(1/x),$$

where $\overline{Q}_{n+k,\,n}(t) = t^n Q_{n+k,\,n}(1/t)$. *Then the Padé approximant error is given by*

$$k \geq 1;\; n \geq \max(0, -k):\quad f(x) - [m + k/n](x) =$$

$$= x^{k+1} \int_0^{1/R} [S_{n.k}(t, x)]^2 \frac{t^{k+1}}{1 - xz} d\mu(x).$$

Lemma 2 [9, p. 25]. *In the set of all polynomials* $Q_n(t)$ *of degree* n *such that* $Q_n(1/x) = 1$, *the polynomial* $S_{n.k}(t, x)$ *minimizes the integral*

$$\int_0^{1/R} [Q_n(t)]^2 \frac{t^{k+1}}{1 - xz} d\mu(x)$$

for all x *belonging to* $]0, R[$.

J. Vinuesa and the author [10] tried (1990, Erice Conference) to extend the above inequalities to the upper half of the Padé table using the well-known property:

If f is a Stieltjes function (we put $f(0) = 1$), then in the inverse

$$f(z) = \frac{1}{1 - zg(z)}, \tag{9}$$

g is also a Stieltjes function and

$$n \geq m:\quad [m/n]_f(z) = \frac{1}{1 - z[n-1/m]_g(z)}. \tag{10}$$

Unfortunately, our coefficients α's and β's in inequalities like (5) and (7) obtained in this way were not optimal and cannot be bounded by 1 as in (6) and (8), respectively.

Finally, a good way to prove the optimal inequalities was found in [11] (1991, Granada Conference). It is based on iterates of formulas (9) and the analysis of terminated continued C-fractions. Moreover, in [11], we obtained rigorous equalities between contiguous Padé approximant errors in the whole Padé table and in the whole complex plane. Theorem 1 appears as an approximation of these equalities.

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