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ON THE COORDINATED APPROXIMATION METHOD FOR NONLINEAR ILL-POSED PROBLEMS

A generalization of the method of coordinated approximation suggested by Yu. Gaponenko [1] for the space $L_2(0, 1)$ is developed for abstract Hilbert spaces. In particular, it is shown that, for $L_2(0, 1)$, some assumptions concerning an exact solution can be weaken.

Наведено узагальнення на абстрактний простір Гільберта узгодженої апроксимації, запропонованої Ю. Л. Гапоненком для простору $L_2(0, 1)$. Зокрема, показано, що для $L_2(0, 1)$ деякі умови відносно точного розв'язку можуть бути послаблені.

1. Introduction. Consider the operator equation

$$A(u) = f, \tag{1}$$

where A is a possibly nonlinear operator over a real Hilbert space H. Suppose that, for an exact right-hand member f, problem (1) has an exact solution u^*, satisfying the a priori estimate

$$\| u^* \| \le R. \tag{2}$$

If the right-hand member $f_\delta \in H$ differs from f at most by δ, i.e., $\|f_\delta - f\| \le \delta$, then we are interested in a regularization method for finding approximate solutions $u_\delta = R_\delta(f_\delta)$ such that the rate of convergence $u_\delta \to u^*$ $(\delta \to 0)$ can be effectively estimated.

The coordinated approximation method. For the case where $H = L_2[0, 1]$, Gaponenko [1] suggested the method of coordinated approximation under the hypothesis that the exact solution $u^*(t)$ satisfies an a priori estimate

$$|u^*(t)| \le R \qquad \forall\, t \in [0, 1]. \tag{3}$$

Gaponenko's constructive method uses many special properties of $L_2[0, 1]$ and is complicated enough, so even its extension to the multidimensional case $L_2(G)$, where $G \subset \mathbb{R}^n$, is rather difficult. Therefore, it may be interesting to describe a general scheme of the method of coordinated approximation in abstract Hilbert spaces. Besides, it will be shown that, when $H = L_2[0, 1]$, Gaponenko's hypothesis (3) can be replaced by the weaker assumption (2). Moreover, when $u^*(t)$ is smooth enough, the convergence of regularized solutions can be improved.

2. Weak convergence in Hilbert spaces. Let (X_1, X_0, H) be a triple of spaces, where $(X_1, \|\cdot\|_1)$, $(X_0, \|\cdot\|_0)$ are real separable Banach spaces and H is a real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|x\| = \langle x, x\rangle^{1/2}$. Further, suppose that X_1 is densely, continuously, and compactly imbedded in X_0, and X_0 is densely and continuously embedded in H, i.e.,

i) $\forall\, x \in X_1 \Rightarrow x \in X_0$, $\|x\|_0 \le C_1 \|x\|_1$; $\forall\, x \in X_0 \Rightarrow x \in H$, $\|x\| \le C_0 \|x\|_0$;

ii) $$\forall\, x \in X_0 \ \exists\, \{x_n\} \subset X_1 : \ \|x_n - x\|_0 \to 0, \ n \to \infty;$$

$$\forall\, w \in H \ \exists\, \{w_n\} \subset X_0 : \ \|w_n - w\|_0 \to 0, \ n \to \infty;$$

iii) every set bonded in the norm of X_1 is relatively compact in X_0.

We introduce in H a variational norm

$$\|x\|_V = \sup\{\langle x, v\rangle:\ v \in X_1;\ \|v\|_1 \le 1\},\quad x \in H. \tag{4}$$

The following theorem shows that the weak convergence in H can be described by means of the variational norm (4):

Theorem 1. *A sequence $\{x_n\} \subset H$ converges weakly to $x \in H$, $x_n \to x$, if and only if the following two conditions hold:*

$$\exists\, R > 0:\ \|x_n\|,\ \|x\| \le R,\quad \forall\, n \ge 1, \tag{5}$$

$$\|x_n - x\|_V \to 0,\quad n \to \infty. \tag{6}$$

Proof. 1) Suppose that $(x_n \to x)$, $n \to \infty$. Then the sequence $\{x_n\}$ is bounded by a constant R. Since $\|x\| \le \underline{\lim}\,\|x_n\| \le R$, condition (5) is satisfied. By definition (4), we can choose $\{v_n\} \subset S_1 = \{x \in X_1:\ \|x\|_1 \le 1\}$ and $\varepsilon_n \to +0$ such that

$$\|x_n - x\|_V = \sup_{v \in S_1}\langle x_n - x, v\rangle \le \langle x_n - x, v_n\rangle + \varepsilon_n. \tag{7}$$

Since $|\langle x_n - x, v_n\rangle| \le \|x_n - x\|\|v_n\| \le 2RC_0C_1\|v_n\|_1 \le 2RC_0C_1$, the sequence $\{\langle x_n - x, v_n\rangle\}$ is compact. Let l be an arbitrary limit point of $\langle x_n - x, v_n\rangle$. Then there exists a subsequence $\langle x_{n_k} - x, v_{n_k}\rangle$ convergent to l. Since $S_1 \subset X_0$ is compact, we can select a convergent subsequence $v_{n_{k'}} \to v \in X_0$. Taking into account that $\langle x_{n_{k'}} - x, v\rangle \to 0$ because $x_{n_{k'}} \to x$ and

$$\langle x_{n_{k'}} - x, v_{n_{k'}} - v\rangle \le \|x_{n_{k'}} - x\|\|v_{n_{k'}} - v\| \le 2RC_0\|v_{n_{k'}} - v\|_0 \to 0,$$

we get $\langle x_{n_{k'}} - x, v_{n_{k'}}\rangle = \langle x_{n_{k'}} - x, v\rangle + \langle x_{n_{k'}} - x, v_{n_{k'}} - v\rangle \to 0$. Thus, we have

$$l = \lim_k \langle x_{n_k} - x, v_{n_k}\rangle = \lim_{k'} \langle x_{n_{k'}} - x, v_{n_{k'}}\rangle = 0.$$

Since l is an arbitrary limit point of $\langle x_n - x, v_n\rangle$, it follows that $\langle x_n - x, v_n\rangle \to 0$ and (7) implies (6).

2) It is evident that X_1 is dense in H; hence, $\forall\, v \in H\ \exists\ \tilde v \in X_1\ (\|\tilde v\|_1 = r)$: $\|v - \tilde v\| \le \delta(r)$, where $\delta(r) \to 0$, $r \to +\infty$. Suppose that $\|x_n - x\|_V \le \varepsilon_n$, where ε_n is a sequence decreasing monotonically to zero. For any fixed $v \in H$, we choose $v_n \in X_1$ such that $\|v_n\|_1 = \varepsilon_n^{-1/2}$ and $\|v_n - v\|_V \le \delta(\varepsilon_n^{-1/2})$. It follows from

$$\langle x_n - x, v\rangle = \langle x_n - x, v_n\rangle + \langle x_n - x, v - v_n\rangle \le$$

$$\le \varepsilon_n^{-1/2}\langle x_n - x, \varepsilon_n^{1/2} v_n\rangle + 2R\delta(\varepsilon_n^{-1/2}) \le$$

$$\le \varepsilon_n^{-1/2}\|x_n - x\|_V + 2R\delta(\varepsilon_n^{-1/2}) \le \varepsilon_n^{1/2} + 2R\delta(\varepsilon_n^{-1/2}) \to 0,\quad n \to \infty,$$

that $x_n \to x$. Theorem 1 is proved.

3. Finite dimensional approximations. Suppose that X_0 and H possess a common basis $\{e_i\}_1^\infty \subset X_0$. Without loss of generality, we can assume that $\langle e_i, e_j\rangle = \delta_{ij}$, $i, j \ge 1$. Consider a bounded linear projection $P_N:\ H \to X_0$ defined by

$$P_N x = \sum_{i=1}^{N} \langle x, e_i \rangle e_i.$$

Since $\{e_i\}$ is a common basis of X_0 and H, we have

$$\forall\, x \in H \quad \|(I - P_N)x\| \to 0, \quad N \to \infty, \tag{8}$$

$$\forall\, x \in X_0 \quad \|(I - P_N)x\|_0 \to 0, \quad N \to \infty, \tag{9}$$

where I is the identity operator. Further,

$$\|P_N x\|_0 \le \sum_{i=1}^{N} |\langle x, e_i \rangle| \, \|e_i\|_0 \le \left(\sum_{i=1}^{N} \|e_i\|_0 \right) \|x\| \equiv \tilde{C}_N \|x\|.$$

Thus,

$$\|P_N x\|_0 \le \tilde{C}_N \|x\|. \tag{10}$$

Theorem 2. *For any* $x \in H$, *the following estimate holds:*

$$\|(I - P_N)x\|_V \le C_0 \varepsilon_N \|x\|, \tag{11}$$

where $\varepsilon_N \to 0$, $N \to \infty$.

Proof. Since

$$|\langle x - P_N x, v \rangle| = |\langle x - P_N x, P_N v + v - P_N v \rangle| = |\langle x, v - P_N v \rangle| \le$$

$$\le \|x\| \, \|(I - P_N)v\| \le C_0 \|x\| \, \|(I - P_N)v\|_0$$

for every $x \in H$ and $v \in S_1 = \{ v \in X_1 : \|v\|_1 \le 1 \}$, we have $\|(I - P_N)x\|_V \le$ $\le C_0 \varepsilon_N \|x\|$, where

$$\varepsilon_N = \sup_{v \in S_1} \|(I - P_N)v\|_0.$$

Since $\|(I - P_N)v\|_0 \to 0$ for every $v \in X_0$, we have $\|(I - P_N)\|_{X_0 \to X_0} \le C$ by the Banach–Steinhauss theorem. Further, because $S_1 \subset X_0$ is compact, for any $\varepsilon > 0$, there exists an $\varepsilon(2C)^{-1}$-finite net of S_1, denote it by $\{v_1, v_2, v_m\}$, such that $\forall\, v \in S_1 \ \exists\, v_i : \|v - v_i\|_0 \le \varepsilon(2C)^{-1}$. On the other hand, there exists $N = N_0(\varepsilon)$ such that

$$\forall\, N \ge N_0 \quad \|(I - P_N)v_i\|_0 < \frac{\varepsilon}{2}, \quad i = 1, 2, \ldots, m.$$

By combining the last two inequalities, we have

$$\forall\, v \in S_1, \ \forall\, N \ge N_0 \quad \|(I - P_N)v\|_0 \le$$

$$\le \|(I - P_N)v_i\|_0 + \|(I - P_N)(v - v_i)\|_0 \le \frac{\varepsilon}{2} + C_{\varepsilon/2C} = \varepsilon.$$

Thus, $\varepsilon_N \to 0$ as $N \to \infty$. Theorem 2 is proved.

In what follows, we suppose that $\{e_i\}_1^\infty \subset X_1$. Then

$$\| P_N x \|_0 \le \sum_{i=1}^{N} |\langle x, e_i\rangle| \, \| e_i \|_0 = \sum_{i=1}^{N} \| e_i \|_0 \| e_i \|_1 \left| \left\langle x, e_i / \| e_i \|_1 \right\rangle \right| \le$$

$$\le \left(\sum_{i=1}^{N} \| e_i \|_0 \| e_i \|_1 \right) \| x \|_V \equiv C_N \| x \|_V.$$

Thus, we get the important inequality

$$\forall x \in H \quad \| P_N x \|_0 \le C_N \| x \|_V. \tag{12}$$

Let $S = \{ x \in H : \|x\| \le R \}$. Since S is a bounded closed set in the Hilbert space H, it is weakly compact. By Theorem 1, $(S, \|\cdot\|_V)$ is a compact metric space.

By setting

$$S_N = P_N(S) = \left\{ x_N = \sum_{i=1}^{N} \langle x, e_i \rangle e_i : x \in S \right\},$$

and

$$S_N^{(L)} = \left\{ x = \sum_{i=1}^{N} c_i e_i, \|x\| \le R;\ c_i \in \left\{ 0, \pm \frac{R}{NL}, \ldots, \pm \frac{(NL-1)R}{NL}, \pm R \right\} \right\},$$

we obtain the following result:

Lemma 1. *$S_N^{(L)}$ is a $C_0 R(\varepsilon_N + C_1 L^{-1})$-finite net for the compact set $(S, \|\cdot\|_V)$.*

Proof. First, we note that, by Theorem 2, $\| x - x_N \|_V = \| x - P_N x \|_V \le C_0 R \varepsilon_N$. Further, since $|\langle x, e_i \rangle| \le \| x \| \le R$, we can choose $C_i^{(L)}$ from the finite set $\{0, \pm R/NL, \ldots, \pm R\}$ such that $| C_i^{(L)} - \langle x, e_i \rangle | \le R/NL$ and $| C_i^{(L)} | < |\langle x, e_i \rangle|$, $i = 1, 2, \ldots, N$. Now let

$$x_N^{(L)} = \sum_{i=1}^{N} C_i^{(L)} e_i.$$

A simple calculation shows that $\| x_N^{(L)} \| \le R$, we have

$$\| x_N^{(L)} - x_N \|_V = \left\| \sum_{i=1}^{N} \left(C_i^{(L)} - \langle x, e_i \rangle \right) e_i \right\|_V \le$$

$$\le \sum_{i=1}^{N} | C_i^{(L)} - \langle x, e_i \rangle | \, \| e_i \|_V \le \frac{R}{NL} \sum_{i=1}^{N} \| e_i \|_V.$$

Since $\langle e_i, v \rangle \le \| e_i \| \| v \| \le C_0 \| v \|_0 \le C_0 C_1 \| v \|_1 \le C_0 C_1$ for any $v \in S_1 = \{ v \in X_1 : \| v \|_1 \le 1 \}$, we get $\| e_i \|_V \le C_0 C_1$; hence, $\| x_N^{(L)} - x_N \|_V \le C_0 C_1 R L^{-1}$ It is clear that $S_N^{(L)}$ is a $C_0 R(\varepsilon_N + C_1 L^{-1})$-finite net for $(S, \|\cdot\|_V)$. Indeed, for any $x \in S$, there exists $x_N^{(L)} \in S_N^{(L)}$ such that

$$\| x_N^{(L)} - x \|_V \le \| x_N^{(L)} - x_N \|_V + \| x_N - x \|_V \le C_0 R(\varepsilon_N + C_1 L^{-1}). \tag{13}$$

Definition. $x_N^{(L)}$ *is called a projection of* $x \in S$ *onto* $S_N^{(L)}$.

4. Coordinated approximation method. In this section, we suppose that the nonlinear operator $\mathcal{A}: H \to H$ satisfies the following hypotheses:

h1) $\mathcal{A}$ is a one-to-one and strongly continuous mapping (i.e., if $x_N \to x$, then $\mathcal{A}x_N \to \mathcal{A}x$);

h2) The modulus of continuity of $\mathcal{A}$ in the compact set $(S, \|\cdot\|_V)$ is supposed to be known, $\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\| \le \omega(\|v_1 - v_2\|_V)$, $\forall\, v_1, v_2 \in S$;

h3) The exact solution u^* of problem (1) satisfies the a priori estimate (2).

First, we choose the regularization parameters $N = N(\delta)$ and $L = L(\delta)$ from the conditions

$$\omega(C_0 R[\varepsilon_N + C_1/l]) > \delta \quad \text{for} \quad n = 1, \dots, N-1,\ l = 1, \dots, L-1, \tag{14}$$

$$\omega(C_0 R[\varepsilon_N + C_1/L]) \le \delta. \tag{15}$$

It is obvious that $N(\delta), L(\delta) \to +\infty$ as $\delta \to 0$.

Let $W(\delta) = \{v \in S_N^{(L)} : \|\mathcal{A}(v) - f_\delta\| \le 2\delta\}$, where $N = N(\delta)$ and $L = L(\delta)$ are the regularization parameters defined above.

Lemma 2. *The set* $W(\delta)$ *is nonempty and* $W(\delta) \to 0$ *as* $\delta \to 0$.

Proof. Denote by $u_N^{(L)}$ a projection of $u^* \in S$ onto $S_N^{(L)}$. It follows from (13) and (15) that

$$\|\mathcal{A}(u_N^{(L)}) - f_\delta\| \le \|\mathcal{A}(u_N^{(L)}) - \mathcal{A}(u^*)\| + \|f - f_\delta\| \le$$

$$\le \omega(C_0 R[\varepsilon_N + C_1/L]) + \delta \le 2\delta.$$

Thus, $u_N^{(L)} \in W(\delta)$. Since $\mathcal{A}: (S, \|\cdot\|_V) \to \mathcal{A}(S)$ is a one-to-one and continuous mapping on the compact set $(S, \|\cdot\|_V)$, by the Tikhonov's lemma [3], the inverse operator $\mathcal{A}^{-1}: \mathcal{A}(S) \to (S, \|\cdot\|_V)$ is continuous. Let $\overline{\omega}(t)$ be the modulus of continuity of $\mathcal{A}^{-1}$, i.e., $\|\mathcal{A}^{-1}(f_1) - \mathcal{A}^{-1}(f_2)\|_V \le \overline{\omega}(\|f_1 - f_2\|)$, $f_1, f_2 \in \mathcal{A}(S)$, where $\overline{\omega}(t) = \overline{\omega}(t, R) \to 0$ as $t \to 0$ and R is fixed. Since $\|\mathcal{A}(v) - \mathcal{A}(\tilde{v})\| \le$ $\le \|\mathcal{A}(v) - f_\delta\| + \|\mathcal{A}(\tilde{v}) - f_\delta\| \le 4\delta$ for any $v, \tilde{v} \in W(\delta)$, we have $\|v - \tilde{v}\|_V \le$ $\le \overline{\omega}(4\delta)$. Thus, $\operatorname{diam} W(\delta) \le \overline{\omega}(4\delta) \to 0$, $\delta \to 0$.

Now we are ready to prove the main theorems.

Theorem 3. *Suppose that all hypotheses h1)–h3) are satisfied. Then, for every* $w^\delta \in W(\delta)$, *the following estimate holds:*

$$\|w^\delta - u^*\|_V \le \operatorname{diam} W(\delta) + C_0 R(\varepsilon_{N(\delta)} + C_1/L(\delta)) \equiv \mu(\delta, R).$$

Proof. For any $w^\delta \in W(\delta)$, we have $\|w^\delta - u_N^{(L)}\|_V \le \operatorname{diam} W(\delta)$. It follows from Lemma 2, estimate (13), and the last inequality that

$$\|w^\delta - u^*\|_V \le \|w^\delta - u_N^{(L)}\|_V + \|u_N^{(L)} - u^*\|_V \le$$

$$\le \operatorname{diam} W(\delta) + C_0 R(\varepsilon_{N(\delta)} + C_1/L(\delta)) \le$$

$$\le \overline{\omega}(4\delta) + C_0 R(\varepsilon_{N(\delta)} + C_1/L(\delta)) \to 0, \quad \delta \to 0.$$

Let $w_k^\delta = P_k w^\delta$ and $u_k^* = P_k u^*$ be the finite dimensional approximations of w^δ and u^*, respectively.

Theorem 3 and estimate (12) lead us to the following practically useful relation:

$$\| w_k^\delta - u_k^* \|_0 \le c_k \| w^\delta - u^* \|_V \le c_k \{ \operatorname{diam} W(\delta) + C_0 R [\varepsilon_{N(\delta)} + C_1 / L(\delta)] \}.$$

Theorem 4. *In addition to hypotheses h1)–h3), assume that* $u^* \in X_0$. *Then* $\| u^\delta - u^* \|_0 \to 0$, $\delta \to 0$, *where* $u^\delta = P_k w^\delta$, *and* $K = K(\delta)$ *satisfies the relations*

$$C_k \mu(\delta, R) \le 1/k, \quad k = \overline{1, k}, \qquad C_k \mu(\delta, R) > 1/k, \quad k = K+1.$$

Proof. It is clear that $K = K(\delta) \to \infty$ as $\delta \to 0$. By letting $u_K^* = P_K u^*$, we have

$$\| u^\delta - u^* \|_0 \le \| u^\delta - P_K u^* \|_0 + \| P_K u^* - u^* \|_0.$$

Since $u^* \to X_0$, we have $\| P_K u^* - u^* \|_0 \to 0$, $\delta \to 0$. Further, by (12),

$$\| u^\delta - P_K u^* \|_0 = \| P_K (w^\delta - u^*) \| \le C_K \| w^\delta - u^* \|_V \le C_{k(\delta)} \mu(\delta, R) \le 1/K(\delta).$$

Thus,

$$\| u^\delta - u^* \|_0 \le 1/k(\delta) + \| P_{k(\delta)} u^* - u^* \|_0 \to 0, \quad \delta \to 0.$$

Now let us return to the case $H = L_2[0,1]$. Define $X_0 \equiv H_0^m[0,1]$ to be a subspace consisting of all functions $u(t)$ in $C^{m-1}[0,1]$ with $u^{m-1}(t)$ absolutely continuous on $[0,1]$, $u^{(m)} \in L_2[0,1]$, and $u^{(i)}(0) = u^{(i)}(1)$, $i = \overline{0, m-1}$. Then X_0 is a Banach space with the norm

$$\| u \|_0 = \sum_{i=0}^{m-1} \max_{0 \le t \le 1} | u^{(i)}(t) | + \left(\int_0^1 \left| u^{(m)}(t) \right|^2 dt \right)^{1/2}.$$

Further, denote by X_1 the space $C_0^l[0,1]$ of l-times continuously differentiable functions satisfying the boundary conditions: $u^{(i)}(0) = u^{(i)}(1)$, $i = \overline{0, l}$. Let

$$\| u \|_1 = \sum_{i=0}^{l} \max_{0 \le t \le 1} | u^{(i)}(t) |.$$

Suppose that $l \ge m+1$. Then X_1 is continuously and compactly imbedded in X_0 and X_0 is continuously imbedded in H. By using well-known facts on Fourier series [2], we can prove that the common basis of X_0 and H consists of trigonometric functions, which also belong to X_1,

$$\{e_n\}_{n=0}^{\infty} = \left\{1, \sqrt{2}\sin 2\pi n t, \sqrt{2}\cos 2\pi n t\right\}.$$

Hence, X_1 is dense in X_0 and X_0 is dense in H. By applying Theorems 3 and 4 we come to the following conclusions:

1. The method of coordinated approximation is still convergent if estimate (3) is replaced by the weaker assumption (2).

2. If the exact solution $u^*(t)$ is smooth enough, i.e., $u^* \in H_0^m[0,1]$, then the regularized solutions u^δ converge to u^* in the norm of $H_0^m[0,1]$.

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