УДК 519.1

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EXISTENCE OF A MULTIPLICATIVE BASIS FOR A FINITELY SPACED MODULE OVER AN AGGREGATE

СКІНЧЕННО ЗОБРАЖУВАНИЙ МОДУЛЬ НАД АГРЕГАТОМ МАЄ МУЛЬТИПЛІКАТИВНИЙ БАЗИС

It is proved that a finitely spaced module over a k-category admits a multiplicative basis (such a module gives rise to a matrix problem, in which the allowed column transformations are determined by a module structure, the row transformations are arbitrary, and the number of canonical matrices is finite).

Доведено, що скінченно зображуваний модуль над k-категорією (який можна зв'язати з матричною задачею, стовпцеві перетворення якої задаються модульною структурою, рядкові довільні та існує лише скінченне число матриць канонічного вигляду) має мультиплікативний базис.

It was proved in [1] that a finite-dimensional algebra, having finitely many isoclasses of indecomposable representations, admits a multiplicative basis. In [2] (Sections 4.10 – 4.12), an analogous hypothesis was formulated for finitely spaced modules over an aggregate and an approach to its proof was proposed. Our objective is to prove this hypothesis. Throughout the paper, k denotes an algebraically closed field.

Let us recall some definitions from [2] (see also [3]).

By definition, an *aggregate* $\mathcal{A}$ over k is a category that satisfies the following conditions:

a) For each X, $Y \in \mathcal{A}$, the set $\mathcal{A}(X, Y)$ is a finite-dimensional vector space over k;

b) The composition maps are bilinear;

c) $\mathcal{A}$ has finite direct sums;

d) Each idempotent $e \in A(X, X)$ has the kernel.

As a consequence, each $X \in A$ is a finite direct sum of indecomposables and the algebra of endomorphisms of each indecomposable is local.

We denote by $\mathcal{J}A$ a spectroid of A, i.e., a full subcategory formed by chosen representatives of the isoclasses of indecomposables, and let $\mathcal{R}_{\mathcal{A}}$ be the radical of $\mathcal{A}$. We suppose that $\mathcal{J}\mathcal{A}$ has finitely many objects. For each $a, b \in \mathcal{J}\mathcal{A}$, the space $\mathcal{R}_{\mathcal{A}}(a, b)$ consists of all irreversible morphisms of $\mathcal{A}(a, b)$, therefore, $\mathcal{A}(a, b) = \mathcal{R}_{\mathcal{A}}(a, b)$ for $a \neq b$, $\mathcal{A}(a, a) = k\,1_a \oplus_k \mathcal{R}_{\mathcal{A}}(a, a)$.

A *module* M over an aggregate $\mathcal{A}$ consists of finite-dimensional vector spaces $M(X)$, one for each object $X \in \mathcal{A}$, and of linear maps $M(f)\colon M(X) \to M(Y)$, $m \mapsto fm$, $f \in \mathcal{A}(X, Y)$, which satisfy the standard axioms: $1_X m = m$, $(f+g)m = fm + gm$, $(gf)m = g(fm)$, $f(\alpha m) = \alpha(fm) = (\alpha f)m$, $\alpha \in k$. It gives a k-linear functor from $\mathcal{A}$ into the category $\operatorname{mod} k$ of finite-dimensional vector spaces over k. A module M over $\mathcal{A}$ is *faithful* if $M(f) \neq 0$ for each nonzero $f \in \mathcal{A}(X, Y)$.

Define the *basis* of $(M, \mathcal{A})$ as a set $\{ m_i^a, f_l^{ba} \}$ consisting of bases $m_1^a, m_2^a, \ldots$ of the spaces $M(a)$, $a \in \mathcal{J}\mathcal{A}$, and bases $f_1^{ba}, f_2^{ba}, \ldots$ of the spaces $\mathcal{R}_{\mathcal{A}}(a, b)$, $a, b \in \mathcal{J}\mathcal{A}$. The maximal rank of $M(f_l^{ba})$ is called the *rank* of a basis. A basis is called a *scalarly multiplicative basis* if it satisfies the following conditions:

a) Each morphism f_l^{ba} is *thin*, i.e. $f_l^{ba} = g + h$ implies $\operatorname{rank} M(f_l^{ba}) \leq \operatorname{rank} M(g)$ or $\operatorname{rank} M(f_l^{ba}) \leq \operatorname{rank} M(h)$ for all $g, h \in \mathcal{A}(a, b)$;

b) Each product $f_l^{ba} m_i^a$ has the form λm_p^b, $\lambda \in k$;

c) $f_l^{ba} m_i^a = \lambda m_p^b$, $f_l^{ba} m_j^a = \mu m_p^b$, and $\lambda, \mu \in k \setminus \{0\}$ imply $i = j$.

We say that the basis is *multiplicative* if each nonzero product $f_l^{ba} m_i^a$ is a basis vector m_p^b.

We denote by M^k the aggregate formed by all triples (V, h, X), where $V \in \operatorname{mod} k$, $X \in \mathcal{A}$, and $h \in \operatorname{Hom}_k(V, M(X))$. A morphism from (V, h, X) to (V', h', X') is defined by the pair of morphisms $\varphi \in \operatorname{Hom}_k(V, V')$ and $\xi \in \mathcal{A}(X, X')$ such that $h'\varphi = M(\xi)h$. We call these triples *spaces* on M. We say that M is *finitely spaced* if M^k has a finite spectroid.

The objective of the paper is to prove the following theorem:

Theorem. *If M is a faithful finitely spaced module over an aggregate $\mathcal{A}$, then $(M, \mathcal{A})$ admits a multiplicative basis of* rank ≤ 2.

We wish to express gratitude to P. Gabriel, Th. Brüstle, T. Guidon, and U. Hassler for discussions and essential corrections.

1. Construction of a scalarly multiplicative basis. In Sections 1 – 3, M always denotes a finitely spaced module over an aggregate $\mathcal{A}$.

As shown in [2] (sections 4.7, 4.8), for each $a \in \mathcal{IA}$, the space $M(a)$ has a dimension $d(a) \leq 3$ and a sequence $m_1, m_2, \ldots, m_{d(a)}$, where

$$m_i \in (\mathcal{R}_{\mathcal{A}}(a, a))^{i-1} M(a) \setminus (\mathcal{R}_{\mathcal{A}}(a, a))^i M(a),$$

is a basis of $M(a)$. It will be called a *triangular basis* because the matrix of each map $M(f)$, $f \in \mathcal{A}(a, a)$, has a lower triangular form. We assume that each basis $m_1^a, \ldots, m_{d(a)}^a$ in a scalarly multiplicative basis is triangular (it is always triangular up to permutations of vectors).

A scalarly multiplicative basis is called *normed* if it satisfies the following condition:

d) $f_l^{ba} m_i^a = \lambda m_p^b$ and $\lambda \notin \{0, 1\}$ imply that $f_l^{ba} m_{i'}^a = m_{p'}^b$ for some $i' < i$.

A scalarly multiplicative basis can be reduced to a normed basis by means of multiplication of f_l^{ba} by scalars.

A scalarly multiplicative basis is called *reduced* if it satisfies condition d) and the following condition:

e) if a morphism $\varphi = \sum_l \lambda_l f_l^{ba}$ is a product of basis morphisms, then

$$\operatorname{rank} M(\varphi) = \sum_{\lambda_l \neq 0} \operatorname{rank} M(f_l^{ba}).$$

At the end of this section, we shall prove that every multiplicative basis of $(M, \mathcal{A})$ is reduced if $\operatorname{char}(k) \neq 2$.

Let $m_1^a, \ldots, m_{d(a)}^a$ be a fixed triangular basis of $M(a)$ for each $a \in \mathcal{IA}$. For m_j^a and m_i^b, we define a linear map $e_{ij}^{ba} : M(a) \to M(b)$ such that $e_{ij}^{ba} m_j^a = m_i^b$ and $e_{ij}^{ba} m_{j'}^a = 0$ for all $j' \neq j$.

Let $f \in \mathcal{R}_{\mathcal{A}}(a, b)$, $a, b \in \mathcal{IA}$. We say that f is a *short morphism* if $f \notin \mathcal{R}_{\mathcal{A}}(c, b)\mathcal{R}_{\mathcal{A}}(a, c)$ for all $c \in \mathcal{IA}$, f is a *prime morphism* if $M(f) = e_{ij}^{ba}$, and f

is a *double morphism* if

$$M(f) = e_{ij}^{ba} + \lambda e_{i'j'}^{ba}, \ e_{ij}^{ba} \notin M(a,b), \ i < i', \ j < j', \ 0 \neq \lambda \in k.$$

The coefficient λ is called the *parameter* of a double morphism.

Proposition 1. *A set* $\{ m_i^a, f_l^{ba} \}$ *is a normed (reduced, respectively) scalarly multiplicative basis if and only if the following two conditions are satisfied*:

1) $m_1^a, m_2^a, \ldots$ *is a triangular basis of* $M(a)$, $a \in \mathcal{JA}$.

2) $f_1^{ba}, f_2^{ba}, \ldots$ *is the set of all prime and double morphisms of* $\mathcal{A}(a,b), a, b \in \in \mathcal{JA}$, *except a single double morphism (a single short double morphism, respectively) if the number of double morphisms is equal to* 3. *Moreover, the number of double morphisms of* $\mathcal{A}(a,b)$ *is equal to* 0, 1, *or* 3, *and, in the last case, there exists a short double morphism.*

The statement of Proposition 1 about a normed scalarly multiplicative basis follows from Lemmas 1 and 5. The complete proof of Proposition 1 will be given in Section 3.

Lemma 1. *If* $d(a) = 2$, *then* $M(a,a) = k1_{M(a)} + ke_{21}^{aa}$. *If* $d(a) = 3$, *then* $M(a,a) = k1_{M(a)} + ke_{21}^{aa} + ke_{32}^{aa}$ *or*

$$M(a,a) = k1_{M(a)} + k(e_{21}^{aa} + \lambda_{aa} e_{32}^{aa}) + ke_{31}^{aa} \tag{1}$$

and $0 \neq \lambda_{aa} \in k$.

The proof of Lemma 1 is obvious.

For every linear map $\varphi : M(a) \to M(b)$, we denote by $\varphi_{ij} \in ke_{ij}^{ba}$ linear maps such that $\varphi = \sum \varphi_{ij}$. We introduce an order relation on $\{1, 2, \ldots, d(b)\} \times \{1, 2, \ldots, d(a)\}$ by $(i,j) \geq (l,r)$ if $i \leq l$ and $j \geq r$. A pair (l,r) is called a *step* of $\varphi \in M(a,b)$ if $\varphi_{lr} \neq 0$ and $\varphi_{ij} = 0$ for all $(i,j) > (l,r)$. A pair (l,r) is called a *step* of $M(a,b)$ if $\psi_{lr} \neq 0$ for some $\psi \in M(a,b)$ and $\varphi_{ij} = 0$ for all $\varphi \in \in M(a,b)$ and all $(i,j) > (l,r)$ ($l \geq r$ because each basis $m_1^a, m_2^a, \ldots$ is triangular).

Lemma 2. *If* $a, b \in \mathcal{JA}$, $a \neq b$, $d(a) = d(b) = 3$, *and* $M(a,b)$ *has two steps* $(1,2)$ *and* $(2,3)$, *then* $M(b,a) = ke_{31}^{ab}$.

Proof. Let $\psi \in M(b,a)$. There is $\varphi \in M(a,b)$ having the steps $(1,2)$ and $(2,3)$. By Lemma 1, there exist $\varepsilon \in M(a,a)$ and $\delta \in M(b,b)$ such that $\varphi' = \varphi\varepsilon + \delta\varphi$ has the steps $(1,1)$, $(2,2)$, and $(3,3)$. The inclusion $\mathcal{A}(b,a)\mathcal{A}(a,b) \subset \subset \mathcal{R}_{\mathcal{A}}(a,a)$ implies

$$M(b,a)M(a,b) \subset M(\mathcal{R}_{\mathcal{A}}(a,a)) = ke_{21}^{aa} \oplus ke_{31}^{aa} \oplus ke_{32}^{aa}.$$

Since $\psi\varphi' \in M(\mathcal{R}_{\mathcal{A}}(a,a))$, all steps of ψ are not higher that $(2,1)$ and $(3,2)$. Since $\psi\varphi \in M(\mathcal{R}_{\mathcal{A}}(a,a))$, we have $\psi \in ke_{31}^{ab}$.

Therefore, $M(b,a) \subset ke_{31}^{ab}$. Assume that $M(b,a) = 0$. Let us examine the space $\mathcal{H}_\lambda = (k^6, h_\lambda, a^2 \oplus b^2) \in M^k$, where

$$k^6 = k \oplus k \oplus k \oplus k \oplus k \oplus k, \ a^2 = a \oplus a, \quad b^2 = b \oplus b, \ \lambda \in k$$

and h_λ is the linear mapping of k^6 into

$$M(a^2 \oplus b^2) = (km_1^a)^2 \oplus (km_2^a)^2 \oplus (km_3^a)^2 \oplus (km_1^b)^2 \oplus (km_2^b)^2 \oplus (km_3^b)^2$$

with the matrix

$$\begin{pmatrix}1\\0\end{pmatrix} \oplus \begin{pmatrix}0\\1\end{pmatrix} \oplus \left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{array}\right)^T \oplus \begin{pmatrix}1\\1\end{pmatrix} \oplus \begin{pmatrix}1\\\lambda\end{pmatrix}.$$

We show that $\mathcal{H}_\lambda \not\simeq \mathcal{H}_\mu$ if $\lambda \neq \mu$. Let (φ, ξ) be an isomorphism $\mathcal{H}_\lambda \to \mathcal{H}_\mu$. The linear mapping $M(\xi)$ has the block matrix (K_{ij}), $i, j \leq 6$, where K_{ij} are 2×2-matrices. By $M(b, a) = 0$ and Lemma 1, we have $K_{ij} = 0$ if $i < j$. Evidently, $K_{11} = K_{22} = K_{33}$, $K_{44} = K_{55} = K_{66}$, and $K_{43} = 0$.

Since $h_\mu \varphi = M(\xi) h_\lambda$, the matrix of the nondegenerate mapping φ also has the block form (Φ_{ij}), $i, j \leq 5$, where the blocks $\Phi_{11}, \Phi_{22}, \Phi_{44}$, and Φ_{55} are 1×1-matrices, the block Φ_{33} is a 2×2-matrix, and $\Phi_{ij} = 0$ if $i < j$. Moreover,

$$\begin{pmatrix}1\\0\end{pmatrix}\Phi_{11} = K_{11}\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}\Phi_{22} = K_{22}\begin{pmatrix}0\\1\end{pmatrix},$$

$$\left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{array}\right)^T \Phi_{33} = (K_{33} \oplus K_{44})\left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{array}\right)^T,$$

$$\begin{pmatrix}1\\1\end{pmatrix}\Phi_{44} = K_{55}\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\\mu\end{pmatrix}\Phi_{55} = K_{66}\begin{pmatrix}1\\\lambda\end{pmatrix}.$$

By the third equality, we obtain $K_{33} = K_{44}$, by the first and second equalities, we get

$$K_{11} = K_{22} = \ldots = K_{66} = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},$$

and, by the forth and fifth equalities, $\alpha = \beta$ and $\lambda = \mu$. We have infinitely many nonisomorphic indecomposable spaces $\mathcal{H}_\lambda$, $\lambda \in k$, on M. This proves Lemma 2.

Let $(l_1, r_1), \ldots, (l_t, r_t)$ be all steps of $M(a, b)$. Set

$$S(a, b) = \sum\nolimits_{(i,j)} k e_{ij}^{ba} \qquad (\text{resp. } \overline{S}(a, b) = \sum\nolimits_{(i,j)} k e_{ij}^{ba}),$$

where the sum is taken over all (i, j) such that there exists a step $(l_p, r_p) > (i, j)$ ($(l_p, r_p) \geq (i, j)$, respectively).

Lemma 3. *Let* $a \neq b$ *and* $M(a, b)$ *have the steps* $(1, 1)$, $(2, 2)$, *and* $(3, 3)$. *Then there is no* $\psi \in M(a, b)$ *such that* $M(a, b) = k\psi + S(a, b)$.

Proof. Assume that there exists $\psi \in M(a, b)$ such that $M(a, b) = k\psi + S(a, b)$. By the form of $M(a, b)$ and $\mathcal{A}(b, a)\mathcal{A}(a, b) \subset \mathcal{R}_\mathcal{A}(a, a)$, we have $M(b, a) \subset k e_{21}^{ab} + k e_{31}^{ab} + k e_{32}^{ab}$.

Let us examine the space $\mathcal{H}_\lambda = (k^3, h_\lambda, a \oplus b)$, where $\lambda \in k$ and h_λ is the linear map from k^3 into

$$M(a \oplus b) = k m_1^a \oplus k m_2^a \oplus k m_3^a \oplus k m_1^b \oplus k m_2^b \oplus k m_3^b$$

with the matrix

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & \lambda\end{array}\right)^T.$$

Let (φ, ξ) be an isomorphism $\mathcal{H}_\lambda \to \mathcal{H}_\mu$. It follows from the conditions imposed on $M(a, a)$, $M(a, b)$, $M(b, a)$ and $M(b, b)$ that the matrix of $M(\xi)$ has form

$$\left(\begin{array}{ccc|ccc} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & 0 & \gamma_1 & 0 & 0 \\ \alpha_4 & \alpha_3 & \alpha_1 & \gamma_3 & \gamma_2 & 0 \\ \hline \delta_1 & 0 & 0 & \beta_1 & 0 & 0 \\ \delta_4 & \delta_2 & 0 & \beta_2 & \beta_1 & 0 \\ \delta_6 & \delta_5 & \delta_3 & \beta_4 & \beta_3 & \beta_1 \end{array}\right).$$

Moreover, $\delta_1 = \delta\varepsilon_1$, $\delta_2 = \delta\varepsilon_2$, and $\delta_3 = \delta\varepsilon_3$, where $\delta \in k$ and ε_1, ε_2, and ε_3 are the diagonal elements of the lower triangular matrix of ψ. By $h_\mu \varphi = M(\xi) h_\lambda$, we find succesively that $\delta = 0$, the mapping φ has the lower triangular matrix with the diagonal $(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 = \beta_1$, and $\lambda = \mu$.

Hence $\mathcal{H}_\lambda \neq \mathcal{H}_\mu$ for $\lambda \neq \mu$ and M is infinitely spaced. We arrive at a contradiction that proves Lemma 3.

Lemma 4. $S(a, b) \subset M(a, b)$.

Proof. We must show that if (l, r) is a step of $M(a, b)$, then

$$S_{lr}(a, b) = \sum_{(i,j)<(l,r)} k e_{ij}^{ba} \subset M(a, b).$$

By Lemma 3, there exists a $\psi \in M(a, b)$ having the step (l, r) but not more than two steps. If ψ and $M(a, b)$ have the steps $(1, 2)$ and $(2, 3)$, then, by Lemma 2, $e_{31}^{ab}\psi \in M(a, a)$ has the unique step $(3, 2)$. Hence,

$$M(a, a) = k1_{M(a)} \oplus k e_{21}^{aa} \oplus k e_{31}^{aa} \oplus k e_{32}^{aa}.$$

In all other cases, by Lemma 1, $S_{lr}(a, b)$ is contained in the space generated by all $\delta\psi\varepsilon$, where $\varepsilon \in M(a, a)$ and $\delta \in M(b, b)$. This proves Lemma 4.

By Lemma 4, we have the following lemma.

Lemma 5. *Let $a, b \in \mathcal{J}\mathcal{A}$, $a \neq b$, and $M(a, b) \neq \overline{S}(a, b)$. Then only three cases can occur* $(\lambda_{ab} \neq 0 \neq \mu_{ab})$:

a) $M(a, b)$ *has two steps* (l_1, r_1) *and* (l_2, r_2), $l_1 < l_2$, *and is equal to*

$$k(e_{l_1 r_1}^{ba} + \lambda_{ab} e_{l_2 r_2}^{ba}) \oplus S(a, b);$$

b) $M(a, b)$ *has the steps* $(1, 1)$, $(2, 2)$, *and* $(3, 3)$ *and is equal to*

$$k(e_{11}^{ba} + \lambda_{ab} e_{22}^{ba}) \oplus k e_{33}^{ba} \oplus S(a, b),$$

or

$$k(e_{11}^{ba} + \lambda_{ab} e_{33}^{ba}) \oplus k e_{22}^{ba} \oplus S(a, b),$$

or

$$k(e_{22}^{ba} + \lambda_{ab} e_{33}^{ba}) \oplus k e_{11}^{ba} \oplus S(a, b);$$

c) $M(a, b)$ *has the steps* $(1, 1)$, $(2, 2)$, *and* $(3, 3)$ *and is equal to*

$$k(e_{11}^{ba} + \lambda_{ab} e_{22}^{ba}) \oplus k(e_{11}^{ba} + \mu_{ab} e_{33}^{ba}) \oplus S(a, b).$$

Remarks. 1) In a normed scalarly multiplicative basis, each long double morphism $\varphi \in \mathcal{A}(a, b)$ is the product of double basis morphisms. Indeed, let $\varphi = \tau\psi$, where $\psi \in \mathcal{R}_{\mathcal{A}}(a, c)$ and $\tau \in \mathcal{R}_{\mathcal{A}}(c, b)$. Then ψ is the unique double morphism of

$\mathcal{A}(a,c)$ (otherwise, φ is the sum of prime morphisms). Therefore, ψ is a basis morphism. Similarly, τ is also a basis morphism.

2) A normed scalarly multiplicative basis is reduced if and only if all long double morphisms are basis morphisms. Indeed, let a long double morphism $\varphi \in \mathcal{A}(a,b)$ be not a basis morphism. Then $\mathcal{A}(a,b)$ has two double basis morphisms and φ is their linear combination. But this contradicts the definition of a reduced basis.

3) Lemma 1 and Lemma 5 imply the statement of Proposition 1 about a normed scalarly multiplicative basis. By Remark 2, to complete the proof of Theorem 1 we must prove that each $\mathcal{A}(a,b)$ $(a, b \in \mathcal{J}\mathcal{A})$ does not contain three long double morphisms.

4) If $\operatorname{char}(k) \neq 2$, then every multiplicative basis is reduced. Indeed, otherwise, there is, by Remark 2, a long double morphism $\varphi \in \mathcal{A}(a,b)$, which is not a basis morphism. By Lemma 5, $\varphi = \psi - \tau$, where ψ and τ are basis long double morphisms of $\mathcal{A}(a,b)$. Hence, $M(\varphi) = e_{ii}^{ba} - e_{jj}^{ba}$. By Remark 3, φ is a product of basis morphisms; hence, $M(\varphi) = e_{ii}^{ba} + e_{jj}^{ba}$ and $\operatorname{char}(k) = 2$.

2. The graph of a scalarly multiplicative basis. In this section, we study some properties of a scalarly multiplicative basis and give the proof of Proposition 1.

Following [2] (Section 4.9), we define a poset $\mathcal{P}$, whose elements are the spaces $a_i = (\mathcal{R}_{\mathcal{A}}(a,a))^{i-1}M(a)$ $(a \in \mathcal{J}\mathcal{A},\ 1 \le i \le d(a))$ and where $a_i \le b_j$ if and only if $\mathcal{A}(b,b)fa_i = b_j$ for some $f \in \mathcal{A}(a,b)$. The elements $a_i \in \mathcal{P}$ are in a one-to-one correspondence with the basis vectors m_i^a of every scalarly multiplicative basis $\{m_i^a, f_l^{ba}\}$, moreover, $a_i < b_j$ if and only if $f_l^{ba} m_i^a = \lambda m_j^a$ for some f_l^{ba} and $0 \neq \lambda \in k$. We decompose the poset $\mathcal{P}$ into disjoint totally ordered subsets $\{a_1, \dots, a_{d(a)}\}$, $(a_1 < a_2 < \dots < a_{d(a)},\ d(a) \le 3)$; each of them is called a *double* if $d(a) = 2$ and a *triple* if $d(a) = 3$.

The following three lemmas were given in [2] without proofs.

Lemma 6 (see [2] (Lemma 4.12.1)). *The union* $\cup \{a_1, a_2, a_3\}$ *of all triples is totally ordered.*

Proof. The elements of a triple are totally ordered.

Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be triples and let some a_i be not comparable with some b_j. We shall construct indecomposable spaces $\mathcal{H}_\lambda = (k^6, h_\lambda, a^2 \oplus b^2)$ on M, $\lambda \in k$, such that $\mathcal{H}_\lambda \not\simeq \mathcal{H}_\mu$ for $\lambda \neq \mu$.

For $i = 3$ and $j = 1$, the spaces $\mathcal{H}_\lambda$ were constructed in the proof of Lemma 2. For arbitrary i and j, $\mathcal{H}_\lambda$ is constructed analogously with the block

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)^T$$

of $h_\lambda : k^6 \to M(a^2 \oplus b^2)$ located in the rows of

$$km_i^a \oplus km_i^a \oplus km_j^b \oplus km_j^b \subset M(a^2 \oplus b^2).$$

Let $(\varphi, \xi) : \mathcal{H}_\lambda \tilde{\to} \mathcal{H}_\mu$ and let (M_{ij}) be the block matrix of $M(\xi)$. Then (M_{ij}) is not upper block-triangular, but we can reduce (M_{ij}) to the upper block-triangular form by means of simultaneous transpositions of vertical and horizontal stripes, since the set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is partially ordered. Hence, M is infinitely spaced. We arrive at a contradiction that proves Lemma 6.

Lemma 7 (see [2] (Lemma 4.9)). *There are no elements* a_i, $a_{i'}$, b_j, *and* $b_{j'}$ *such that* $a_i \neq a_{i'}$, $b_j \neq b_{j'}$, a_i *is not comparable to* $b_{j'}$, *and* b_j *is not comparable to* $a_{i'}$. *There are no elements* a_i, $a_{i'}$, b_j, $b_{j'}$, c_l, *and* $c_{l'}$ *such that* $a_i \neq a_{i'}$, $b_j \neq b_{j'}$, $c_l \neq c_{l'}$, a_i *is not comparable to* $b_{j'}$, b_j *is not comparable to* $c_{l'}$, *and* c_l *is not comparable to* $a_{i'}$.

Proof. In the first case, we set

$$\mathcal{H}_\lambda = (ke_1 \oplus ke_2,\ h_\lambda,\ a \oplus b) \in M^k,$$

where $h_\lambda e_1 = m_i^a + m_{j'}^b$ and $h_\lambda e_2 = m_j^b + \lambda m_{i'}^a$. In the second case, we set

$$\mathcal{H}_\lambda == (ke_1 \oplus ke_2 \oplus ke_3,\ h_\lambda,\ a \oplus b \oplus c),$$

where $h_\lambda e_1 = m_i^a + m_{j'}^b$, $h_\lambda e_2 = m_j^b + m_{l'}^c$, and $h_\lambda e_3 = m_l^c + \lambda m_{i'}^a$. Obviously, $\mathcal{H}_\lambda \not\simeq \mathcal{H}_\mu$ for $\lambda \neq \mu$.

Lemma 8. (see [2] (Lemma 4.12.2)). *Each triple contains at least two elements comparable with all elements of all doubles.*

Proof. Assume that Lemma 8 is not true for a triple $\{a_1, a_2, a_3\}$ and doubles $\{b_1, b_2\}$ and $\{c_1, c_2\}$.

***Case* 1.** Assume that $b \neq c$. For definiteness, we suppose that a_2 is not comparable to b_1 and a_3 is not comparable to c_1.

For each representation $\mathcal{H}$

$$\begin{array}{ccccccccccc} & & & & & & k^{r_1} & & & & \\ & & & & & & A_1 \downarrow & & & & \\ k^{r_4} & \xrightarrow{B_2} & k^{t_2} & \xleftarrow{B_1} & k^{r_2} & \xrightarrow{A_2} & k^{t_1} & \xleftarrow{A_3} & k^{r_3} \xrightarrow{C_1} & k^{t_3} & \xleftarrow{C_2} k^{r_5} \end{array}$$

of the quiver $\tilde{E}_7$ (see [2], (Section 6.3)), we construct the space

$$\overline{\mathcal{H}} = (k^{r_1 + \ldots + r_5}, h,\ a^{t_1} \oplus b^{t_2} \oplus c^{t_3}) \in M^k,$$

where

$$h = A_1 \oplus \begin{pmatrix} A_2 \\ B_1 \end{pmatrix} \oplus \begin{pmatrix} A_3 \\ C_1 \end{pmatrix} \oplus B_2 \oplus C_2$$

is a linear mapping of $k^{r_1 + \ldots + r_5}$ into

$$M(a^{t_1} \oplus b^{t_2} \oplus c^{t_3}) = (km_1^a)^{t_1} \oplus [(km_2^a)^{t_1} \oplus$$

$$\oplus\ (km_1^b)^{t_2}] \oplus [(km_3^a)^{t_1} \oplus (km_1^c)^{t_3}] \oplus (km_2^b)^{t_2} \oplus (km_2^c)^{t_3}.$$

The functor $\mathcal{H} \mapsto \overline{\mathcal{H}}$ on the representations $\mathcal{H}$ with injective A_1, A_2, A_3, B_2, and C_2 preserves indecomposability and *heteromorphism* (i.e., $\mathcal{H} \simeq \mathcal{H}'$ if $\overline{\mathcal{H}} \simeq \simeq \overline{\mathcal{H}}'$). Indeed, let $(\varphi, \xi)\colon \overline{\mathcal{H}} \tilde{\to} \overline{\mathcal{H}}'$. The nondegenerate linear maps φ and $M(\xi)$ have the block forms (Φ_{ij}), $i, j \leq 5$, and (K_{ij}), $i, j \leq 7$. The equality $h'\varphi = M(\xi)h$ implies $A_1'\Phi_{11} = K_{11}A_1$,

$$\begin{pmatrix} A_2' \\ B_1' \end{pmatrix}\Phi_{22} = \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{pmatrix}\begin{pmatrix} A_2 \\ B_1 \end{pmatrix}, \qquad \begin{pmatrix} A_3' \\ C_1' \end{pmatrix}\Phi_{33} = \begin{pmatrix} K_{44} & K_{45} \\ K_{54} & K_{55} \end{pmatrix}\begin{pmatrix} A_3 \\ C_1 \end{pmatrix},$$

$$B_2'\Phi_{44} = K_{66}B_2, \quad C_2'\Phi_{55} = K_{77}C_2.$$

Since $\{a_1, a_2, a_3\}$ is a triple and $\{b_1, b_2\}$ and $\{c_1, c_2\}$ are doubles, we have $K_{11} = K_{22} = K_{44}$, $K_{33} = K_{66}$, and $K_{55} = K_{77}$. Since a_2 is not comparable to b_1 and a_3 is not comparable to c_1, we have $K_{23} = 0$, $K_{32} = 0$, $K_{45} = 0$, and $K_{54} = 0$. Hence, the diagonal blocks of (Φ_{ij}) and (K_{ij}) determine a morphism $\mathcal{H} \to \mathcal{H}'$.

We shall show that this morphism is an isomorphism, i.e., the diagonal blocks Φ_{ii} and K_{ii} are invertible. By strengthening the partial order relation in $\{a_1, a_2, a_3, b_1, b_2, c_1, c_2\}$, we obtain a total order relation $\ll$ such that $a_2 \ll b_1$ and $a_3 \ll c_1$ (these pairs are not comparable with respect to $<$).

We transpose the horizontal stripes of the matrices of h and h' according to the new order. Then we transpose the vertical stripes to get lower trapezoidal matrices. Correspondingly, we transpose the blocks of (Φ_{ij}) and (K_{ij}). Then the new matrix (K_{ij}) has a lower triangular form. The upper nonzero blocks of vertical stripes are the injective maps A_1, A_2, A_3, B_2, and C_2 (since $a_2 \ll b_1$ and $a_3 \ll c_1$). It follows from $h'\varphi = M(\xi)h$ that (Φ_{ij}) also has a lower triangular form. Hence, the diagonal blocks Φ_{ii} and K_{ii} are invertible and $\mathcal{H} \simeq \mathcal{H}'$.

But the quiver $\tilde{E}_7$ admits an infinite set of nonisomorphic indecomposable representations of the form $\mathcal{H}$ with injective A_1, A_2, A_3, B_2, and C_2 (and surjective B_1 and C_1, which will be used in case 2). These representations are determined by the matrices

$$(A_1 | A_2 | A_3) = \left(\begin{array}{cc|ccc|ccc} 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

$$(B_1 | B_2) = (C_1 | C_2) = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

and they are nonisomorphic for different $\alpha \in k$. This contradicts the assumption that M is finitely spaced.

***Case* 2.** Assume that $b = c$. By Lemma 7, if a_i is not comparable to b_1 and a_j is not comparable to b_2, then $i = j$. Let a_2 and a_3 be not comparable to b_1. Then $a_1 < b_1$ and $a_3 < b_2$.

As in case 1, for each representation $\mathcal{H}$ of the quiver $\tilde{E}_7$ with injective A_1, A_2, A_3, B_2, and C_2 and surjective B_1 and C_1, we construct the space $\hat{\mathcal{H}} = (k^{r_1 + \ldots + r_5}, h, a^{t_1} \oplus b^{t_2 + t_3}) \in M^k$, where

$$h = A_1 \oplus \begin{pmatrix} A_2 \\ B_1 \end{pmatrix} \oplus \begin{pmatrix} A_3 \\ C_1 \end{pmatrix} \oplus B_2 \oplus C_2$$

is a linear mapping of $k^{r_1 + \ldots + r_5}$ into

$$M(a^{t_1} \oplus b^{t_2 + t_3}) = (km_1^a)^{t_1} \oplus [(km_2^a)^{t_1} \oplus (km_1^b)^{t_2}] \oplus$$

$$\oplus [(km_3^a)^{t_1} \oplus (km_1^b)^{t_3}] \oplus (km_2^b)^{t_2} \oplus (km_2^b)^{t_3}.$$

Let (φ, ξ), $\hat{\mathcal{H}} \xrightarrow{\sim} \hat{\mathcal{H}}'$. It follows from the order relation for $\{a_1, a_2, a_3, b_1, b_2\}$

that all blocks over the diagonal of the block matrix $K = (K_{ij})_{i,j=1,2,\dots,7}$ of the mapping $M(\xi)$ are zero except the blocks $K_{35} = K_{67}$. Let us prove that they are zero, too.

Indeed, by comparing the blocks with index $(2, 3)$ in the equality $h'\varphi = M(\xi)h$, we obtain $A_2'\Phi_{23} = 0$ and $\Phi_{23} = 0$ since A_2' is injective. By comparing the blocks with index $(3, 3)$, we obtain $B_1'\Phi_{23} = K_{35}C_1$ and $K_{35} = 0$ since C_1 is surjective.

Hence K is the lower block-triangular matrix. Therefore Φ also is a lower block-triangular matrix the diagonal blocks K_{ii} and Φ_{ii} of which are invertible, $\mathcal{H} \simeq \mathcal{H}'$. This proves our lemma.

Now fix a normed scalarly multiplicative basis $\{m_i^a, f_l^{ba}\}$ and define the oriented graph Γ, the set of vertices Γ_0 of which is the poset $\mathcal{P}$ and there is an arrow $a_p \to b_q$ $(a_p, b_q \in \Gamma)$ if and only if $M(f_l^{ba}) = \lambda e_{qp}^{ba} + \mu e_{q'p'}^{ba}$ for some short double morphism f_l^{ba} (then there is an arrow $a_{p'} \to b_{q'}$ and we shall say that the arrows $a_p \to b_q$ and $a_{p'} \to b_{q'}$ are *connected*). An arrow $a_p \to b_q$ will be called a *weak arrow* if $\mathcal{A}(a, b)$ contains three double morphisms. Each weak arrow is connected with two arrows. The others will be called *strong arrows*, each of them is connected exactly with one arrow.

Lemma 9. *Let $a_i < b_j < c_r$ and $a_i \to c_r$ be an arrow. Then $a \neq b \neq c \neq a$, $i = r$, the spaces $\mathcal{A}(a, b)$, $\mathcal{A}(b, c)$ and $\mathcal{A}(a, c)$ contain exactly 1, 1 and 3 double morphisms respectively, and there exists a pair of oriented paths $(a_i \to \dots \to b_j \to \dots \to c_i,\ a_{i'} \to \dots \to b_{j'} \to \dots \to c_{i'})$ consisting of connected strong arrows, and a pair of connected weak arrows $(a_i \to c_i, a_{i''} \to c_{i''})$, $i' \neq i''$. In the case of a reduced scalarly multiplicative basis, there is no other arrow from $\{a_l\}$ to $\{c_l\}$.*

Proof. Since $a_i < b_j < c_r$, there are morphisms $g \in \mathcal{A}(a, b)$ and $h \in \mathcal{A}(b, c)$ such that $M(g) = \alpha e_{ji}^{ba} + \beta e_{j'i'}^{ba}$ and $M(h) = \gamma e_{rj}^{cb} + \delta e_{r''j''}^{cb}$ $(\alpha, \beta, \gamma, \delta \in k$ and $\alpha \neq 0 \neq \gamma)$. If hg is a prime morphism, then $M(hg) = \alpha\gamma e_{ri}^{ca}$ contradicts the existence of the arrow $a_i \to c_r$. Hence hg is a double morphism, $\beta \neq 0 \neq \delta$, $j' = j''$ and g and h are the unique double morphisms of $\mathcal{A}(a, b)$ and $\mathcal{A}(b, c)$ respectively. The space $\mathcal{A}(a, c)$ contains the double morphism hg and the short double morphism corresponding to the arrow $a_i \to c_r$, hence $M(a, c)$ has the form from item c) of Lemma 5.

If the basis is reduced then by Remark 2 of Sect. 1, the double morphism hg is a basis morphism and there is only one pair of connected arrows from $\{a_l\}$ to $\{c_l\}$. This proves our lemma.

***Proof of Proposition* 1.** By Remark 3 of Sect. 1, we must prove that each space $\mathcal{A}(a, c)$ $(a, c \in \mathcal{J}\mathcal{A})$ does not contain three long double morphisms. By contradiction let $f_1, f_2, f_3 \in \mathcal{A}(a, c)$ be three long double morphisms and let $f_r = h_r g_r$, where g_r is a short double morphism and $r = 1, 2, 3$. The morphisms g_1, g_2 and g_3 correspond to the pairs of connected arrows $(a_1 \to x_i,\ a_2 \to x_{i'})$, $(a_1 \to y_j,\ a_3 \to y_{j'})$ and $(a_2 \to z_l,\ a_3 \to z_{l'})$.

Let $x_i < y_j$. By putting $(a_i, b_j, c_r) = (a_1, x_i, y_j)$ in Lemma 9, we obtain that $\mathcal{A}(a, y)$ contains three double morphisms. By putting $(a_i, b_j, c_r) = (a_1, y_j, c_1)$ in Lemma 9, we have that $\mathcal{A}(a, y)$ contains exactly one double morphism.

Hence x_i is not comparable to y_j. Similarly $x_{i'}$ is not comparable to z_l and $y_{j'}$ is not comparable to $z_{l'}$. This contradicts Lemma 7 and proves Proposition 1.

We shall now assume that the graph Γ is obtained from a reduced scalarly multiplicative basis.

Lemma 10. *If two arrows start from (stop at) the same vertex, then the arrows connected with them start from (stop at) different vertices.*

Proof. By contradiction, let $b_j \leftarrow a_i \rightarrow c_r$ and $b_{j'} \leftarrow a_{i'} \rightarrow c_{r'}$ be connected arrows. If $b_j < c_r$, then $a_i < b_j < c_r$ and, by Lemma 9, the arrows connected with $a_i \rightarrow b_j$ and $a_i \rightarrow c_r$ must start from different vertices, but they start from $a_{i'}$. Analogously $b_{j'}$ is not comparable to $c_{r'}$. This contradicts Lemma 7.

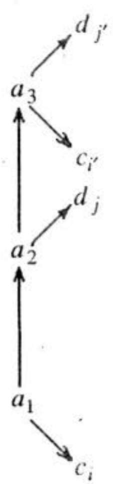
Lemma 11. *There are no two arrows starting from (stopping at) the same vertex of a double. There are no three arrows starting from (stopping at) the same vertex of a triple.*

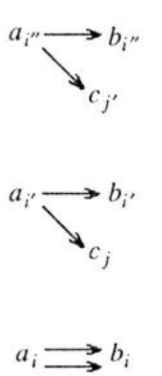
The proof follows from Lemma 10.

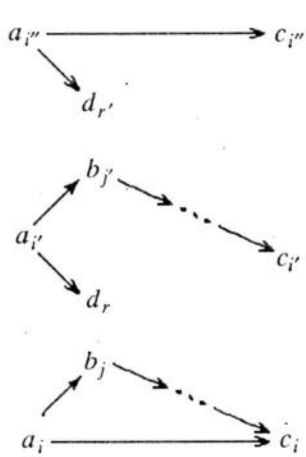
Lemma 12. *There are at most two different pairs of connected arrows starting from (stopping at) the same triple.*

Proof. By contradiction, let there be three pairs of connected arrows from a triple $\{a_1, a_2, a_3\}$ to $\{b_i\}$, $\{c_i\}$, $\{d_i\}$. Since there exist at most two pairs of connected arrows from a triple to a triple, then there are no three coinciding objects among a, b, c, d. Hence there exist five possibilities up to a permutation of b, c, d: 1) $a = b \neq \neq c \neq d$, $a \neq d$; 2) $a = b \neq c = d$; 3) $a \neq b = d \neq c$, $a \neq c$; 4) a, b, c, d are distinct and there are two arrows $a_i \rightarrow b_j$ and $a_i \rightarrow c_r$, $b_j < c_r$; 5) a, b, c, d are distinct and for each pair of arrows $a_i \rightarrow x$, $a_i \rightarrow y$, the vertices x and y are incomparable.

By Lemmas 9 – 11, we have the following subgraphs of Γ in cases 1, 3 and 4:

1) 

2) 

4) 

Consider these cases.

1) If $c_i < a_2$ or $d_j < a_3$, then by Lemma 9, $\mathcal{A}(a, a)$ contains three double

morphisms, which is a contradiction. If $a_2 < c_i$ or $a_3 < d_j$, then by Lemma 9, there is an arrow $a_2 \to c_i$ or $a_3 \to d_j$, in contradiction with Lemma 11. Hence a_2 is incomparable with c_i and a_3 is incomparable with d_j, which is impossible by Lemma 8.

2) This case is similar to the previous one.

3) The inequality $b_{i'} < c_j$ is impossible, by Lemma 9, because $\mathcal{A}(a, b)$ contains three double morphisms. The inequality $b_{i'} > c_j$ is impossible, by Lemma 9, because there are four arrows from $\{a_l\}$ to $\{b_l\}$. Hence $b_{i'}$ is incomparable with c_i. Analogously $b_{i''}$ is not comparable to $c_{j'}$ in contradiction with Lemma 7.

4) The inequalities $c_{i'} < d_r$ and $c_{i''} < d_{r'}$ are impossible, by Lemma 9, because $\mathcal{A}(a, c)$ contains three double morphisms. If $d_r < c_{i'}$ or $d_{r'} < c_{i''}$, then the double morphism $\lambda e^{ca}_{i'i'} + \mu e^{ca}_{i''i''}$ $(\lambda \neq 0 \neq \mu)$ is a product of double morphisms in $\mathcal{A}(a, d)$ and $\mathcal{A}(d, c)$, hence $\mathcal{A}(a, c)$ contains two long double morphisms in contradiction with the arrows $a_i \to c_i$ and $a_{i''} \to c_{i''}$. Hence $c_{i'}$ is not comparable to d_r and $c_{i''}$ is not comparable to $d_{r'}$, in contradiction with Lemma 7.

5) This case is impossible by Lemma 7. The proof of Lemma 12 is thus complete.

3. A construction of a multiplicative basis. In this section we shall prove following proposition.

Proposition 2. *From every reduced scalarly multiplicative basis, we can obtain a reduced scalarly multiplicative basis by means of multiplications of the basis vectors by non-zero elements of* k.

Let Γ be the graph of a reduced scalarly multiplicative basis $\{ m^a_i, f^{ba}_l \}$ and let Γ_1 be the set of its arrows. An integral function $z : \Gamma_1 \to \mathbb{Z}$ will be called a *weight function* and its value at an arrow will be called the *weight of the arrow* if:

a) $z(\alpha_1) = -z(\alpha_2)$ for each pair of connected arrows α_1, α_2;

b) the sum of the weights of all arrows stopping at a vertex $v \in \Gamma_0$ is equal to the sum of the weights of all arrows starting from v (this sum will be called the *weight of* v and will be denoted by $z(v)$).

Lemma 13. *There exists no non-zero weight function.*

Proof. By contradiction let $z : \Gamma_1 \to \mathbb{Z}$ be a non-zero weight function. An arrow α will be called *nondegenerate* if $z(\alpha) \neq 0$.

Let $v_1 < \ldots < v_m$ be the set of all vertices of the triples of Γ. For each vertex v_i, we denote by $v_{i'}$, $v_{i''}$ the two vertices such that $\{v_i, v_{i'}, v_{i''}\}$ is a triple.

By an *elementary path of length* s we shall mean a sequence of arrows of the form

$$v_p \xrightarrow{\lambda_1} u_1 \xrightarrow{\lambda_2} u_1 \longrightarrow \ldots \longrightarrow u_{s-1} \xrightarrow{\lambda_s} v_q, \qquad (2)$$

where $u_1, \ldots, u_{s-1}$ are vertices of doubles (they may be absent, i. e., a path may consist of exactly one arrow) and $z(\lambda_1) \neq 0$. Then by Lemma 11 and item b) of the definition of a weight function, $z(\lambda_1) = z(\lambda_2) = \ldots = z(\lambda_s)$, this non-zero integer we shall call the *weight of path* (2). We shall say that the elementary path (2) *avoids* a vertex v_i if $p < i < q$. Now we establish some properties of elementary paths:

A. The intersection of two elementary paths does not contain any vertex of a double.

B. Each nondegenerate arrow is contained in an elementary path.

C. If a vertex v_i is avoided by an elementary path (2) having length at least 2, then the v_i is incomparable with some vertex u_l in this path. Otherwise, $v_p < u_1 < \ldots$ $\ldots < u_{s-1} < v_q$ implies one of the following conditions: $v_p < v_i < u_1$ or $u_j <$ $< v_i < u_{j+1}$ for some j or $u_{s-1} < v_i < v_q$. This contradicts Lemma 9 because the vertices $u_1, \ldots, u_{s-1}$ are contained in doubles.

D. If a vertex of a triple is avoided by an elementary path of length at least 2, then all other vertices of this triple can not be avoided by any elementary path having length ≥ 2. This follows from property C and Lemma 8.

E. The sum of the weights of all elementary paths avoiding a vertex v_i is equal to $-z(v_i)$. Indeed, this is obvious for v_1 because, by property B, only arrows having weight 0 can stop at v_1. If property E is true for v_i, then the sum of the weights of all elementary paths avoiding v_i and starting from v_i is equal to 0. But the set of these paths coincides with the set of all elementary paths avoiding v_{i+1} and stopping at v_{i+1}. Hence property E is true for v_{i+1}.

F. Let a triple $\{b_1, b_2, b_3\}$ satisfy the following conditions: 1) there is no nondegenerate arrow starting from $a < b_1$; 2) there is a pair of connected degenerate strong arrows starting from (b_1, b_2) or (b_1, b_3), 3) there is a pair of connected nondegenerate weak arrows starting from (b_2, b_3). Then there exists a triple $\{a_1, a_2, a_3\}$ satisfying the same conditions and $a_1 < b_1$. Indeed, let for definiteness the pair of connected degenerate strong arrows start from (b_1, b_2). From $z(b_1) = 0$, $z(b_2) = -z(b_3) \neq 0$ and properties D and E, it follows that b_2 or b_3 is avoided by a nondegenerate arrow. Let b_3 be avoided by a nondegenerate arrow $a_i \to c_j$. Then $a_i < b_3 < c_j$. By Lemma 9, there exists a path $a_i \to \ldots \to b_3 \to \ldots \to c_j$ consisting of strong arrows. But by Lemma 12, there is only a weak arrow starting from b_3. Hence b_2 is avoided by some nondegenerate arrow $a_i \to c_j$. By Lemma 9, it is a weak arrow, $i = j$ and there is a path $a_i \to \ldots \to b_2 \to \ldots \to c_i$ consisting of strong arrows. But there is only one strong arrow starting from b_2 and it is connected with an arrow starting from b_1. Hence the arrows connected with $a_i \to \ldots \to b_2 \to \ldots \to c_i$ compose the path $a_{i'} \to \ldots \to b_1 \to \ldots \to c_{i'}$. The triple $\{a_1, a_2, a_3\}$ satisfies our requirements.

Let c_l be the vertex such that there is a nondegenerate arrow starting from c_l and there is no nondegenerate arrow starting from $b < c_l$. Then there is no nondegenerate arrow stopping at c_l, hence $z(c_l) = 0$ and there are two arrows starting from c_l and having the weights n and $-n$, moreover $l = 1$ and the arrows connected with them start from c_2 and c_3. Since $z(c_2) = -z(c_3) = \pm n \neq 0$, the vertices c_2 and c_3 are avoided by elementary paths, and one of them is a nondegenerate arrow. Let for definiteness c_2 be avoided by a nondegenerate arrow $b_i \to d_j$. By Lemma 9, $i = j$ and there is a path $b_i \to \ldots \to c_2 \to \ldots \to d_i$. Since there exists exactly one arrow starting from c_2 and this arrow is connected with an arrow starting from c_1, we have that the arrows connected with $b_i \to \ldots \to c_2 \to \ldots \to d_i$ compose the path $b_{i'} \to \ldots \to c_1 \to \ldots \to d_{i'}$. Since $b_{i'} < c_1$, there is no nondegenerate arrow starting from $b_{i'}$. Hence the arrow $b_i \to d_i$ is connected with the arrow $b_{i''} \to d_{i''}$, where $i' \neq i''$, and $i' = 1$. By applying property F to the triple $\{b_1, b_2, b_3\}$, we obtain a triple $\{a_1, a_2, a_3\}$. By applying property F to the triple $\{a_1, a_2, a_3\}$, we obtain another triple and so on. This contradicts the finiteness of the graph Γ. This proves our Lemma.

***Proof of Proposition* 2.** We number all vertices and all arrows of the graph Γ:

$$\Gamma_0 = \{a_1, a_2, \dots, a_r\}, \quad \Gamma_1 = \{f_{11}, f_{12}, \dots, f_{s1}, f_{s2}\},$$

where $f_{j1}\colon a_{p(j1)} \to a_{q(j1)}$ and $f_{j2}\colon a_{p(j2)} \to a_{q(j2)}$ are two connected arrows and $a_{p(j1)} < a_{p(j2)}$. Let the basis vector m_i correspond to the vertex a_i and let the double morphism f_i correspond to the pair (f_{j1}, f_{j2}). Then $f_j m_{p(j1)} = m_{q(j1)}$ and $f_j m_{p(j2)} = = \lambda_j m_{q(j2)}$, where λ_j is the parameter of a double morphism f_j.

By changes of the basis vectors

$$m_i = x_i m_i', \quad 0 \neq x_i \in k, \tag{3}$$

we obtain a new set of double morphisms: $f_j' = x_{p(j1)} x_{q(j1)}^{-1} f_j$, $1 \le j \le s$, with the parameters $\lambda_j' = \lambda_j x_{p(j1)} x_{q(j1)}^{-1} x_{p(j2)}^{-1} x_{q(j2)}$.

The change (3) gives a multiplicative basis if $\lambda_1' = \lambda_2' = \dots = \lambda_s' = 1$, i.e., if $x_1, x_2, \dots, x_r$ satisfy the system of equations

$$\lambda_j x_{p(j1)} x_{p(j2)}^{-1} = x_{q(j1)} x_{q(j2)}^{-1}, \quad 1 \le j \le s. \tag{4}$$

We shall solve the system by elimination: solve the first equation for some x_i and substitute the result in other equations. This amounts to the multiplication of each of them by a rational power of the first equation. Futher we solve the second equation of the obtained system for some x_j and substitute the result in other equations... There are two possibilities:

1. After the sth step, we obtain the solution $(x_1, \dots, x_t) \in (k\setminus\{0\})^t$ of (4).

2. After the $(t-1)$th step $(1 < t \le s)$, we obtain a system, the tth equation of which does not contain unknowns. In this case, the tth equation of (4), up to scalar multiples λ_t, is the product of rational powers of the 1th, ..., $(t-1)$th equations. It means that there exist integers $z_1, \dots, z_t$ such that $z_t \neq 0$ and the equality

$$(x_{p(11)} x_{p(12)}^{-1})^{z_1} \dots (x_{p(t1)} x_{p(t2)}^{-1})^{z_t} = (x_{q(11)} x_{q(12)}^{-1})^{z_1} \dots (x_{q(t1)} x_{q(t2)}^{-1})^{z_t} \tag{5}$$

is the identity, i.e., each x_i has the same exponents at the two sides of (5).

Define the integer function $z\colon \Gamma_1 \to \mathbb{Z}$ by $z(f_{j1}) = -z(f_{j2}) = z_j$ for $j \le t$ and $z(f_{j1}) = z(f_{j2}) = 0$ for $j > t$. Since x_i corresponds to the vertex a_i of Γ, we have by (5) that this function is a non-zero weight function, which contradicts Lemma 13. Hence case 2 is impossible. This finishes the proof of Proposition 2.

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Received 23. 06. 92