UDC 517.53

P. Koosis, prof. (McGill Univ., Montreal)

A RELATION BETWEEN TWO RESULTS CONCERNING ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

СПІВВІДНОШЕННЯ МІЖ ДВОМА РЕЗУЛЬТАТАМИ ПРО ЦІЛІ ФУНКЦІЇ ЕКСПОНЕНЦІАЛЬНОГО ТИПУ

It is shown that the Beurling – Malliavin multiplier theorem for the entire functions of exponential type can be derived from certain estimates for polynomials on the complex plane.

Показано, що теорема Бьорлінга – Малявіна про мультиплікатори для цілих функцій експоненціального типу може бути виведена з деяких оцінок поліномів на комплексній площині.

The sum $\sum_{-\infty}^{\infty} \log^+|P(n)|/(1+n^2)$ may be regarded as a discrete analog of the integral $\int_{-\infty}^{\infty} \left(\log^+|P(t)|/(1+t^2)\right) dt$. It is well known that the size of a *polynomial* $P(z)$ is controlled on the whole complex plane by the latter expression; it is thus not too surprising that, with suitable precautions, the former one can also be used for the same purpose. The following theorem is indeed true:

Theorem. *There exist numerical constants* $\eta_0 > 0$ *and* k *such that, for any polynomial* $P(z)$ *with* $\sum_{-\infty}^{\infty} \log^+|P(n)|/(1+n^2) = \eta \le \eta_0$, *the relation* $|P(z)| \le \le C_\eta e^{k\eta|z|}$ *holds for all complex* z *with* C_η *depending on* η *but not on* P.

This result can be found in [1, p. 520]. In this book, the restriction to *small* values of $\eta > 0$ is indeed necessary, as is shown by examples. For *even* polynomials $P(z)$ with $P(0) = 1$, a theorem of this sort was already published in 1966 (see [2]), and the main work is in the proof for this case. From there, the passage to general polynomials is rather easy. Treatment of the special case is straightforward in principle, but made intricate by various technical difficulties.

The establishment of the result was originally motivated by a desire to deduce from it a new proof of the multiplier theorem due to Beurling and Malliavin [3]. However, up to now, this project has not been realized. The purpose of the present article is to demonstrate the method of deduction. In passing, we shall obtain some auxiliary propositions of independent interest.

I thank Henrik Pedersen for having drawn my attention to some mistakes and obscurities in a preliminary version of this paper.

1. Extension of the Result to the Entire Functions of Small Exponential Type. The idea of such an extension is proposed as Problem 24 in [1, p. 518]; there, the reader is asked to imitate the proof of the result for polynomials. One can also, however, arrive at the extension directly.

Lemma. *Let* $f(z) = \prod_1^{\infty}\left(1 - z^2/\lambda_k^2\right)$, *where* $\lambda_k > 0$, *is of exponential type* α. *Then, for all sufficiently large integers* N, $f_N(z) = \prod_{\lambda_k \le N}\left(1 - z^2/\lambda_k^2\right)$ *satisfies*

$$\sum_1^{\infty} \frac{\log^+|f_N(n)|}{n^2} \le \sum_1^{\infty} \frac{\log^+|f(n)|}{n^2} + C\alpha + o(1).$$

Here, C *is a numerical constant and the term* $o(1)$ *tends to zero as* $N \to \infty$.

Proof. Let $\nu(t)$ denote the number of λ_k (counting multiplicities) in $(0, t]$ and set $\nu_N(t) = \nu(t) - \nu(N)$ for $t \ge N$ with $\nu_N(t) = 0$ for $0 \le t \le N$. The last function is, in fact, zero on an interval $[0, N+\varepsilon)$, where $\varepsilon > 0$.

We have

$$\log|f_N(x)| = \log|f(x)| - \int_N^\infty \log\left|1 - x^2/t^2\right| d\nu_N(t).$$

Integration by parts converts the right-hand side to

$$\log|f(x)| + \int_N^\infty \frac{2x^2}{t^2 - x^2}\,\frac{\nu_N(t)}{t}\,dt$$

because $\nu(t)$ is $O(t)$ for large t. Here, the *integral* — call it $g_N(x)$ — is positive and increasing for $0 \le x \le N$, so

$$\sum_1^{N-1} \frac{g_N(n)}{n^2} \le 4 \int_0^N \frac{g_N(x)}{x^2}\,dx,$$

and

$$\sum_1^{N-1} \frac{\log^+|f_N(n)|}{n^2} \le \sum_1^{N-1} \frac{\log^+|f(n)|}{n^2} + 4\int_0^N \frac{g_N(x)}{x^2}\,dx. \tag{1}$$

Since $f(z)$ is of exponential type α, the standard application of Jensen's formula gives $\nu(t)/t \le e\alpha + o(1)$ for $t \to \infty$ (see [1, p. 5], problem 1). Thence,

$$\int_0^N \frac{g_N(x)}{x^2}\,dx = \int_0^N \int_N^\infty \frac{2}{t^2 - x^2}\,\frac{\nu_N(t)}{t}\,dt\,dx =$$

$$= \int_N^\infty \frac{\nu_N(t)}{t} \log\left|\frac{t+N}{t-N}\right| \frac{dt}{t} \le \frac{\pi^2}{4}(e\alpha + o(1)), \tag{2}$$

with the term $o(1)$ tending to zero as $N \to \infty$. (We have used the relation

$$\int_1^\infty \log\left(\frac{\tau+1}{\tau-1}\right)\frac{d\tau}{\tau} = \frac{\pi^2}{4}.)$$

To estimate $\sum_N^\infty \left(\log^+|f_N(n)|\right)/n^2$, note that

$$\log|f_N(x)| = \int_0^N \log\left|\frac{x^2}{t^2} - 1\right| d\nu(t)$$

is increasing for $x \ge N$; hence, in $[N, \infty)$, $\log|f_N(x)|$ is ≥ 0 exactly on an interval of the form $[x_0, \infty)$, where $x_0 \ge N$. Let M be the first integer $> x_0$. Then, by virtue of the fact that $\log|f_N(x)|$ increases on $[N, \infty)$,

$$\sum_N^\infty \frac{\log^+|f_N(n)|}{n^2} = \sum_M^\infty \frac{\log|f_N(n)|}{n^2} \le$$

$$\le 4\int_M^\infty \frac{\log|f_N(x)|}{x^2}\,dx = 4\int_M^\infty \int_0^N \log\left(\frac{x^2}{t^2} - 1\right) d\nu(t)\,\frac{dx}{x^2}.$$

The last expression, after doing its inner integral by parts, becomes

$$4\nu(N)\int_M^\infty \log\left(\frac{x^2}{N^2}-1\right)\frac{dx}{x^2} + 4\int_M^\infty\int_0^N \frac{2}{x^2-t^2}\frac{\nu(t)}{t}\,dt\,dx.$$

The first term is

$$4\,\frac{\nu(N)}{N}\int_{M/N}^\infty \log(\xi^2-1)\frac{d\xi}{\xi^2} \le$$

$$\le \frac{4\nu(N)}{N}\int_{\sqrt{2}}^\infty \log(\xi^2-1)\frac{d\xi}{\xi^2} \le (4e\alpha+o(1))\log(3+2\sqrt{2})$$

with $o(1)$ tending to zero as $N\to\infty$. The second term is

$$4\int_0^N \frac{\nu(t)}{t}\log\left|\frac{t+M}{t-M}\right|\frac{dt}{t} \le 4\int_0^M \frac{\nu(t)}{t}\log\left|\frac{t+M}{t-M}\right|\frac{dt}{t} =$$

$$= 4\int_0^1 \frac{\nu(M\tau)}{M\tau}\log\left|\frac{\tau+1}{\tau-1}\right|\frac{d\tau}{\tau} \le \pi^2(e\alpha+o(1)),$$

where $o(1)$ also tends to zero as $N\to\infty$ (because $M\ge N$ and

$$\int_0^1 \log\left(\frac{1+\tau}{1-\tau}\right)\frac{d\tau}{\tau} = \frac{\pi^2}{4}).$$

Thus,

$$\sum_N^\infty \frac{\log^+|f_N(n)|}{n^2} \le \left(\pi^2+4\log\left(3+2\sqrt{2}\right)\right)(e\alpha+o(1)).$$

Combining this with (2) and (1) yields the lemma.

Theorem. *Let $f(t)$ be an entire function of exponential type α. Suppose that*

$$\sum_{-\infty}^\infty \frac{\log^+|f(n)|}{1+n^2} = \eta.$$

Provided that α and η are both less than a certain numerical constant c_0, we have, for all z,

$$|f(z)| \le C_{\alpha,\eta}\,e^{\kappa(\alpha+\eta)|z|}$$

with a numerical constant κ and $C_{\alpha,\eta}$ depending on α and η but not on f.

Proof. First, let $f(z)$ be even and have only real zeros and let $f(0)=1$. Then the condition in the hypothesis makes $\sum_1^\infty(\log^+|f(n)|)/n^2 \le 2\eta$; so, by the lemma,

$$\sum_1^\infty \frac{\log^+|f_N(n)|}{n^2} \le C\alpha + 2\eta + \varepsilon_N \quad (\text{with } \varepsilon_N\to 0 \text{ as } N\to\infty)$$

for the polynomials $f_N(z)$ considered there. Thence,

$$\sum_{-\infty}^\infty \frac{\log^+|f_N(n)|}{1+n^2} \le 2C\alpha + 4\eta + 2\varepsilon_N;$$

hence, if $2C\alpha + 4\eta$ is less than the number η_0 appearing in the theorem cited in the introduction, we have, by that result, for sufficiently large N,

$$|f_N(z)| \le Ke^{(2C\alpha+4\eta+2\varepsilon_N)k|z|},$$

with K depending only on $2C\alpha + 4\eta + 2\varepsilon_N$. Letting $N \to \infty$, we get an estimate of the desired form $f(z)$.

When $f(z)$ is even and equal to one at the origin but with *complex* zeros, we can construct the even function $g(z)$ with *real* zeros having the same moduli as those of f, and $g(0) = 1$. Then $g(z)$ is of exponential type and, indeed, $|f(z)| \le |g(i|z|)|$, whereas $|g(x)| \le |f(x)|$ on $\mathbb{R}$. Therefore, the result just obtained implies that $g(z)$ and, *hence*, $f(z)$ satisfy the required inequality when the hypothesis holds for f.

In the *general* case, we may take $f(z)$ to be *real* on $\mathbb{R}$ since both $f(z) + \overline{f(\bar{z})}$ and $(f(z) - \overline{f(\bar{z})})/2i$ have that property. Then, for any $\eta > 0$, one has a constant M_η such that $\sum_{-\infty}^{\infty} (\log u(n))/(1+n^2) < 3\eta$ for $u(z)$ equal either to

$$1 + z^2(f(z) + f(-z))^2/M_\eta \quad \text{or to} \quad 1 + (f(z) - f(-z))^2/M_\eta$$

whenever the hypothesis holds for f (for details, see [1, p. 519 – 522]). Both these functions $u(z)$ are entire, of exponential type $\le 2\alpha$, even, and equal to 1 at the origin. Consequently, when η and α are small enough, our estimate holds *for them*. An estimate of the same form then holds for $f(z)$. The theorem is proved.

2. A Weak Multiplier Theorem. The principal result of this section is contained in the following lemma:

Lemma. *Let $f(z)$ be an entire function of exponential type α with*

$$\int_{-\infty}^{\infty} (\log^+ |f(x)|/(1+x^2))\, dx < \infty.$$

If α is sufficiently small, then there exists an entire function $\varphi(z) \not\equiv 0$ of exponential type $\le \pi$ with $|\varphi(x)|$ and $|f(x)\varphi(x)|$ bounded on $\mathbb{R}$.

Proof. We prove this lemma by using the Fourier analysis and duality. We first reduce our situation to the case of an entire function $g(z)$ with modulus ≥ 1 on $\mathbb{R}$, having all its zeros on $\mathcal{J}z < 0$. For this purpose, let $G(z) = f(z)\overline{f(\bar{z})}$; this function is entire, of exponential type $\le 2\alpha$, and ≥ 1 (sic!) on $\mathbb{R}$. We also have $\int_{-\infty}^{\infty} (\log G(x)/(1+x^2))\, dx < \infty$; thus, by the well-known Akhiezer theorem (see [1, p. 55 – 58; 4, p. 125, 132; 5, p. 567]), one can write $G(z) = g(z)\overline{g(\bar{z})}$ with $g(z)$ being an entire function of exponential type $\le \alpha$ (the *same* on the upper and lower half planes, see [1, p. 66]), and having all its zeros on $\mathcal{J}z < 0$. Since $|g(x)| = \sqrt{1 + |f(x)|^2}$ on $\mathbb{R}$, the lemma will follow if we can construct an entire function $\varphi(z) \not\equiv 0$ of exponential type $\le \pi$ with $|\varphi(x)g(x)|$ bounded on $\mathbb{R}$.

We have $\int_{-\infty}^{\infty} (\log|g(x)|/(1+x^2))\, dx < \infty$, whence, for $\mathcal{J}z > 0$,

$$\log|g(z)| \le \alpha \mathcal{J}z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathcal{J}z \log|g(t)|}{|z-t|^2}\, dt$$

and, similarly, for $x \in \mathbb{R}$,

$$\log|g(x)| \le \alpha + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|g(t+i)|}{(x-t)^2+1}\,dt$$

(see [1, p. 38, 47 – 52; 4, p. 92, 93; 5, p. 311]). Substituting $x = n$ into the second relation, dividing by $1 + n^2$, and adding, we find, by using the first formula and the relation $\sum_{-\infty}^{\infty} 1/(((t-n)^2+1)(n^2+1)) \le \text{const}/(t^2+1)$, that

$$\sum_{-\infty}^{\infty} \frac{\log|g(n)|}{1+n^2} < \infty, \tag{3}$$

with $\log|g(t)| \ge 0$ on $\mathbb{R}$.

Denote the zeros of $g(z)$ by λ_k; we have $\mathcal{J}\lambda_k < 0$ for each k, and, without loss of generality, the λ_k may be taken as distinct. Indeed, if this is not so, we may split each multiple zero λ_k of $g(z)$ into simple ones very close to it (by adding different small negative imaginary quantities to λ_k without, however, altering the corresponding exponential factors in the Hadamard factorization of g) and then, after multiplication by a suitable constant, the new function will have the modulus $\ge |g(x)|$ on $\mathbb{R}$ and still satisfy (3). This new function can then be used instead of $g(z)$ in what follows.

I say that if α (the type of $f(z)$) is *small* enough, the $e^{i\lambda_k t}$ *cannot be complete* in $L_2(-\pi, \pi)$. For each λ_k, the $e^{i\lambda_k t}$ has, on $(-\pi, \pi)$, the Fourier series

$$\sum_{-\infty}^{\infty} (-1)^n \frac{\sin \pi\lambda_k}{\pi} \frac{1}{\lambda_k - n} e^{int}.$$

Therefore, if the exponentials *were* complete in $L_2(-\pi, \pi)$, the functions of n equal to $(-1)^n/(\lambda_k - n)$ *and, hence, those equal to* $1/(\lambda_k - n)$ would, by Parseval's theorem, be complete in $l_2(\mathbb{Z})$.

But this cannot be true when α is small. Otherwise, there would be a sequence of finite sums $s_r(n)$, each of the form $\sum_k a_k/(\lambda_k - n)$, such that $s_r(n) \xrightarrow[r]{} \delta(n)$ in $l_2(\mathbb{Z})$ (the Kronecker δ is equal to 1 for $n = 0$ and to 0 for $n \ne 0$). Each $s_r(n)$ can be expressed as $g_r(n)/g(n)$ with an entire function $g_r(z)$ of exponential type $\le \alpha$; $g_r(z)$ is obtained by dividing $g(z)$ by the factors $\lambda_k - z$ corresponding to the denominators in the sum $s_r(n)$ and then multiplying the quotient by the other linear factors, fewer in number. We thus have

$$\sum_{-\infty}^{\infty} \left|(g_r(n)/g(n)) - \delta(n)\right|^2 \xrightarrow[r]{} 0.$$

This implies, firstly, that $|g_r(n)| \le \text{const}|g(n)|$ on $\mathbb{Z}$ and, secondly, by dominated convergence, the definition of $\delta(n)$, and (3), that

$$\sum_{-\infty}^{\infty} \frac{\log^+|g_r(n)/g(0)|}{1+n^2} \xrightarrow[r]{} 0.$$

For small $\alpha \ge 0$, this and the theorem in Section 1 imply that $|g_r(z)| \le$

$\leq K_\gamma |g(0)| e^{\kappa\gamma|z|}$ when r is large. Here, κ is a numerical constant, γ may be taken to be any number $> \alpha$, and K_γ depends only on γ. A subsequence of $g_r(z)$ thus tends to an entire function $h(z)$ of exponential type $\leq \kappa\gamma$ with $h(n) = \delta(n) g(0)$ on $\mathbb{Z}$. This, however, is *impossible* for $\kappa\gamma < 1/e$, since $h(z)/g(0)$, although 1 at 0, would vanish at every $n \neq 0$ in $\mathbb{Z}$ (see the proof of the lemma in Section 1). This means that, for small α, the $e^{i\lambda_k t}$ are not complete in $L_2(-\pi, \pi)$, as claimed.

In this case, we have a *nonzero* $\Psi(t) \in L_2(-\pi, \pi)$ with $\int_{-\pi}^{\pi} e^{i\lambda_k t}\Psi(t)\,dt = 0$ at each λ_k. The function $\psi(z) = \int_{-\pi}^{\pi} e^{izt}\Psi(t)\,dt$, entire, of exponential type $\leq \pi$, and $\not\equiv 0$. Thus, it vanishes at each zero of $g(z)$, so the ratio $\varphi(z) = \psi(z)/g(z)$ is entire and of exponential type by the Lindelöf theorem [1, p. 22]. On $\mathbb{R}$, this ratio is *bounded* (for $\psi(x)$, this is *clearly* true) and $|g(x)| \geq 1$. We can thence conclude that $\varphi(z)$ is, in fact, of exponential type $\leq \pi$ (see [4, p. 127; 5, p. 207 – 208, 315, 605]). The product $\varphi(x) g(x) = \psi(x)$ is bounded for real x; therefore, in view of $\psi(x) \not\equiv 0$, the theorem is proved.

Remark. The following observations (prompted by a question of Peter Jones) really fall outside the scope of the present discussion but their inclusion here is perhaps nevertheless worthwhile.

Given any $W(n) \geq 1$ *such that* $\sum_{-\infty}^{\infty} (\log W(n)/(1+n^2)) < \infty$, *there are* c_n, *not all zero, with* $\sum_{-\infty}^{\infty} |c_n| W(n) < \infty$ *(and, hence, in particular,* $|c_n| \leq \text{const}/W(n)$*), such that* $\sum_{-\infty}^{\infty} c_n e^{in\lambda}$ *vanishes on the interval* $-h \leq \lambda \leq h$, *where* $h > 0$.

Verification of this fact uses the idea from the proof presented above. There is no loss of generality in supposing that $W(n) \to \infty$ for $n \to \pm\infty$, since otherwise, one may replace $W(n)$ by $(1+n^2)W(n)$ in what follows.

If $h > 0$ is *small enough*, the finite linear combinations of $e^{i\lambda n}/W(n)$ with $-h \leq \lambda \leq h$ *cannot be uniformly dense* in $C_0(\mathbb{Z})$. Otherwise, there would be a sequence of finite sums $g_r(z)$, each of the form $\sum_{-h \leq \lambda \leq h} a_\lambda e^{i\lambda z}$, with $g_r(n)/W(n)$ tending to $\delta(n)$ (the Kronecker δ-function) *uniformly on* $\mathbb{Z}$ as $r \to \infty$. From this, one arrives at a contradiction for small $h > 0$ by arguing just as in the proof given above but using the condition on $\log W(n)$ instead of (3).

Since $W(n) \to \infty$ as $n \to \pm\infty$, the result just established gives us, by duality, a sequence of b_n, not all zero, with $\sum_{-\infty}^{\infty} |b_n| < \infty$ and $\sum_{-\infty}^{\infty} (e^{i\lambda n}/W(n))\, b_n = 0$ for $-h \leq \lambda \leq h$. Our statement thus holds with $c_n = b_n/W(n)$.

Suppose now that we *have* this sequence $\{c_n\}$. Let us construct a complex measure μ on $\mathbb{Z}$ by setting $\mu(\{n\}) = c_n$. Taking the convolution of μ with the function $\Delta(x) = \max(1 - 4|x|, 0)$, we obtain a nonzero $u(x)$ in $L_1(-\infty, \infty)$, *vanishing* on each interval $n + \frac{1}{4} \leq x \leq n + \frac{3}{4}$, $n \in \mathbb{Z}$, and with $|u(x)| \leq \text{const}/W(n)$ if $|x - n| \leq \frac{1}{4}$ for such n. The Fourier transform $\int_{-\infty}^{\infty} e^{i\lambda x} u(x)\,dx$ *also* vanishes for $-h \leq \lambda \leq h$.

No regularity of $W(n)$ is required for these results under the condition that $\sum_{-\infty}^{\infty} (\log W(n)/(1+n^2))\,dx < \infty$.

One can give a *necessary and sufficient* condition on $W(n) \geq 1$ for the conclusion of the statement given above to hold. This condition is not altogether explicit, and we restrict our discussion to the functions $W(n)$ tending to ∞ as $n \to \pm\infty$ in order to save time..

Given any $h > 0$, we let $W_h(z)$ denote, for each complex z, the supremum of $|\varphi(z)|$ for the entire functions $\varphi(z)$ of exponential type $\le h$, bounded on the real axis and with $|\varphi(n)/W(n)| \le 1$ for $n \in \mathbb{Z}$.

Provided that $W(n) \ge 1$ *tends to* ∞ as $n \to \pm\infty$, *a sequence of* c_n, *not all zero and with* $\sum_{-\infty}^{\infty} |c_n| W(n) < \infty$ *and* $\sum_{-\infty}^{\infty} c_n e^{in\lambda}$ *vanishing on an interval of positive length, exists if and only if* $\sum_{-\infty}^{\infty} (\log W_h(n)/(1+n^2)) < \infty$ *for sufficiently small values of* $h > 0$.

Proof of the *sufficiency* is similar to that of our first observation; it suffices to note that the functions $g_r(z)$ appearing therein actually satisfy (by definition!) the relation $|g_r(n)| \le \text{const}\, W_h(n)$ for $n \in \mathbb{Z}$.

The *necessity* follows from classical results. If a sequence $\{c_n\}$ having the stipulated properties exists, the function $\sum_{-\infty}^{\infty} c_n e^{ina} e^{in\lambda}$, under a suitable choice of the parameter a, vanishes on the interval $-h \le \lambda \le h$, $h > 0$; finite linear combinations of $e^{i\lambda n}/W(n)$ with λ from this interval are, therefore, *not* uniformly dense in $C_0(\mathbb{Z})$. A result of Mergelian [1, p. 174], thence, implies (a fortiori!) that $\int_{-\infty}^{\infty} (\log W_h(x)/(1 + x^2))\, dx < \infty$. From this, one can easily derive, by using the argument similar to that in [1, p. 523 – 524], that $\sum_{-\infty}^{\infty} (\log W_h(n)/(1+n^2)) < \infty$. (Cf. the proof of (11) in Section 4 below and the proof of (3).)

3. A Lemma on Poisson Integrals. Assume that, in the lemma in Section 2, one can take the type α of $f(z)$ to be *arbitrarily large*. This gives the Beurling – Malliavin theorem on the multiplier. Such an extension is possible. For this purpose, we need a result concerning the Poisson integral

$$U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{J} z U(t)}{|z - t|^2}\, dt$$

constructed, for $\mathcal{J} z > 0$, from a *positive* function $U(t) \not\equiv 0$ with $\int_{-\infty}^{\infty} (U(t)/(1 + t^2))\, dt < \infty$.

For each $x \in \mathbb{R}$ and $y > 0$, the ratio $U(x+iy)/y$ is a *strictly decreasing* function of y tending (by dominated convergence) to 0 as $y \to \infty$. Given any fixed $a > 0$, there exists, hence, a definite $Y(x) \ge 0$ such that $U(x+iy) < ay$ for $y > Y(x)$, while $U(x+iy) \ge ay$ for $0 \le y \le Y(x)$ (it is possible that $Y(x) = 0$ if $U(t)$ vanishes at x).

The set

$$\mathcal{D}_a = \{(x, y);\ y > Y(x)\}$$

is thus a domain on the upper half plane and $\Omega_a = \{\mathcal{J} z \ge 0\} \sim \mathcal{D}_a$ is a closed region lying above and on the real axis, whose interior may consist of several components. Concerning Ω_a, we get the important lemma from [6].

Lemma (Beurling and Malliavin, 1967). *For* $a > 0$, *we have*

$$\iint_{\Omega_a} \frac{dx\,dy}{1+x^2+y^2} < \infty.$$

For the reader's convenience, we sketch the proof. Fixing a, consider the function $V(z) = a\mathcal{J}z - U(z)$, harmonic and >0 in $\mathcal{D}_a$ and *zero* on its boundary $y = Y(x)$. Fix any $y_0 > 0$, for which $V(iy_0) > 0$.

For each $r > Y(0)$, there exists an open arc $\sigma(r)$ of the circle $|z| = r$ lying entirely in $\mathcal{D}_a$ with endpoints on the curve $y = Y(x)$ and on the opposite sides of the y-axis. The *union* of these $\sigma(r)$ is a domain $\mathcal{D} \subseteq \mathcal{D}_a$ (perhaps properly). We set $\Omega = \{\mathcal{J}z \geq 0\} \sim \mathcal{D}$. Then $\Omega \supseteq \Omega_a$ and we proceed to show that $\iint_\Omega dx\,dy/(1 + x^2 + y^2) < \infty$, which implies the lemma.

If $R > y_0$, we denote by $\mathcal{D}(R)$ the part of $\mathcal{D}$ lying *within* the arc $\sigma(R)$ making $iy_0 \in \mathcal{D}(R)$. The harmonic function $V(z)$ is $\leq aR$ on $\sigma(R)$ and *zero* on the rest of $\partial\mathcal{D}(R)$; thus, we have

$$V(iy_0) \leq aR\,\omega_R(\sigma(R), iy_0), \tag{4}$$

where $\omega_R(\ ,\ )$ is a harmonic measure for $\mathcal{D}(R)$.

Writing $r\theta(r)$ for the *length* of $\sigma(r)$, we have, by the Ahlfors–Carleman relation [7, p. 102],

$$\omega_R(\sigma(R), iy_0) \leq \text{const}\exp\left(-\pi\int_{y_0}^{R}\frac{dr}{r\theta(r)}\right).$$

The endpoints of every arc $\sigma(r)$ are of the form $re^{i\varphi(r)}$ and $re^{i(\pi-\psi(r))}$ with $\varphi(r)$ and $\psi(r)$ both ≥ 0 and $<\pi/2$. Thence, $\theta(r) = \pi - \varphi(r) - \psi(r)$, and we can use the relation $1/\theta(r) \geq 1/\pi + (\varphi(r) + \psi(r))/\pi^2$ when estimating the integral to the right. In this way, we obtain

$$\omega_R(\sigma(R), iy_0) \leq \frac{\text{const}\,y_0}{R}\exp\left(-\frac{1}{\pi}\int_{y_0}^{R}\frac{(\varphi(r)+\psi(r))}{r}\,dr\right).$$

The integral here is just $\iint_{\Omega\cap\{y_0<|z|<R\}} dx\,dy/(x^2+y^2)$; so, if $\iint_\Omega dx\,dy/(1 + x^2 + y^2) = \infty$, we must have $\omega_R(\sigma(R), iy_0) = o(1/R)$ as $R \to \infty$. Substituting this into (4), we get $V(iy_0) = 0$, arriving at a contradiction. The lemma is proved.

4. Construction of Two Majorants. We now take any fixed entire function $F(z)$ of exponential type with $|F(x)| \geq 1$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty}(\log|F(x)|/(1+x^2))\,dx < \infty$. The function $\log|F(x)|$ is then continuously differentiable and, indeed, real analytic at the points $x \in \mathbb{R}$. There is no loss of generality in supposing that all the zeros of $F(z)$ lie on $\mathcal{J}z < 0$ [1, p. 54; 4, p. 90], and this property is henceforth *assumed*.

We first show how to construct, for any $h > 0$, a *majorant* $W(x)$ of $|F(x)|$ with $|\log W(x) - \log W(x')| \leq h|x - x'|$ on $\mathbb{R}$ and also $\int_{-\infty}^{\infty}(\log|W(x)|/(1+x^2))\,dx < \infty$.

Let us start by considering the open set

$$O = \{x \in \mathbb{R};\ \log|F(\xi)| - \log|F(x)| > h(\xi - x)\ \text{for some}\ \xi > x\},$$

outside of which $d \log|F(x)|/dx$ is everywhere $\leq h$; O is a finite or countable union of disjoint open intervals. All of these intervals must have finite length because, otherwise, there would be a sequence of ξ_k tending to ∞ with $\liminf_{k\to\infty} (\log|F(\xi_k)|/\xi_k \geq h$, and this would contradict the well known fact that the functions $F(z)$ under consideration have zero exponential growth along the real axis [1, p. 174; 4, p. 97; 5, p. 315].

We thus have $O = \bigcup_k I_k$ with the disjoint finite intervals $I_k = (a_k, b_k)$. Their construction is best visualized by imagining the parallel rays of light, all of slope h, shining *downwards* upon the graph of $\log|F(x)|$ vs x. Certain disjoint portions of this graph will thus be *cast in shadow*, and the intervals I_k lie precisely *under* these.

From this observation, it is clear that, for each $I_k = (a_k, b_k)$, we have

$$\log|F(b_k)| = \log|F(a_k)| + h(b_k - a_k). \tag{5}$$

Let us now define a function $\omega_+(x)$ by setting

$$\omega_+(x) = \begin{cases} \log|F(x)| & \text{for } x \notin O, \\ \log|F(a_k)| + h(x - a_k) & \text{if } a_k < x < b_k. \end{cases}$$

Then $\omega_+(x)$ is continuous, piecewise smooth, and $\geq \log|F(x)|$ on $\mathbb{R}$. All the discontinuities of $\omega'_+(x)$ are at the *left endpoints* a_k of the intervals of O; elsewhere, $\omega'_+(x)$ exists and is $\leq h$. At any a_k, both $\omega'_+(a_k-)$ and $\omega'_+(a_k+)$ exist, with the *former* equal to the derivative of $\log|F(x)|$ at a_k and, hence, $\leq h$; the *latter* is equal to h.

In addition, we have the following lemma:

Lemma.

$$\int_{-\infty}^{\infty} \frac{\omega_+(x)}{1 + x^2}\, dx < \infty.$$

Proof. Since $F(z)$ has all its zeros on $\mathfrak{I}z < 0$, we may just as well assume $|F(x + iy)|$ to be an *increasing* function of $y \geq 0$ for each real x; in any case, this will be so if we replace $F(z)$ by $e^{-icz}F(z)$ with sufficiently large $c > 0$. (This can be verified by the logarithmic differentiation of the Hadamard product; see [4, p. 226; 5, p. 457].)

We have

$$\int_{\mathbb{R}\sim O} \frac{\omega_+(x)}{1 + x^2}\, dx = \int_{\mathbb{R}\sim O} \frac{\log|F(x)|}{1 + x^2}\, dx < \infty.$$

Hence, it is only necessary to show that

$$\int_{O} \frac{\omega_+(x)}{1 + x^2}\, dx < \infty.$$

For this purpose, we apply the lemma in Section 3 to the function

$$U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathfrak{I}z \log|F(t)|}{|z - t|^2}\, dt,$$

harmonic and (without loss of generality) strictly positive in $\mathfrak{I}z > 0$. Note that, for a suitable choice of the number $A > 0$, we have

$$\log|F(z)| = A\mathcal{I}z + U(z) \tag{6}$$

on the same half plane (see [1, p. 47; 4, p. 92; 5, p. 311].

Consider an arbitrary interval $I_k = (a_k, b_k)$. Since $\log|F(b_k + iy)|$ increases for $y > 0$, we have

$$U(b_k + iy_k) \geq \frac{1}{2}\log|F(b_k)| = A\, y_k$$

for

$$y_k = \frac{1}{2A}\log|F(b_k)| \tag{7}$$

by (6). From (5), we get $A y_k \geq h(b_k - a_k)/2$; hence, the Harnack theorem gives us a constant c depending on h and A with

$$U(x + iy_k) \geq cA\, y_k \quad \text{for} \quad a_k \leq x \leq b_k.$$

The rectangle of height y_k with a base on the interval I_k belongs, therefore, to the set Ω_a appearing in the lemma in Section 3 if a is taken equal to cA. By this lemma, we thus have

$$\sum_k \int_0^{y_k}\int_{a_k}^{b_k} \frac{dxdy}{1 + x^2 + y^2} < \infty.$$

When $a_k \geq 0$, the denominator in the corresponding integral appearing on the left is, by virtue of (7), $\leq 1 + b_k^2 + (\log|F(b_k)|)^2/4A^2 \leq 1 + \text{const}\, b_k^2$, $F(z)$ being of exponential type. Referring again to (7), we see that

$$\sum_{a_k \geq 0} \frac{(b_k - a_k)\log|F(b_k)|}{1 + b_k^2} < \infty. \tag{8}$$

This and (5) imply, in particular, that $\sum_{a_k \geq 0}(b_k - a_k)^2/(1 + b_k^2) < \infty$, but then there can only *finitely many* $b_k \geq 1$ with $b_k > 2a_k$. Thence, by (8),

$$\sum_{b_k \geq 1} \frac{(b_k - a_k)\log|F(b_k)|}{1 + a_k^2} < \infty.$$

Therefore, by the definition of $\omega_+(x)$, we get

$$\sum_{b_k \geq 1} \int_{I_k} \frac{\omega_+(x)}{1 + x^2}\, dx < \infty.$$

The corresponding sum over the I_k with $a_k \leq -1$ seems to be convergent, and the remaining I_k (if there are any) lie on $[-1, 1]$, where $\omega_+(x)$ is certainly bounded. Thus,

$$\int_O \frac{\omega_+(x)}{1 + x^2}\, dx = \sum_k \int_{I_k} \frac{\omega_+(x)}{1 + x^2}\, dx < \infty.$$

The lemma is proved.

Our next step is to obtain a continuous piecewise smooth $\omega_-(x) \geq \log|F(x)|$ with both $\omega_-'(x-)$ and $\omega_-'(x+)$, being not less than $-h$, by analogy with the construction of $\omega_+(x)$. We may obtain the function $\omega_-(x)$ even by applying the latter procedure directly to $\log|F(-x)|$ instead of $\log|F(x)|$ and then changing the sign of x. It is readily seen that

$$\int_{-\infty}^{\infty} \frac{\omega_-(x)}{1+x^2}\, dx < \infty;$$

for this, it suffices to repeat the proof of the lemma with $\overline{F(-\bar{z})}$ instead of $F(z)$.

With $\omega_+(x)$ and $\omega_-(x)$ at hand, we finally take

$$\omega(x) = \max(\omega_+(x), \omega_-(x))$$

and set $W(x) = e^{\omega(x)}$. Thus, we obtain the following statement:

Theorem. *For the function $W(x)$ just defined, we have*

$$|F(x)| \le W(x), \quad x \in \mathbb{R},$$

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\, dx < \infty,$$

and

$$|\log W(x) - \log W(x')| \le h|x - x'| \quad \textit{on } \mathbb{R}.$$

Proof. The first two relations follow directly from the corresponding properties of $\omega_+(x)$ and $\omega_-(x)$.

The last one is a geometrically evident consequence of the inequalities $\omega'_+(x) \le h$ and $\omega'_-(x) \ge -h$ and the definition of $\omega(x)$; let us, nevertheless, give a formal proof in order that there be no doubt.

For any given x_0, $\omega(x_0) = \max(\omega_+(x_0), \omega_-(x_0))$ is either *equal* to $\log|F(x_0)|$ or exceed this quantity. In the former case, $\omega_-(x_0)$ and $\omega_+(x_0)$ should be both *equal* to $\log|F(x_0)|$; thus, x_0 lies outside the two open sets where any of the first two functions is greater than the last one. This makes $d\log|F(x)|/dx$ both $\le h$ and $\ge -h$ at x_0 (see the description of O above). At such point x_0, $\omega'_-(x+)$ has *the same* value as the preceding derivative, while $\omega'_+(x+)$ is *at least* as large, being, however, $\le h$. We see that $\omega'(x_0+)$ exists and lies between $-h$ and h.

When $\omega(x_0) > \log|F(x_0)|$ and $\omega(x)$ coincides either with $\omega_+(x)$ or with $\omega_-(x)$ on a *neighborhood* of x_0, $\omega'(x_0)$ exists and is either h or $-h$. The remaining possibility here is that $\omega(x) = \omega_-(x)$ for $x_0 - \eta < x \le x_0$ and $\omega(x) = \omega_+(x)$ for $x_0 \le x < < x_0 + \eta$, where $\eta > 0$. Then $\omega'(x_0+)$ exists and is equal to h.

The continuous function $\omega(x)$ thus has a right-hand derivative $\omega'(x+)$ at every point x, with $-h \le \omega'(x+) \le h$. This implies that $|\omega(x) - \omega(x')| \le h|x - x'|$ on $\mathbb{R}$. The theorem is proved.

The construction just carried out enables us to realize another one, interesting in its own right.

Theorem. *If $F(z)$ is an entire function of exponential type with $|F(x)| \ge 1$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} (\log F(x)/(1+x^2))\, dx < \infty$, then there exist entire functions $H(z)$ of arbitrarily small exponential type with $|H(x)| \ge |F(x)|$ on $\mathbb{R}$ and*

$$\int_{-\infty}^{\infty} (\log H(x)/(1+x^2))\, dx < \infty.$$

Proof. Fixing any $h > 0$, we construct the majorant $W(x)$ of $|F(x)|$ having the properties ensured by the preceding theorem and show how to obtain an entire function $H(z)$ of exponential type $\leq 2h$ satisfying the condition imposed above on $\log|H(x)|$, with $|H(x)| \geq W(x)$ on $\mathbb{R}$. The procedure has been used elsewhere [8, p. 302 – 303], and we explain it here for the reader's convenience.

We set

$$\Omega(x) = \pi(x^2+1)(W(x))^2,$$

and then define $M(z_0)$ (for any complex z_0) as the *supremum* of $|f(z_0)|$ for the entire functions $f(z)$ of exponential type $\leq h$, bounded on $\mathbb{R}$, with

$$\int_{-\infty}^{\infty} (|f(x)|^2/\Omega(x))\, dx \leq 1. \tag{9}$$

For these f, we have [1, p. 47 – 52; 4, p. 92 – 93; 5, p. 311]

$$\log|f(z)| \leq h|\mathcal{J}z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\mathcal{J}z|\,\log|f(t)|}{|z-t|^2}\, dt; \tag{10}$$

hence, by (9) and the inequality between the arithmetic and geometric means, we get

$$\log|f(z)| \leq h|\mathcal{J}z| + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\mathcal{J}z|}{|z-t|^2}\log\left(\frac{\Omega(t)}{\pi}\right) dt,$$

provided that $|\mathcal{J}z| \geq 1$. Furthermore, we have

$$\log|f(x)| \leq h + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(t+i)|}{(x-t)^2+1}\, dt$$

for $x \in \mathbb{R}$, so, by the preceding relation and Fubini's theorem,

$$\log|f(x)| \leq 2h + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log(\Omega(t)/\pi)}{(x-t)^2+4}\, dt, \qquad x \in \mathbb{R}.$$

This holds for all f of the considered kind satisfying (9); hence, by the definition of M,

$$\log M(x) \leq 2h + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log\left((t^2+1)(W(t))^2\right) dt}{(x-t)^2+4}$$

on $\mathbb{R}$ and, thence,

$$\int_{-\infty}^{\infty} \frac{\log M(x)}{1+x^2}\, dx < \infty. \tag{11}$$

Substituting $\log|f(t)| \leq \log M(t)$ on the right-hand side of (10), we see, by the last relation, that the collection of our entire functions $f(z)$ satisfying (9) is a normal family on the complex plane. This means that, to get the supremum $M(z_0)$, it is not necessary to use the *whole* collection of the functions f; it suffices to take an arbitrary subset of it, dense therein in the norm

$$\sqrt{\int_{-\infty}^{\infty} (|f(t)|^2 / \Omega(t))\, dt}\,.$$

One can now show that $(M(x))^2$ coincides on $\mathbb{R}$ with an entire function of exponential type $\le 2h$. Take any countable subset in our collection of functions $f(z)$ *dense* therein in the above-mentioned norm. By applying the Schmidt orthogonalization process to this subset, we arrive at a *complete* sequence $\{p_n(z)\}$ of entire functions of exponential type $\le h$, bounded on $\mathbb{R}$, and *orthonormal* for the inner product $\int_{-\infty}^{\infty} (p(x)\overline{q(x)} / \Omega(x))\, dx$. According to the observation just made, $M(x)$ can be then obtained as a supremum of finite linear combinations of $p_n(x)$ satisfying (9). From this, a simple computation based on the Schwarz inequality shows that

$$(M(x))^2 = \sum_n |p_n(x)|^2, \quad x \in \mathbb{R}.$$

For every N, $G_N(z) = \sum_{n \le N} p_n(z)\overline{p_n(\bar{z})}$ is an entire function bounded on $\mathbb{R}$ of exponential type $\le 2h$. Moreover, $G_N(x) \le (M(x))^2$; thus, by an analog of (10),

$$\log|G_N(z)| \le 2h|\mathcal{J}z| + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{|\mathcal{J}z| \log M(t)}{|z-t|^2}\, dt \tag{12}$$

for all N. This and (11) imply that $G_N(z)$ form a normal family in the complex plane. When $N \to \infty$, they converge on $\mathbb{R}$ (to $(M(x))^2$); hence, they tend everywhere to an entire function $G(z)$ also satisfying (12), and it can be thus readily verified that $G(z)$ is of exponential type $\le 2h$. (To estimate the growth of $G(z)$ inside the sectors of the form $|\arg z| < \delta$ and $|\arg z - \pi| < \delta$, we first use (12) to get uniform estimates for $G_N(z)$ on the *boundaries* of these sectors and then apply Phragmén – Lindelöf twoker to derive uniform estimates *inside* them.)

Since $G(x) = (M(x))^2$, we have

$$\int_{-\infty}^{\infty} (\log G(x) / (1 + x^2))\, dx < \infty$$

by virtue of (11).

Finally, it remains to show that a suitable multiple of $(M(x))^2$ dominates $W(x)$; *that is where we can use the property*

$$|\log W(x) - \log W(x')| \le h|x - x'|. \tag{13}$$

Fixing any $x_0 \in \mathbb{R}$, we consider the test function

$$f_0(z) = \cos h\sqrt{(z - x_0)^2 - R_0^2}$$

with $R_0 = (\log W(x_0)) / \sqrt{2}h$. Since the Taylor series of $\cos w$ involves only even powers of w, $f_0(z)$ is an entire function of exponential type h bounded on the real axis. We have $\log f_0(x_0) \le (\log W(x_0)) / \sqrt{2}$. So, $\log|f_0(x)| \le \log W(x)$ for $|x - x_0| \le R_0$ by (13). The same is true when $|x - x_0| > R_0$ because then $|f_0(x)| \le 1 \le W(x)$. Thus, $|f_0(x)| \le W(x)$ on $\mathbb{R}$; hence, by the definition of $\Omega(x)$, we get

$$\int_{-\infty}^{\infty} (|f_0(x)|^2 / \Omega(x))\, dx \le 1.$$

Thence,

$$M(x_0) \geq f_0(x_0) \geq \frac{1}{2}(W(x_0))^{1/\sqrt{2}}$$

and, finally,

$$G(x) = (M(x))^2 \geq \frac{1}{4}(W(x))^{\sqrt{2}}$$

on $\mathbb{R}$, making $4G(x) \geq W(x) \geq |F(x)|$ there since $|F(x)| \geq 1$. The theorem follows if we take $H(z) = 4G(z)$.

5. Theorem on the Multiplier.

Theorem (Beurling and Malliavin [3]). *If $f(z)$ is an entire function of exponential type and $\int_{-\infty}^{\infty} (\log^+|f(x)| / (1 + x^2))\, dx < \infty$, then, for any $\eta > 0$, there exists an entire function $\psi(z) \not\equiv 0$ of exponential type $\leq \eta$ with $(1 + |f(x)|)\psi(x)$ bounded on $\mathbb{R}$.*

Proof. Let $G(z) = 1 + f(z)\overline{f(\bar{z})}$; then $G(z)$ is an entire function of exponential type, and $G(x) = 1 + |f(x)|^2 \geq 1$ on $\mathbb{R}$. We also have $\int_{-\infty}^{\infty} (\log G(x)/(1 + x^2))\, dx < \infty$. Therefore, the Akhiezer theorem, already used in Section 2, gives us an entire function $g(z)$ of exponential type with all its zeros on $\mathcal{J}z < 0$, such that $G(z) = g(z)\overline{g(\bar{z})}$. Given $\eta > 0$, we set

$$F(z) = g\left(\frac{\pi z}{\eta}\right);$$

$F(z)$ satisfies the hypothesis of the second theorem in Section 4; hence, for any $h > 0$, there exists an entire function $H(z)$ of exponential type $\leq 2h$ with $|H(x)| \geq |F(x)|$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} (\log|H(x)| / (1 + x^2))\, dx < \infty$.

Taking $h > 0$ small enough, we get from the lemma in Section 2 an entire function $\varphi(z) \not\equiv 0$ of exponential type $\leq \pi$ with $H(x)\varphi(x)$ bounded on $\mathbb{R}$. The product $F(x)\varphi(x)$ is thus bounded on $\mathbb{R}$ and the desired conclusion holds with $\psi(z) = \varphi(\eta z/\pi)$. The theorem is proved.

Remark. Beurling and Malliavin also proved in [3] that if $W(x) \geq 1$ has the last two properties enumerated in the first theorem of Section 4, then there are entire functions $\psi(z) \not\equiv 0$ of arbitrarily small exponential type with $W(x)\psi(x)$ bounded on $\mathbb{R}$. At the end of [8], it was shown that this result *follows* from the theorem just proved; for this purpose, the construction realized to establish the second theorem in Section 4 was used. By the first theorem in Section 4, we now see that the result just stated also *implies* the theorem in the present section. These two results are thus equivalent.

1. *Koosis P.* The logarithmic integral. I. – Cambridge: Univ. press, 1988.
2. *Koosis P.* Weighted polynomial approximation on arithmetic progressions of intervals or points // Acta Math. – 1966. – **16**. – P. 223 – 277.
3. *Beurling A, Malliavin P.* On Fourier transforms of measures with compact support // Ibid. – 1962. – **107**. – P. 291 – 309.
4. *Boas R.* Entire functions. – New York: Acad. press, 1954.
5. *Левин Б. Я.* Распределение корней целых функций. – М: Гостехиздат, 1956.
6. *Beurling A, Malliavin P.* On the closure of characters and the zeros of entire functions // Acta Math. – 1967. – **118**. – P. 79 – 93.
7. *Fuchs W.* Topics in the theory of functions of one complex variable. – Princeton: Van Nostrand, 1967.
8. *Koosis P.* Harmonic estimation in certain slit regions and a theorem of Beurling and Malliavin // Acta Math. – 1979. – **142**. – P. 275 – 304.

Received 29. 07. 93