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# TAME AND WILD SUBSPACE PROBLEMS

# РУЧНІ ТА ДИКІ ЗАДАЧІ ПРО ПІДПРОСТОРИ

Let $B$ be a finite-dimensional algebra over an algebraically closed field $k$, $\mathcal{B}_d = \operatorname{Spec} k[\mathcal{B}_d]$ the affine algebraic scheme whose $R$-points are the $B \otimes_k k[\mathcal{B}_d]$-module structures on $R^d$, and $M_d$ the canonical $B \otimes_k k[\mathcal{B}_d]$-module supported by $k[\mathcal{B}_d]^d$. Further, let us say that an affine subscheme $\mathcal{V}$ of $\mathcal{B}_d$ is *classtrue* if the functor $F_{\mathcal{V}}: X \mapsto M_d \otimes_{k[\mathcal{B}]} X$ induces an injection between the sets of isomorphism classes of indecomposable finite-dimensional modules over $k[\mathcal{V}]$ and $B$. If $\mathcal{B}_d$ contains a classtrue plane for some $d$, then the schemes $\mathcal{B}_e$ contain classtrue subschemes of arbitrary dimensions. Otherwise, each $\mathcal{B}_d$ contains a finite number of classtrue punctures straight lines $\mathcal{L}(d, i)$ such that: For each $n$, almost each indecomposable $B$-module of dimension $n$ is isomorphic to some $F_{\mathcal{L}(d,i)}(X)$; furthermore, $F_{\mathcal{L}(d,i)}(X)$ is not isomorphic to $F_{\mathcal{L}(l,j)}(Y)$ if $(d, i) \neq (l, j)$ and $X \neq 0$. The proof uses a reduction to subspace problems, for which an inductive algorithm permits us to prove corresponding statements.

Нехай $B$ – скінченновимірна алгебра над алгебраїчно замкненим полем $k$, $\mathcal{B}_d = \operatorname{Spec} k[\mathcal{B}_d]$ — афінна алгебраїчна схема, $R$-точки якої є $B \otimes_k k[\mathcal{B}_d]$-модульними структурами на $R^d$, і $M_d$ — канонічний $B \otimes_k k[\mathcal{B}_d]$-модуль на $k[\mathcal{B}_d]^d$. Афінну підсхему $\mathcal{V}$ схеми $\mathcal{B}_d$ будемо називати вірною, якщо функтор $F_{\mathcal{V}}: X \mapsto M_d \otimes_{k[\mathcal{B}]} X$ індукує ін'єкцію між множинами класів ізоморфності нерозкладних скінченновимірних модулів над $k[\mathcal{V}]$ і $B$. Якщо $\mathcal{B}_d$ містить вірну площину для деякого $d$, то схеми $\mathcal{B}_e$ містять вірні підсхеми довільної розмірності. У противному разі кожна $\mathcal{B}_d$ містить скінченну кількість вірних перфорованих прямих $\mathcal{L}(d, i)$, для яких для будь-якого $n$ майже кожний нерозкладний $B$-модуль розмірності $n$ ізоморфний деякому $F_{\mathcal{L}(d,i)}(X)$, причому модуль $F_{\mathcal{L}(d,i)}(X)$ не ізоморфний $F_{\mathcal{L}(l,j)}(Y)$, якщо $(d, i) \neq (l, j)$ та $X \neq 0$. Доведення використовує редукцію до задач про підпростори, для яких індуктивний алгоритм дає змогу довести відповідні твердження.

**1. Notations, terminology, objective.** Throughout the paper, $k$ denotes an algebraically closed field.

By $\mathcal{A}$ we denote a $k$-category, i. e. a category whose morphism sets $\mathcal{A}(X, Y)$ are endowed with vector space structures over $k$ such that the composition maps are bilinear. Furthermore, we suppose that $\mathcal{A}$ is an *aggregate* (over $k$), i. e. that the spaced $\mathcal{A}(X, Y)$ have finite dimensions over $k$, that $\mathcal{A}$ has finite direct sums and that each idempotent $e \in \mathcal{A}(X, X)$ has a kernel. As a consequence, each $X \in \mathcal{A}$ is a finite direct sum of indecomposables, and the algebra of endomorphisms of each indecomposable is local. We shall denote by $\mathfrak{S}$ a *spectroid* of $\mathcal{A}$, i. e. the full subcategory formed by chosen representatives of the isoclasses of indecomposables, by $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathfrak{S}}$ the *radicals of* $\mathcal{A}$ and $\mathfrak{S}$.

Typical examples of aggregates are provided by the category $\operatorname{proj} A$ of finitely generated projective *right* modules over a finite-dimensional algebra $A$, or by the category $\operatorname{mod} A$ of all finite-dimensional *right* $A$-modules. The aggregate $\operatorname{proj} A$ has a finite spectroid, $\operatorname{mod} A$ in general not.

A *pointwise finite*(left) *module* $M$ over $\mathcal{A}$ is by definition a $k$-linear functor from $\mathcal{A}$ to $\operatorname{mod} k$. For instance, in the examples considered above, each $N \in \operatorname{mod} A^{\mathrm{op}}$ yields a module $P \mapsto P \otimes_A N$ over $\operatorname{proj} A$, each $L \in \operatorname{mod} A$ a series of modules $X \mapsto \mathrm{Ext}^n_A(L, X)$ over mod $A$.

With each module $M$ over $\mathcal{A}$ we associate a new aggregate $M^k$ whose objects are the *M-spaces*, i. e. the triples $(V, f, X)$ formed by a space $V \in \mathrm{mod}\, k$, an object $X \in \mathcal{A}$ and a linear map $f: V \to M(X)$. A morphism from $(V, f, X)$ to $(V', f', X')$ is determined by morphisms $\varphi: V \to V'$ and $\xi: X \to X'$ such that $f'\varphi = M(\xi)f$.

Let $\mathcal{L} = (K, J, \ldots)$ be a *bond* on $M$, i. e. a *finite* set of submodules. We say that $(V, f, X) \in M^k$ *avoids* $\mathcal{L}$ if $f^{-1}(L(X)) = \{0\}$ for each $L \in \mathcal{L}$. The triples which avoid $\mathcal{L}$ form a full subaggregate of $M^k$ which we denote by $M^k_{\mathcal{L}} = M^k_{K, J, \ldots}$.

When $V$ and $X$ are fixed, the triples $(V, f, X) \in M^k$ may be identified with the points of the space $\mathrm{Hom}_k(V, M(X))$. The triples avoiding $\mathcal{L}$ then correspond to the points of a (Zariski-)open subset $\mathrm{Hom}^{\mathcal{L}}_k(V, M(X))$ which inherits from $\mathrm{Hom}_k(V, M(X))$ the structure of an algebraic variety. Our objective is to examine the "number of parameters" occurring in an algebraic family of maps $f \in \mathrm{Hom}^{\mathcal{L}}_k(V, M(X))$ such that the triples $(V, f, X)$ are indecomposable and pairwise nonisomorphic.

**2. Formulation of the main theorems.**

**2. 1.** With the notations introduced above, let $e = (e_0, \ldots, e_t)$ be a *coordinate system* of an affine subspace $S$ of $\mathrm{Hom}_k(V, M(X))$, i. e. a sequence of vectors $e_i \in \mathrm{Hom}_k(V, M(X))$ such that the map

$$k^t \to \mathrm{Hom}_k(V, M(X)), \quad x \to e_0 + x_1 e_1 + \ldots + x_t e_t$$

induces a bijection $k^t \cong S$. Then $e$ provides a functor $F_e: \mathrm{rep}\, Q^t \to M^k$, where $\mathrm{rep}\, Q^t$ is the aggregate formed by the finite-dimensional representations of the quiver $Q^t$ with 1 vertex and $t$ arrows: $F_e$ maps a sequence $a \in \mathrm{rep}\, Q^t$ of $t$ endomorphisms $a_i: W \to W$ onto the triple $(W \otimes V, f_e(a), W \otimes X)$, where $W \otimes X \in \mathcal{A}$ represents the functor $\mathrm{Hom}_k(W, \mathcal{A}(X, ?))$ (hence, $k^n \otimes X \cong X^n$) and

$$f_e(a) = \mathbb{1}_W \otimes e_0 + a_1 \otimes e_1 + \ldots + a_t \otimes e_t: W \otimes V \to W \otimes M(X) \cong M(W \otimes X).$$

The functor $F_e$ behaves well towards affine subspaces $S' \subset S$. Let $e'$ be a coordinate system of $S'$, where $e'_0 = e_0 + \sum_{i=1}^t T_{0i} e_i$ and $e'_j = \sum_{i=1}^t T_{ji} e_i$, $1 \le j \le s$. We then have $F_{e'} = F_e \circ \Phi$, where $\Phi: \mathrm{rep}\, Q^s \to \mathrm{rep}\, Q^t$ is the functor $a' \to a$ defined by $a_i = T_{0i} \mathbb{1}_W + \sum_{j=1}^s T_{ji} a'_j$, $1 \le i \le t$. In the case $S' = S$, $\Phi$ is an automorphism.

**2. 2.** Let now $R$ be an affine subspace of $\mathrm{Hom}_k(W, W)^t$ with coordinate system $d = (d_0, d_1, \ldots, d_s)$, where $d_j = (d_{j1}, \ldots, d_{jt})$. Then $d$ provides a functor $\Phi_d: \mathrm{rep}\, Q^s \to \mathrm{rep}\, Q^t$ which maps $c \in \mathrm{Hom}_k(U, U)^s$ onto $b \in \mathrm{Hom}_k(U \otimes W, U \otimes W)^t$, where $b_i = \mathbb{1}_U \otimes d_{0i} + c_1 \otimes d_{1i} + \ldots + c_s \otimes d_{si}$. A simple calculation shows that $F_e \circ \Phi_d = F_f$, where $f$ is a coordinate system of a subspace of $\mathrm{Hom}_k(W \otimes V, M(W \otimes X))$ and is defined by

$$f_0 = 1_W \otimes e_0 + d_{01} \otimes e_1 + \ldots + d_{0t} \otimes e_t$$

and

$$f_j = d_{j1} \otimes e_1 + \ldots + d_{jt} \otimes e_t, \quad 1 \le j \le s.$$

All compositions $\Phi_g \circ \Phi_d$ have the form $\Phi_h$. In the case $W = k$ and $d_{ji} = T_{ji} \in k \cong$

$\cong \mathrm{Hom}_k(k, k)$, $\Phi_d$ coincides with the functor $\Phi$ of 2.1.

***Example* 1.** Consider the affine subspace $R$ of $\mathrm{Hom}_k(k^{s+1}, k^{s+1})^2$ formed by the pairs of matrices

$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|cc} 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & x_s \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let $d$ be the coordinate system of $R$ for which $x_i$ is the $i$-th coordinate of the above pair. The associated functor $\Phi_d$: $\mathrm{rep}\, Q^s \to \mathrm{rep}\, Q^2$ maps $c \in \mathrm{Hom}_k(X, X)^s$ onto the pair $b \in \mathrm{Hom}_k(X^{s+1}, X^{s+1})^2$ represented by the matrices

$$\left[\begin{array}{ccc|cc} 0 & \mathbb{1}_x & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbb{1}_x \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|cc} 0 & c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & c_s \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

It follows that $\Phi_d$ factors through the full subaggregate $\mathrm{rep}_0 Q^2$ of $\mathrm{rep}\, Q^2$ formed by the pairs of nilpotent simultaneously trigonalizable endomorphisms. A simple calculation shows that $\Phi_d$ preserves indecomposability and *heteromorphism* ($c, c' \in \mathrm{rep}\, Q^s$ are isomorphic if so are the images $\Phi_d(c), \Phi_d(c')$).

***Example* 2** [1]. Consider the affine subspace $U$ of $\mathrm{Hom}_k(k^4, k^4)^2$ formed by the pairs of matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $g$ is the coordinate system of $U$ for which $x_i$ is the $i$-th coordinate, the associated functor $\Phi_g$: $\mathrm{rep}\, Q^2 \to \mathrm{rep}\, Q^2$ factors through the full subaggregate $\mathrm{rep}_0^c Q^2$ of $\mathrm{rep}_0 Q^2$ formed by the pairs of *commuting nilpotent* matrices. The functor $\Phi_g$ preserves indecomposability and heteromorphism.

**2. 3.** We now come back to the module $M$ restrained by a bond $\mathcal{L}$.

**Definition.** *Let $S$ be an affine subspace of dimension $t$ of* $\mathrm{Hom}_k(V, M(X))$, *and $e$ a coordinate system of $S$. We say that $S$ is $\mathcal{L}$-reliable if the functor $F_e$:* $\mathrm{rep}\, Q^t \to M^k$ *factors through $M_{\mathcal{L}}^k$ and preserves indecomposability and heteromorphism.*

**Lemma.** *Suppose that $t = 2$, that $(V, e_0, X)$ avoids $\mathcal{L}$, and that the restriction* $F_e | \mathrm{rep}_0^c Q^2$ *preserves indecomposability and heteromorphism. Then, for each $s \in$* $\mathbb{N}$, *there exists a $U \in$* mod $k$, *a $Y \in \mathcal{A}$, and an $\mathcal{L}$-reliable subspace of* $\mathrm{Hom}_k(U, M(Y))$ *of dimension $s$.*

***Proof.*** Let us set $W = k^{s+1}$ and choose $d$ as in Example 1 and $g$ as in Example 2. Then we have $F_e \circ \Phi_g \circ \Phi_d = F_f$, where $f$ is a coordinate system of an affine subspace $T$ of dimension $s$ of $\mathrm{Hom}_k(V^{4(s+1)}, M(X^{4(s+1)}))$. Since $F_e | \mathrm{rep}_0^c Q^2$ and the functor $\mathrm{rep}\, Q^s \to \mathrm{rep}_0^c Q^2$ induced by $\Phi_g \circ \Phi_d$ preserve indecomposability and heteromorphism, so does $F_f$.

It suffices now to show that $F_e$ maps $\mathrm{rep}_0 Q^2$ into $M_{\mathcal{L}}^k$. For this purpose, we call a sequence

$$0 \to (W', g', Y') \to (W, g, Y) \to (W'', g'', Y'') \to 0$$

of $M^k$ *short exact* if the induced sequences

$$0 \to W' \to W \to W'' \to 0 \text{ and } 0 \to Y' \to Y \to Y'' \to 0$$

are exact in $\operatorname{mod} k$ and split exact in $\mathcal{A}$ respectively. Now it is clear that $F_e$: $\operatorname{rep} Q^2 \to M^k$ preserves short exact sequences and that $M_{\mathcal{L}}^k$ is closed in $M^k$ under extensions (in the sequence above, $(W', g', Y') \in M_{\mathcal{L}}^k$ and $(W'', g'', Y'') \in M_{\mathcal{L}}^k$ imply $(W, g, Y) \in M_{\mathcal{L}}^k$). It follows that $F_e^{-1}\left(M_{\mathcal{L}}^k\right)$ is closed under extensions; therefore it contains $\operatorname{rep}_0 Q^2$, which is the smallest full subaggregate of $\operatorname{rep} Q^2$, closed under extensions and containing $([0], [0]) \in F_e^{-1}\left(M_{\mathcal{L}}^k\right)$.

**2. 4. Definition.** *The module $M$ over $\mathcal{A}$ is called $\mathcal{L}$-wild if, for some $V$ and $X$, there exists an $\mathcal{L}$-reliable affine subspace $S \subset \operatorname{Hom}_k(V, M(X))$ of dimension 2. It is called absolutely wild if it is $\mathcal{L}$-wild for all proper $\mathcal{L}$, i. e. for all $\mathcal{L}$ such that $M \notin \mathcal{L}$.*

Our objective is to examine the pairs $(M, \mathcal{L})$ such that $M$ is not $\mathcal{L}$-wild. For this we need the following further notion. Assume that the submodules $L \in \mathcal{L}$ contain the radical $\mathcal{R}M$ of $M$, consider $\overline{M} = M / \mathcal{R}M$ as a module over $\overline{\mathcal{A}} = \mathcal{A} / \mathcal{R}_{\mathcal{A}}$ and denote by $\overline{\mathcal{L}}$ the set of submodules $\overline{L} = L / \mathcal{R}M$ of $\overline{M}\,(L \in \mathcal{L})$. We say that $M$ is $\mathcal{L}$-*semisimple* if the obvious functor $M_{\mathcal{L}}^k \to \overline{M}_{\overline{\mathcal{L}}}^k$ is an epivalence (i. e. induces surjections on the morphism spaces, detects isomorphisms and hits each isoclass of $\overline{M}_{\overline{\mathcal{L}}}^k$).

**First main theorem.** *Let $M$ be a pointwise finite module over an aggregate $\mathcal{A}$ with finite spectroid. Then $M$ is absolutely wild or $\mathcal{L}$-semisimple for some proper $\mathcal{L}$.*

**2. 5.** For each subset $C \subset k$, we denote by $\operatorname{rep}_C Q^1$ the full subaggregate of $\operatorname{rep} Q^1$ formed by the endomorphisms with eigenvalues in $C$. It is clear that $\operatorname{rep}_C Q^1$ is closed in $\operatorname{rep} Q^1$ under extensions. The converse is true: Each full subaggregate of $\operatorname{rep} Q^1$ which is closed under extensions coincides with some $\operatorname{rep}_C Q^1$.

We apply these considerations to *punched lines of $M$*, i. e. to subsets of some $\operatorname{Hom}_k^{\mathcal{L}}\left(V, M(X)\right)$ of the form $S \setminus E$, where $S$ is a *line* (affine subspace of dimension 1) of $\operatorname{Hom}_k(V, M(X))$ and $E$ a finite subset of $S$. If $e = (e_0, e_1)$ is a coordinate system of $S$, the scalars $\lambda \in k$ such that $e_0 + \lambda e_1 \in S \setminus E$ form a cofinite subset $C$ of $k$. With these notations, the considerations developed above show that $F_e$ maps $\operatorname{rep}_C Q^1$ into $M_{\mathcal{L}}^k$. Accordingly, we say that the punched line $S \setminus E \subset \operatorname{Hom}_k^{\mathcal{L}}\left(V, M(X)\right)$ is $\mathcal{L}$-*reliable* if the functor $\operatorname{rep}_C Q^1 \to M_{\mathcal{L}}^k$ induced by $F_e$ preserves indecomposability and heteromorphism.

In the second main theorem below, we say that an $M$-space $(W, g, Y)$ is *produced* by the punched line $S \setminus E \subset \operatorname{Hom}_k\left(V, M(X)\right)$ if it is isomorphic to some image $F_e(k^n, \lambda \mathbb{1}_n + J_n)$, where $J_n$ is a nilpotent Jordan-block, $n \geq 1$ and $\lambda \in C$. This means that there are isomorphisms $w\colon W \xrightarrow{\sim} V^n$ and $y\colon Y \xrightarrow{\sim} X^n$ such that $M(y) g w^{-1}$ is the linear map $V^n \to M(X^n)$ described by the matrix with $n$ diagonal blocks $e_0 + \lambda e_1$:

$$\left[\begin{array}{cccc:c} e_0+\lambda e_1 & e_1 & 0 & 0 & \\ 0 & e_0+\lambda e_1 & e_1 & 0 & \\ 0 & 0 & e_0+\lambda e_1 & e_1 & \\ 0 & 0 & 0 & e_0+\lambda e_1 & \\ \hdashline & & & & \end{array}\right].$$

We also say that a set $\mathcal{P}$ of punched lines is *locally finite* if, for each $X \in \mathcal{A}$, $\mathcal{P}$ contains only finitely many punched lines of the form $S \setminus E \subset \operatorname{Hom}_k(V, M(Y))$, where $Y \tilde{\to} X$.

**Second main theorem.** *If $M$ is not $\mathcal{L}$-wild, there is a locally finite set $\mathcal{P}$ of $\mathcal{L}$-reliable punched lines such that:*

*a) for each $X \in \mathcal{A}$, the set of isoclasses of indecomposable $M$-spaces $(V, f, X)$ which avoid $\mathcal{L}$ and are not produced by a punched line of $\mathcal{P}$ is finite;*

*b) distinct punched lines of $\mathcal{P}$ produce non-isomorphic $M$-spaces.*

The perspicuous description of the indecomposable $M$-spaces given by the second main theorem confirms us in calling $M$ *$\mathcal{L}$-tame* (or simply *tame* in case $\mathcal{L} = \varnothing$) if it is not $\mathcal{L}$-wild.

The second main theorem also shows that $M$ is $\mathcal{L}$-wild whenever it admits a "two-parametric family" of pairwise non-isomorphic indecomposable $M$-spaces avoiding $\mathcal{L}$. Thus, to prove wildness, $\mathcal{L}$-reliability is not needed even in the weak form of Lemma 2. 3. We owe the following example to Th. Brüstle: Suppose that the spectroid $\mathfrak{S}$ of $\mathcal{A}$ has only one point $w$, that $M(w) = k^4$, and that $\mathfrak{S}(w, w)$ is the subalgebra of $k^{4\times4}$ generated by the matrices

$$t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which act on $k^4$ by matrix-multiplication. Then the $M$-spaces $(k^2, f_{\lambda\mu}, w)$, where $f_{\lambda\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda & \mu \end{bmatrix}^T$ and $\lambda, \mu \in k$, are indecomposable and pairwise non-isomorphic. Hence, $M$ is wild. But the action of the functor $F$: rep $Q^2 \to M^k$ associated with the plane $\{ f_{\lambda\mu}: \lambda, \mu \in k \}$ is already erratic on the 2-dimensional representations of $Q^2$.

**2. 6.** Finally, we consider a *finite-dimensional* $k$-algebra $B$ and the tensor-algebra $\otimes B = k \oplus B \oplus B \otimes_k B \oplus \ldots$ . We identify mod $B$ with a full subcategory of mod $\otimes B$ by the aid of the surjective canonical homomorphism $\otimes B \to B$. Accordingly, if the right $\otimes B$-module structures on a finite-dimensional vector space $V$ are interpreted as points of $\operatorname{Hom}_k(V \otimes_k B, V)$, the $B$-module structures on $V$ are identified with the points of an algebraic subvariety $\mathcal{M}_B(V)$ of $\operatorname{Hom}_k(V \otimes_k B, V)$.

As in 2. 1, each coordinate system $e = (e_0, \ldots, e_t)$ of an affine subspace $S \subset \subset \operatorname{Hom}_k(V \otimes_k B, V)$ gives rise to a functor $F_e$: rep $Q^t \to$ mod $\otimes B$ which maps a sequence $a = (a_1, \ldots, a_t)$ of $t$ endomorphisms $a_i: W \to W$ onto the space $W \otimes_k V$ equipped with the $\otimes B$-module structure

$$\mathbb{1}_W \otimes e_0 + a_1 \otimes e_1 + \ldots + a_t \otimes e_t\colon W \otimes V \otimes B \to W \otimes V.$$

We say that $S$ is $B$*-reliable* if $F_e$ factors through $\operatorname{mod} B$ and preserves indecomposability and heteromorphism.

In the case $t = 1$, we also consider *punched lines* $S \setminus E$, where $E$ is a finite subset of $S$. Setting $C = \{\lambda \in k\colon e_0 + \lambda e_1 \in S \setminus E\}$ as in 2. 5, we say that $S \setminus E$ is $B$*-reliable* if $F_e | \operatorname{rep}_C Q^1\colon \operatorname{rep}_C Q^1 \to \operatorname{mod} \otimes B$ factors through $\operatorname{mod} B$ and preserves indecomposability and heteromorphism. Under these conditions, the indecomposable $B$-modules isomorphic to $F_e(k^n, \lambda\mathbb{1}_n + J_n)$, where $n \geq 1$ and $\lambda \in C$, are called *produced* by $S \setminus E$.

**Third main theorem.** *If $B$ is a finite-dimensional $k$-algebra, one and only one of the following two statements holds*:

*a) $B$ is wild, i. e. there exists a $B$-reliable plane*;

*b) There exists a family of $B$-reliable punched lines $S_i \setminus E_i \subset \operatorname{Hom}_k(V_i \otimes_k B, V_i)$, $i \in I$, with the following properties: For each $d \in \mathbb{N}$, the number of $i \in I$ satisfying $d = \dim V_i$ is finite, and almost all isoclasses of indecomposable $B$-modules of dimension $d$ consist of modules produced by the $S_i \setminus E_i$; furthermore, if $i \neq j$, no indecomposable produced by $S_i \setminus E_i$ can be produced by $S_j \setminus E_j$.*

In case $b$), the algebra $B$ is called *tame*.

A typical example is given by the quotient $B = k[x, y]/x^3, x^2y, xy^2, y^3$ of the polynomial algebra $k[x, y]$ and by the space $V = k^{1\times 4}$ (formed by rows with 4 entries in $k$). A $B$-reliable plane $\{e_{a,b}\colon a, b \in k\}$ of $\operatorname{Hom}_k(V \otimes_k B, V)$ is then described by the matrices

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&0&a\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&b\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&a\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&b\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

(The endomorphisms $v \mapsto e_{a,b}(v \otimes z)$, where $z$ runs through the residue-classes of $1, x, y, x^2, xy, y^2$, are obtained by multiplication with the given matrices; compare with 2. 2, example 2.)

**2. 7.** Our third main theorem raises the question of the factorization of the functor $F_e\colon \operatorname{rep} Q^t \to \operatorname{mod} \otimes B$ of 2.6 through $\operatorname{mod} B$. The answer is surprisingly simple. Let $b_0 = 1_B, b_1, \ldots, b_n$ be a basis of the vector space $B$ and $b_i b_j = \sum_{l=0}^n c_{ij}^l b_l$, $1 \leq i, j \leq n$, the multiplication law. Let us further set $e_{pi}(v) = e_p(v \otimes b_i)$ for all $v \in V, p$ and $i \geq 0$ (2. 6). Then $F_e(W, a)$ lies in $\operatorname{mod} B$ if and only if $\sum_{p=0}^t a_p \otimes e_{p0} = \mathbb{1}_W \otimes \mathbb{1}_V$ and

$$\left(\sum_{q=0}^t a_q \otimes e_{qj}\right)\left(\sum_{p=0}^t a_p \otimes e_{pi}\right) = \sum_{l=0}^n c_{ij}^l \left(\sum_{s=0}^t a_s \otimes e_{sl}\right)$$

for all $i, j \geq 1$, where $a_0 = \mathbb{1}_W$. This condition is satisfied *for all* $(W, a) \in \operatorname{rep}^c Q^t$, *i.e. for all* $(W, a)$ *with commuting endomorphisms* $a_1, \ldots, a_t$, *if and only if* $e_{00} = \mathbb{1}_V, e_{10} = \ldots = e_{t0} = 0$ *and*

$$e_{0j} e_{0i} = \sum_{l=0}^n c_{ij}^l e_{0l}, \quad e_{0j} e_{pi} + e_{pj} e_{0i} = \sum_{l=0}^n c_{ij}^l e_{pl},$$

$$e_{pj}e_{pi} = 0, \quad e_{qj}e_{pi} + e_{pj}e_{qi} = 0$$

for all $i, j \geq 1$ and all $p, q$ such that $1 \leq p < q$. These equations simply mean that *the affine subspace* $S$ of $\operatorname{Hom}_k(V \otimes B, V)$ *is contained in the algebraic variety* $\mathcal{M}_B(V)$ (2. 6). Accordingly, *if* $S$ *is a line*, we have $\operatorname{rep}^c Q^1 = \operatorname{rep} Q^1$, and $F_e$ *factors through* $\operatorname{mod} B$ *if and only if* $S \subset \mathcal{M}_B(V)$.

If we require that $F_e(W, a) \in \operatorname{mod} B$ for all $(W, a) \in \operatorname{rep} Q^t$, we must further impose the conditions $e_{qj}e_{pi} = 0$ for all $i, j \geq 1$ and all $p, q$ such that $1 \leq p < q$. Thus, $F_e : \operatorname{rep} Q^t \to \operatorname{mod} \otimes B$ *factors through* $\operatorname{mod} B$ *if and only if* $S \subset \mathcal{M}_B(V)$ *and* $F_e(k^{1\times 2}, a(p, q)) \in \operatorname{mod} B$ *for all* $p, q$ *such that* $1 \leq p < q$; here we set $a(p, q)_s = 0$ if $s \neq p, q$, whereas $a(p, q)_p$ and $a(p, q)_q$ are the multiplications by the matrices $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Of course, we can also interpret the equations displayed above by saying that $F_e$ factors through $\operatorname{mod} B$ if and only if $F_e(W, a) \in \operatorname{mod} B$ holds for one single $(W, a)$ such that the endomorphisms $\mathbb{1}_W$, $a_i$, and $a_i a_j$, $1 \leq i, j \leq t$, are linearly independent. In the case $t = 2$, for instance, we can choose $W = k^{1\times 3}$ and

$$a_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

**2. 8.** The functor $F_e : \operatorname{rep} Q^t \to \operatorname{mod} B$ admits the following more traditional interpretation. Let $C_t = k\langle x_1, \ldots, x_t\rangle$ denote the free associative algebra generated by $x_1, \ldots, x_t$. The free left $C_t$-module $M_t = C_t \otimes_k V$ is then equipped with a right $\otimes B$-module structure defined by the map

$$C_t \otimes V \otimes B \xrightarrow[\mathbb{1}\otimes e_0 + \dot{x}_1 \otimes e_1 + \ldots + \dot{x}_t \otimes e_t]{} C_t \otimes V,$$

where, for each $c \in C_t$, $\dot{c}$ denotes the map $C_t \to C_t$, $y \mapsto yc$. The $C_t - \otimes B$-bimodule thus obtained gives rise to a functor

$$\operatorname{rep} Q^t \to \operatorname{mod} \otimes B, \quad (W, a) \mapsto W \otimes_{C_t} M_t$$

which is isomorphic to $F_e$. (We define a right $C_t$-module structure on $W$ by setting $wx_i = a_i(w)$, $\forall w \in W$.) The argument produced in 2. 8 shows that this functors factors through $\operatorname{mod} B$ if and only if the right $\otimes B$-module structure on $M_t$ factors through $B$.

Thus, our third main theorem improves results conjectures by Donovan and Freislich [2] and proves by Drozd [3] and Crawley–Boevey [4, 5] with the sophisticated technique of Roiter's boxes [6].

**3. Preparative lemmas.**

**3. 1. Lemma.** *The module* $M : X \mapsto X^3$ *over the aggregate* $\mathcal{A} = \operatorname{mod} k$ *is absolutely wild.*

***Proof.*** We must show that $M$ is $\mathcal{L}$-wild for all proper $\mathcal{L}$. For this, we may assume that $\mathcal{L} = \{L_1, \ldots, L_r\}$ consists of maximal submodules of $M$, and hence, that there exist scalars $\lambda_i, \mu_i, \nu_i$ such that

$$L_i(X) = \{v \in X^3 : \lambda_i v_1 + \mu_i v_2 + \nu_i v_3 = 0\}.$$

Transforming $\mathcal{L}$ by an automorphism of $M$ (i. e., by an invertible $3\times 3$-matrix) if necessary, we may assume furthermore that $\lambda_i \neq 0$ for all $i$. Under these assumptions, we consider the plane $S \subset \mathrm{Hom}_k(k, M(k)) \cong k^3$ formed by the columns $[1 \quad a \quad b]^{\mathrm{T}}$. If $e_0, e_1, e_2$ are the natural basis columns, the functor $F_e$: rep $Q^2 \to M^k$ maps $(A, B) \in (k^{n \times n})^2$ onto the linear map $k^n \to M(k^n) = k^{3n}$ represented by the matrix $[\,\mathbb{1} \quad A^{\mathrm{T}} \quad B^{\mathrm{T}}\,]^{\mathrm{T}}$. We infer that $F_e$ is fully faithful. Moreover, since nilpotent simultaneously trigonalizable matrices $A, B$ give rise to invertible matrices $\lambda_i \mathbb{1}_n + \mu_i A + \nu_i B$, $F_e$ maps $\mathrm{rep}_0\, Q^2$ into $M^k_{\mathcal{L}}$. By Lemma 2. 3, $M$ is $\mathcal{L}$-wild.

**3. 2. Lemma.** *The module* $M: (X, Y) \mapsto X^2 \oplus Y^2$ *over the aggregate* $\mathcal{A} = \mathrm{mod}\, k \times \mathrm{mod}\, k$ *is absolutely wild.*

***Proof.*** The group of automorphisms of $M$ is now identified with $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$. This group acts on the finite sets of proper submodules. We may therefore suppose that, for each $L \in \mathcal{L}$, one of the columns $[1 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$ and $[0 \;\; 0 \;\; 1 \;\; 0]^{\mathrm{T}}$ does not belong to $L(k) \subset M(k) = k^2 \oplus k^2 \cong k^4$. The plane $S \subset \mathrm{Hom}_k(k, M(k))$ attached to the matrices $[1 \quad a \quad 1 \quad b]^{\mathrm{T}}$ with coordinates $a, b$ then provides a fully faithful functor $F_e$: rep $Q^2 \to M^k$ which maps $\mathrm{rep}_0\, Q^2$ into $M^k_{\mathcal{L}}$.

**3. 3.** For each natural number $t \geq 1$, we define as follows a module $M_t$ over a spectroid $\mathfrak{L}_t$ with two points $x$ and $y$. Denoting by $k[e, f]$ the algebra of polynomials in 2 indeterminates $e$ and $f$, we set $\mathfrak{L}_t(x, x) = k\mathbb{1}_x$, $\mathfrak{L}_t(y, y) = k\mathbb{1}_y$, $\mathfrak{L}_t(x, y) = \bigoplus_{i=0}^{t-1} ke^{t-1-i} f^i$, $\mathfrak{L}_t(y, x) = 0$ and $M_t(x) = ke \oplus kf$, $M_t(y) = \bigoplus_{j=0}^{t} ke^{t-j} f^j$. The structural map from $\mathfrak{L}_t(x, y) \otimes M_t(x)$ to $M_t(y)$ is induced by the multiplication of polynomials.

For instance, if $t = 4$, $\mathfrak{L}_t$ is identified with the $k$-category of paths of the quiver $x \substack{\to\\ \to\\ \to\\ \to} y$, and the linear maps $M_t(x) \to M_t(y)$ associated with the 4 arrows are represented in the natural bases by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Of course, we can interpret $\mathfrak{L}_t$ as the spectroid of an aggregate $\mathcal{A}_t$ whose objects are the formal direct sums $x^p \oplus y^q$, and $M_t$ can be extended to $\mathcal{A}_t$ by setting $M_t(x^p \oplus y^q) = M_t(x)^p \oplus M_t(y)^q$.

**Lemma.** *The module* $M_t$ *over the aggregate* $\mathcal{A}_t$ *is absolutely wild.*

***Proof.*** We may suppose that $\mathcal{L}$ consists of maximal submodules $L_1, \ldots, L_r$ of $M_t$, where $L_j(y) = M_t(y)$ and $L_j(x) = \{ue + vf : \lambda_i u + \mu_i v = 0\}$ for some $(\lambda_i, \mu_i) \in k^2 \setminus (0, 0)$. Because of the obvious equivariant action of $\mathrm{GL}_2(k)$ on $\mathfrak{L}_t$ and $M_t$, we may suppose that $\lambda_i \neq 0$ for all $i$. Under these assumptions, we consider the plane $S \subset \mathrm{Hom}_k(k^2, M_t(x^2 \oplus y))$ formed by the maps $k^2 \to M_t(x) \oplus M_t(x) \oplus M_t(y)$ represented by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & b & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$
$$\quad e \qquad\quad f \qquad e^t \quad e^{t-1}f \quad ef^{t-1} \quad f^t$$

Choosing $a$ and $b$ as coordinates of these matrices, we obtain a functor $F_e$: rep $Q^2 \to$ $\to M_t^k$ whose restriction $F_e | \text{rep}_0 Q^2$ factors through $M_{tL}^k$, preserves indecomposability and detects isomorphisms.

**3. 4.** The examples produced in 3. 3 admit the following variations. We denote by $\overline{\mathfrak{S}}_t$ the spectroid with one point $x$, endomorphism algebra $\overline{\mathfrak{S}}_t(x, x) = k\mathbb{1}_x \oplus ke^{t-1} \oplus$ $\oplus ke^{t-2}f \oplus \dots \oplus kf^{t-1}$, radical $ke^{t-1} \oplus \dots \oplus kf^{t-1}$ and radical square zero. The formal direct sums $x^p$ give rise to an aggregate $\overline{\mathcal{A}}_t$.

We further denote by $\overline{M}_t$ the $\overline{\mathcal{A}}_t$-module with stalk $\overline{M}_t(x) = ke \oplus kf \oplus ke^t \oplus$ $\oplus ke^{t-1}f \oplus \dots \oplus kf^t$ and radical $ke^t \oplus \dots \oplus kf^t$ whose structural map $\overline{\mathfrak{S}}_t(x, x) \otimes$ $\otimes (ke \oplus kf) \to \overline{M}_t(x)$ is induced by the multiplication of $k[e, f]$.

**Lemma.** *The module $\overline{M}_t$ over the aggregate $\overline{\mathcal{A}}_t$ is absolutely wild.*

***Proof.*** Use the affine plane of $\text{Hom}_k(k, \overline{M}_t(x))$ formed by the maps represented by the matrices

$$\begin{matrix} [1 & a & 0 & 0 & \dots & 0 & b]^{\text{T}}. \\ e & f & e^t & e^{t-1}f & & ef^{t-1} & f^t \end{matrix}$$

***Remark.*** Let $L$ denote the submodule $(X, Y) \mapsto X^2$ of the module $M: (X, Y) \mapsto$ $\mapsto X^2 \oplus Y$ over $\text{mod}\, k \times \text{mod}\, k$. Then $M$ is $\emptyset$-wild but not $\{L\}$-wild.

**3. 5.** We now turn to the general case of a pointwise finite $\mathcal{A}$-module $M$. Our objective is to compare the representation types of $M$ and of its factor-modules $M / N$. For this sake, *we first suppose in* 3. 5 *and* 3. 6 *that $N$ is a simple module located at some $s \in \mathfrak{S}$* ($\dim N(s) = 1, N(x) = 0$ if $x \in \mathfrak{S}$ and $x \neq s$).

Let $(V, \overline{e}, X)$ be a space over $\overline{M} := M / N$, and $e: V \to M(X)$ a factorization of $\overline{e}: V \to \overline{M}(X)$. We call *transporter* $T_e$ of $V$ into $N(s)$ the set of all maps $V \to$ $N(s)$ induced by morphisms $\mu \in \mathcal{R}_{\mathcal{A}}(X, s)$ such that $\text{Im}\, M(\mu)e \subset N(s)$. We choose some basis $g_1, \dots, g_n$ of a supplement $U$ of $T_e$ in $\text{Hom}_k(V, N(s))$, set

$$V' := \text{Hom}_k(V, N(s)) = T_e \oplus U,$$

and denote by $g$ the induced composition

$$V \xrightarrow{[g_1 \cdots g_n]^{\text{T}}} N(s)^n \xrightarrow{\text{incl.}} M(s)^n \xrightarrow{\sim} M(s^n).$$

Setting $d = [eg]^{\text{T}}$, we thus obtain an $M$-space $(V, d, X \oplus s^n)$ which, up to isomorphism, does not depend on the basis $g_1, \dots, g_n$ of $U$.

**Lemma 1.** *$(V, d, X \oplus s^n)$ avoids each submodule $L$ of $M$ such that $L \cap N =$ $= 0$.*

***Proof.*** Clearly, $e^{-1}(L(X)) \subset K := \bigcap_{\tau \in T_e} \text{Ker}\, \tau$. Since $T_e$ and $g_1, \dots, g_n$ generate $V' = \text{Hom}_k(V, N(s))$, we infer that $\bigcap_i (K \cap \text{Ker}\, g_i) = 0$, and hence, that $d = [e \quad g]^{\text{T}}$ avoids $L$.

**Lemma 2.** *If $(V, \overline{e}, X) \in \overline{M}^k$ is indecomposable, then so is $(V, d, X \oplus s^n) \in$ $\in M^k$.*

***Proof.*** We may of course suppose that $V \neq 0$. Let us further assume that $(V, d,$ $X \oplus s^n) \in \in M^k$ is decomposable. Since $(V, \overline{d}, X \oplus s^n) \in \overline{M}^k$ is the direct sum of

$(V, \bar{e}, X)$ and $(0, 0, s^n)$, $(V, d, X \oplus s^n)$ admits a direct summand of the form $(0, 0, s)$ and a retraction $(0, \rho)$: $(V, d, X \oplus s^n) \to (0, 0, s)$, where $\rho \in \mathcal{A}(X \oplus s^n, s)$. Since $(V, \bar{e}, X) \in \overline{M}^k$ has no direct summand of the form $(0, 0, s)$, $\rho \mid X$ cannot be a retraction. It follows that $\rho \mid s^n$ is a retraction, i. e., that $\rho \mid s^n = a_1\pi_1 + \ldots + a_n\pi_n + \mathcal{k}$, where the $\pi_i$ denote the canonical projections $s^n \to s$, the scalars $a_i$ are not all zero, and $\mathcal{k}$ is radical. This yields

$$0 = M(\rho)d = M(\rho \mid X)e + M(\rho \mid s^n)g = M(\rho \mid X)e + \sum_{i=n}^{n} a_i g_i,$$

where $M(\rho \mid X)e \in T_e$. This provides the wanted contradiction, since $g_1, \ldots, g_n$ is a basis of a supplement of $T_e$.

**3. 6. Lemma.** *Consider fixed maps* $e_0, e_1, e_2 \in \mathrm{Hom}_k(V, M(X))$ *and variable spaces* $W \in \mathrm{mod}\, k$ *equipped with commuting endomorphisms* $a$, $b$. *Let further* $e(a, b)$: $W \otimes V \to W \otimes M(X) \xrightarrow{\sim} M(W \otimes X)$ *denote the map* $\mathbb{1}_W \otimes e_0 + a \otimes e_1 + b \otimes e_2$ *and* $T_{e(a,b)}$ *denote the associated transporter of* $W \otimes V$ *into* $N(s)$. *Then there is a nonzero polynomial* $p$ *in two indeterminates and a fixed subspace* $U$ *of* $V' = \mathrm{Hom}_k(V, N(s))$ *such that*

$$\mathrm{Hom}_k(W \otimes V, N(s)) \xrightarrow{\sim} W^{\mathrm{T}} \otimes V' = T_{e(a,b)} \oplus W^{\mathrm{T}} \otimes U$$

*whenever* $p(a, b)$ *is invertible*.

By $W^{\mathrm{T}}$ we denote the dual of the vector space $W$.

***Proof.*** Let us denote by $u$ and $v$ the compositions

$$\mathcal{R}_{\mathcal{A}}(W \otimes X, s) \xrightarrow{\text{can.}} \mathrm{Hom}_k(M(W \otimes X), M(s)) \xrightarrow{e(a,b)^*} \mathrm{Hom}_k(W \otimes V, M(s))$$

and

$$\mathrm{Hom}_k(W \otimes V, N(s)) \xrightarrow{\text{incl.}} \mathrm{Hom}_k(W \otimes V, M(s)) \xrightarrow{\text{can.}} \mathrm{Coker}\, u,$$

where we set $f^* = \mathrm{Hom}_k(f, M(s))$. The transporter $T_{e(a,b)}$ then equals $\mathrm{Ker}\, v$. On the other hand, $u$ and $v$ are identified with the compositions

$$W^{\mathrm{T}} \otimes \mathcal{R}_{\mathcal{A}}(X, s) \xrightarrow{\mathbb{1}\otimes\text{can.}} W^{\mathrm{T}} \otimes \mathrm{Hom}_k(M(X), M(s)) \xrightarrow{\mathbb{1}\otimes e_0^* + a^{\mathrm{T}}\otimes e_1^* + b^{\mathrm{T}}\otimes e_2^*}$$

$$\xrightarrow{\mathbb{1}\otimes e_0^* + a^{\mathrm{T}}\otimes e_1^* + b^{\mathrm{T}}\otimes e_2^*} W^{\mathrm{T}} \otimes \mathrm{Hom}_k(V, M(s))$$

and

$$W^{\mathrm{T}} \otimes \mathrm{Hom}_k(V, N(s)) \xrightarrow{\mathbb{1}\otimes\text{incl.}} W^{\mathrm{T}} \otimes \mathrm{Hom}_k(V, M(s)) \xrightarrow{\text{can.}} \mathrm{Coker}\, u.$$

Interpreting $a^{\mathrm{T}}$ and $b^{\mathrm{T}}$ as multiplications by $x$ and $y$ in $W^{\mathrm{T}}$ equipped with a module structure over $\Lambda = k[x, y]$, we obtain a description of $u$ and $v$ as tensor products $W^{\mathrm{T}} \otimes_{\Lambda} u_0$ and $W^{\mathrm{T}} \otimes_{\Lambda} v_0$, where $u_0$ and $v_0$ are the $\Lambda$-linear compositions

$$\Lambda \otimes \mathcal{R}_{\mathcal{A}}(X, s) \xrightarrow{\mathbb{1}\otimes\text{can.}} \Lambda \otimes \mathrm{Hom}_k(M(X), M(s)) \xrightarrow{\mathbb{1}\otimes e_0^* + x\otimes e_1^* + y\otimes e_2^*}$$

$$\xrightarrow{\mathbb{1}\otimes e_0^* + x\otimes e_1^* + y\otimes e_2^*} \Lambda \otimes \mathrm{Hom}_k(V, M(s))$$

and

$$\Lambda \otimes \mathrm{Hom}_k(V, N(s)) \xrightarrow{\mathbb{1}\otimes\text{incl.}} \Lambda \otimes \mathrm{Hom}_k(V, M(s)) \xrightarrow{\text{can.}} \mathrm{Coker}\, u_0.$$

Now, there is a nonzero polynomial $q \in k[x, y]$ such that the kernels, images and cokernels of $\Lambda[q^{-1}] \otimes_{\Lambda} u_0$ and $\Lambda[q^{-1}] \otimes_{\Lambda} v_0$ are free. This implies that

$$T_{e(a,\,b)} = \operatorname{Ker} v \xrightarrow{\sim} W^{\mathrm{T}} \otimes_{\Lambda[q^{-1}]} \operatorname{Ker}(\Lambda[q^{-1}] \otimes {}_{\Lambda} v_0) \xrightarrow{\sim} W^{\mathrm{T}} \otimes {}_{\Lambda}\operatorname{Ker} v_0,$$

whenever $q(a, b)$ is invertible.

To conclude, we choose arbitrary scalars $\xi, \eta \in k$ satisfying $q(\xi, \eta) = 0$ and an arbitrary supplement $U$ of $T_{e(\xi,\eta)}$ in $\operatorname{Hom}_k(V, N(s))$. The canonical map

$$w_0 \colon \operatorname{Ker} v_0 \oplus \Lambda \otimes U \longrightarrow \Lambda \otimes \operatorname{Hom}_k(V, N(s))$$

then becomes bijective if we "specialize" $x, y$ to $\xi, \eta$. Hence, there is a nonzero polynomial $r$ such that $\Lambda[r^{-1}] \otimes_{\Lambda} w_0$ is bijective. So we may finally set $p = qr$.

**3. 7.** We now return to the case of an *arbitrary submodule* $N$ of $M$ and denote by $\overline{\mathcal{L}} = \{L/N \colon L \in \mathcal{L} \text{ and } L \supset N\}$ the bond on $\overline{M} = M/N$ induced by a bond $\mathcal{L}$ on $M$.

**Proposition.** *$M$ is $\mathcal{L}$-wild if $M/N$ is $\overline{\mathcal{L}}$-wild.*

***Proof.*** For each $L \in \mathcal{L}$ not containing $N$, let $s_L \in \mathfrak{S}$ be such that $L(s_L)$ does not contain $N(s_L)$. Let further $\overline{e} = (\overline{e}_0, \overline{e}_1, \overline{e}_2)$ be a coordinate-system of an $\overline{\mathcal{L}}$-reliable plane in $\operatorname{Hom}_k(V, \overline{M}(X))$, and $e = (e_0, e_1, e_2)$ a system of factorizations of the $\overline{e}_i$ through $M(X)$. Restricting $\mathfrak{S}$ to the finite full subspectroid formed by the support of $X$ and all points $s_L$, and proceeding by induction on the length of $N$, we are reduced to the case where $N$ is simple and located at some $s$. Let then $p \in k[x, y]$ and $U \subset \operatorname{Hom}_k(V, N(s))$ be chosen according to Lemma 3. 6. Let finally $g_1, \ldots, g_n$ denote a basis of $U$, $g \colon V \to N(s)^n \subset M(s^n)$ the induced map, and $\operatorname{rep}_p^c Q^2$ the full subcategory of $\operatorname{rep} Q^2$ formed by the $(W, a, b)$ such that $a, b$ commute and that $p(a, b)$ is invertible. Setting

$$d_0 = \begin{bmatrix} e_0 & g \end{bmatrix}^{\mathrm{T}} \in \operatorname{Hom}_k(V, M(X \oplus s^n))$$

and $d_1 = \begin{bmatrix} e_1 & 0 \end{bmatrix}^{\mathrm{T}}, d_2 = \begin{bmatrix} e_2 & 0 \end{bmatrix}^{\mathrm{T}}$, we prove that *the restriction*

$$F_d \,|\, \operatorname{rep}_p^c Q^2 \colon \operatorname{rep}_p^c Q^2 \longrightarrow M^k$$

*preserves indecomposability and heteromorphism and factors through* $M_{\mathcal{L}}^k$. Our proposition will then follow from Lemma 2. 3 applied to a coordinate system $(d_0 + \xi d_1 + \eta d_2, d_1, d_2)$, where $(\xi, \eta) \in k^2$ satisfies $p(\xi, \eta) \neq 0$.

The composition

$$\operatorname{rep} Q^2 \xrightarrow{F_d} M^k \xrightarrow{\text{can.}} \overline{M}^k$$

maps $(W, a, b)$ into $F_{\overline{e}}(W, a, b) \oplus (0, 0, W \otimes s^n)$. Since $F_{\overline{e}}$ preserves heteromorphism, so do $F_d$ and $F_d \,|\, \operatorname{rep}_p^c Q^2$.

In order to prove the remaining two statements, we consider some $(W, a, b) \in \operatorname{rep}_p^c Q^2$ and set

$$\overline{e}(a, b) = \mathbb{1} \otimes \overline{e}_0 + a \otimes \overline{e}_1 + b \otimes \overline{e}_2 \colon W \otimes V \longrightarrow W \otimes \overline{M}(X) \xrightarrow{\sim} \overline{M}(W \otimes X),$$

$$e(a, b) = \mathbb{1} \otimes e_0 + a \otimes e_1 + b \otimes e_2 \colon W \otimes V \longrightarrow W \otimes M(X) \xrightarrow{\sim} M(W \otimes X).$$

On account of Lemma 3. 6, $W^{\mathrm{T}} \otimes U$ is a supplement of the transporter $T_{e(\mathrm{a},\,\mathrm{b})}$ of $W \otimes V$ into $N(s)$. The $M$-space $(W \otimes V, \begin{bmatrix} e(a, b) & \varphi \end{bmatrix}^{\mathrm{T}}, W \otimes X \oplus W \otimes s^n)$ provided by a basis $\varphi_1, \ldots, \varphi_m$ of $W^{\mathrm{T}}$ and the associated map

$$\varphi\colon W \otimes V \longrightarrow N(s)^{m \times n}, \quad w \otimes v \mapsto [\varphi_i(w) g_j(v)]$$

avoids $\mathcal{L}$ by Lemma 1 of 3. 5. By Lemma 2 it is indecomposable if so is $(W, a, b)$. It is isomorphic to $F_d(W, a, b)$ as shown in the next diagram

$$\begin{array}{ccc} W \otimes V & \xrightarrow{\ \mathbb{1} \otimes g\ } & W \otimes N(s)^m \\ & \searrow^{\varphi} & \downarrow \iota \\ & & N(s)^{m \times n} \end{array}, \quad \iota(w \otimes z) = [\varphi_i(w) z_j].$$

**4. Proof of the first main theorem.**

**4. 1. Lemma.** *Let $\mathfrak{J}$ be an ideal of an aggregate $\mathcal{A}$ with spectroid $\mathfrak{S}$, $M$ a pointwise finite left module over $\mathcal{A}$, $N$ the annihilator of $\mathfrak{J}$ in $M$, and $\tilde{M}$ the module $M / \mathfrak{J}M$ over $\tilde{\mathcal{A}} = \mathcal{A}/\mathfrak{J}$. We further suppose that the induced maps $\mathfrak{J}(x, y) \to \mathrm{Hom}_k((M/N)(x), (\mathfrak{J}M)(y))$ are surjective for all $x, y \in \mathfrak{S}$. Then:*

a) *either $\mathfrak{J}^2 M = 0$, the induced functor $P\colon M_N^k \to \tilde{M}^k_{N/\mathfrak{J}M}$ is quasi-surjective, and the indecomposables annihilated by $P$ are isomorphic to some $(0, 0, s)$, where $s \in \mathfrak{S}$, $M(s) = 0$ and $\mathbb{1}_s \in \mathfrak{J}$;*

b) *or $\mathfrak{J}$ contains the identity $\mathbb{1}_t$ of one point $t \in \mathfrak{S}$ such that $\dim M(t) = 1$, the induced functor $Q\colon M_N^k \to \tilde{M}^k$ is quasi-surjective, and the indecomposables annihilated by $Q$ are isomorphic to $(0, 0, t)$ or to some $(0, 0, s)$, where $s \in \mathfrak{S}$, $M(s) = 0$ and $\mathbb{1}_s \in \mathfrak{J}$.*

The proof of the first main theorem uses Statement a) only. Statement b) will be used in Section 9.

***Proof.*** We first show that $Q$ induces surjections of the morphism spaces. Let $(V, f, X)$ and $(V', f', X')$ be two objects of $M_N^k$, and $\varphi \in \mathrm{Hom}_k(V, V')$, $\xi \in \mathcal{A}(X, X')$ two morphisms which induce a morphism $(\varphi, \bar{\xi})\colon (V, \tilde{f}, X) \to (V', \tilde{f}', X')$ of $\tilde{M}^k_{N/\mathfrak{J}M}$. By definition, we then have $M(x)f - f'\varphi = ig$ for some $g \in \mathrm{Hom}_k(V, (\mathfrak{J}M)(X'))$, where $i\colon (\mathfrak{J}M)(X') \to M(X')$ denotes the inclusion. Since $(V, f, X)$ avoids $N$, the obvious maps

$$\mathfrak{J}(X, X') \longrightarrow \mathrm{Hom}_k((M/N)(X), (\mathfrak{J}M)(X')) \longrightarrow \mathrm{Hom}_k(V, (\mathfrak{J}M)(X'))$$

are both surjective and $g$ is the image of some $\eta \in \mathfrak{J}(X, X')$. This means that $ig = M(\eta)f$ and implies $M(\xi - \eta)f = f'\varphi$. We infer that $(\varphi, \bar{\xi})\colon (V, \tilde{f}, X) \to (V', \tilde{f}', X')$ is the image of $(\varphi, \xi - \eta)\colon (V, f, X) \to (V', f', X')$.

Now, in case $\mathfrak{J}^2 M = 0$, $P$ maps $M_N^k$ into $\tilde{M}^k_{N/\mathfrak{J}M}$, and $P$ is surjective on the objects. This implies a).

In the case $\mathfrak{J}^2 M \neq 0$, $\mathfrak{S}$ admits a point $t$ such that $(\mathfrak{J}M)(t)$ is not contained in $N(t)$. The image of $\mathrm{Hom}_k((M/N)(t), (\mathfrak{J}M)(t))$ in $\mathrm{End}_k M(t)$ then contains an idempotent of rank 1. A pre-image of this idempotent in $\mathfrak{J}(t, t)$ must be invertible in $\mathfrak{S}(t, t)$, because $\mathfrak{S}(t, t)$ is local. We infer that $\mathbb{1}_t \in \mathfrak{J}$ and that $\dim M(t) = 1$. The last statement of b) now follows from the fact that $\mathfrak{S}$ contains no point $r \neq t$ such that $\mathbb{1}_r \in \mathfrak{J}$ and $M(r) \neq 0$. Otherwise, there would be morphisms $\sigma \in \mathfrak{J}(t, r)$ and $\rho \in$

$\in \mathcal{I}(t, t)$ such that $M(\rho\sigma) = 1_{M(t)}$, and the simple $\mathfrak{L}(t, t)$-module $M(t)$ would not be annihilated by the radical. So it remains to prove that $Q$ hits each isoclass of $\tilde{M}^k$. Indeed, for each $\tilde{M}$-space $(V, \tilde{f}, X)$, we can choose a factorization $f: V \to M(X)$ of $\tilde{f}$ and an isomorphism $g: V \tilde{\to} M(t)^d$, where $d = \dim V$; then $(V, [f\ g]^T, X \oplus t^d)$ avoids $N$, and its image in $\tilde{M}^k$ is isomorphic to $(V, \tilde{f}, X)$.

**4.2.** ***Remarks.*** a) The assumptions of our lemma remain valid if we factor the annihilator of $M$ out of $\mathcal{A}$. Hence, we might restrict ourselves to the case where $M$ is faithful. In this case, the maps

$$\mathcal{I}(x, y) \to \mathrm{Hom}_k((M/N)(x), (\mathcal{I}M)(y))$$

are bijective. In subcase b) it follows that $\mathcal{I}(x, y)$ is identified with $\mathfrak{L}(t, y) \otimes_k \mathfrak{L}(x, t)$. In both subcases, $\mathcal{I}$ can be completely "described" in terms of the vector spaces $I(x) = (\mathcal{I}M)(x) \subset N(x) \subset M(x)$ (where $x \neq t$ in case b). Accordingly, formal examples are constructed with ease.

b) Our concrete examples are the following. We start with a morphism $\mu \in \mathfrak{L}(s, t)$ such that $M(\mu): M(s) \to M(t)$ has rank 1. Setting $S = \mathrm{Im}\, M(\mu)$, we denote by $C_S$ the submodule of $\mathcal{A}(?, t)$ which consists of the morphisms $\xi; X \to t$ of $\mathcal{A}$ mapping $M(X)$ into $S$. Then we claim that *the assumptions of our lemma are satisfied by the ideal* $\mathcal{I}$ *generated by any submodule* $C$ *of* $C_S$ *which contains* $\mu$. Indeed, for all $x, y \in \mathfrak{L}$, the composition of $\mathfrak{L}$ maps $\mathfrak{L}(t, y) \otimes_k C(x)$ onto $\mathcal{I}(x, y)$, and $N(x)$ is the annihilator of $C(x)$ in $M(x)$. Hence, the obvious map $(M/N)(x) \to \mathrm{Hom}_k(C(x), S)$ is injective, and the transposed map $C(x) \to \mathrm{Hom}_k((M/N)(x), S)$ is surjective. Taking into account that $(\mathcal{I}M)(y)$ is the image of $\mathfrak{L}(t, y) \otimes_k S$, we infer that the double-headed arrows of the diagram

$$\begin{array}{ccc}
\mathfrak{L}(t, y) \otimes C(x) \twoheadrightarrow \mathfrak{L}(t, y) \otimes \mathrm{Hom}_k\left(\left(\frac{M}{N}\right)(x), S\right) & \tilde{\to} & \mathrm{Hom}_k\left(\left(\frac{M}{N}\right)(x), \mathfrak{L}(t, y) \otimes C(x)\right) \\
\downarrow & & \downarrow \\
\mathcal{I}(x, y) & \longrightarrow & \mathrm{Hom}_k\left((M/N)(x), (\mathcal{I}M)(y)\right)
\end{array}$$

are surjective. Hence, so is the lower arrow.

**4.3.** Let us now consider pairs $(\mathcal{A}, M)$ formed by an aggregate $\mathcal{A}$ and a pointwise finite $\mathcal{A}$-module $M$. We say that two such pairs $(\mathcal{A}, M)$ and $(\mathcal{A}', M')$ are *equivalent* if there exist a $k$-linear equivalence $E: \mathcal{A} \longrightarrow \mathcal{A}'$ and an isomorphism $M \tilde{\to} M'E$. And we say that the $\mathcal{A}$-module $M$ is *climacteric* if the pair $(\mathcal{A}/\mathcal{N}_M, M)$, where $\mathcal{N}_M$ denotes the annihilator of $M$ in $\mathcal{A}$, is equivalent to one of the absolutely wild pairs examined in 3.1, 3.2, 3.3, and 3.4.

**Lemma.** *Let* $M$ *be a pointwise finite module over an aggregate* $\mathcal{A}$ *with finite spectroid* $\mathfrak{L}$. *If* $M$ *is not semisimple and has no climacteric quotient,* $\mathfrak{L}$ *admits a morphism* $\mu \in \mathcal{R}_{\mathfrak{L}}(x, y)$ *such that* $M(\mu): M(x) \to M(y)$ *has rank* 1 *and* $M(\lambda\mu) = 0 = M(\mu\nu)$ *for all* $\lambda \in \mathcal{R}_{\mathfrak{L}}(y, z)$, $\nu \in \mathcal{R}_{\mathfrak{L}}(z, x)$, *and* $z \in \mathfrak{L}$.

***Proof.*** a) *Reduction to the case of height* 2: Let us assume that $M$ has height $h > 2$, and that the proposition is true for modules of height 2. We then denote by $S_i M$ the annihilator of $\mathcal{R}^i_{\mathcal{A}}$ in $M$. Thus $\overline{M} = M/S_{h-2}M$ has height 2. If it admits a climacteric quotient, then so does $M$. Otherwise, there is a $\rho \in \mathcal{R}_{\mathcal{A}}(x, y)$ such that $\overline{M}(\rho)$ has rank 1 and vanishes on $(\mathcal{R}\overline{M})(x)$. Since $\rho\overline{M}(x) \neq 0$, we have $\sigma\rho M(x) \neq 0$ for some $\sigma \in \mathcal{R}^{h-2}_{\mathcal{A}}(y, z)$. On the other hand, $\sigma\rho \in \mathcal{R}^{h-1}_{\mathcal{A}}(x, z)$ annihilates $(\mathcal{R}M)(x)$, and $M(\sigma\rho)$ admits a factorization

$$M(x) \xrightarrow{\rho_*} M(y)/(S_{h-2}M)(y) \xrightarrow{\sigma_*} M(z).$$

where $\rho_*$ is induced by $\rho$ and $\sigma_*$ by $\sigma$. We infer that $M(\sigma\rho)$ has rank 1.

b) Finally, we suppose that $M$ *has height* 2. Factoring out the annihilator of $M$ in $\mathcal{A}$ if necessary, we may suppose that the module $M$ is faithful. We then consider 4 cases.

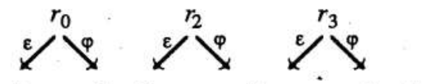
If $M/S_1M$ has an isotypic component of dimension 1 supported, say, by $x \in \mathfrak{A}$, then each nonzero radical morphism $\mu: x \to y$ of $\mathfrak{A}$ suits.

If $M/S_1M$ has an isotypic component of dimension $\geq 3$, then $M$ has a climacteric quotient of type 3. 1.

If $M/S_1M$ has at least 2 isotypic components of dimension 2, then $M$ has a climacteric quotient of type 3. 2.

If $M/S_1M$ is isotypic of dimension 2 and supported by $x \in \mathfrak{A}$, then we choose any $y \in \mathfrak{A}$ such that $\mathcal{R}_{\mathcal{A}}(x, y) \neq 0$ and consider two subclasses. If $M(\mu)$ has rank 1 for some $\mu \in \mathcal{R}_{\mathcal{A}}(x, y)$, then $\mu$ suits. If $M(\rho)$ has rank 2 for all nonzero $\rho \in \mathcal{R}_{\mathcal{A}}(x, y)$, we denote by $M'$ the sum of the isotypic components of $S_1M$ not supported by $y$. Then $N = M/M'$ has a quotient of type 3. 3 or 3. 4 according as $x \neq y$ or $x = y$:

*To prove this*, we choose two vectors $e, f \in N(x)$ whose classes modulo $S_1N$ form a basis of $(N/S_1N)(x)$. The module structure of $N$ then provides two maps $\varepsilon$, $\varphi: \mathcal{R}_{\mathcal{A}}(x, y) \overset{\to}{\to} (S_1N)(y)$ defined by $\varepsilon(\rho) = \rho e$ and $\varphi(\rho) = \rho f$. Since $M(\rho)$ has rank 2 for each $\rho \neq 0$, $a\varepsilon + b\varphi$ is injective for all $(a, b) \in k^2 \setminus (0, 0)$. By Kronecker's classification of pairs of linear maps, we can therefore choose bases $n = (n_0, \ldots, n_t)$ of $(S_1N)(y)$ and $r = (r_i)_{i \in I}$ of $\mathcal{R}_{\mathcal{A}}(x, y)$, where $I \subset \{0, 1, \ldots, t-1\}$, such that $r_ie = \varepsilon(r_i) = n_i$ and $r_if = \varphi(r_i) = n_{i+1}$ for all $i \in I$. A typical example is




where $t = 5$ and $I = \{0, 2, 3\}$.

Now we choose natural numbers $a < b$ such that $\{x \in \mathbb{N}: a \leq x < b\} \subset I$ and $a - 1 \notin I$, $b \notin I$ (for instance $a = 2, b = 4$ in the case of our diagram). Factoring out the basis vectors $n_i$ for $i < a$ and for $b < i$, we obtain a quotient $N'$ of $N$ such that $(N'/S_1N')(x) \xrightarrow{\sim} ke \oplus kf$ and $(S_1N')(y) \xrightarrow{\sim} \bigoplus_{a \leq i \leq b} kn_i$. If $\mathcal{N}$ denotes the annihilator of $N'$, the pair $(\mathcal{A}/\mathcal{N}, N')$ is equivalent to one of the pairs $(\mathcal{A}_{b-a}, M_{b-a})$ or $(\overline{\mathcal{A}}_{b-a}, \overline{M}_{b-a})$ examined in 3.3 and 3.4.

**4. 4.** ***Proof of the first main theorem*** **(2.4).** We proceed by induction on the length of $M$. If $M$ is not semisimple and has no climacteric quotient, we choose a morphism $\mu \in \mathcal{R}_{\mathfrak{A}}(x, y)$ according to Lemma 4. 3 and denote by $\mathfrak{I}$ the ideal of $\mathcal{A}$ generated by $\mu$. Then the annihilator $N$ of $\mathfrak{I}$ in $M$ is a maximal submodule of $M$, and $M/N$ is supported by $x$. By 4. 2 b), the assumptions of 4. 1 a) are satisfied. If $\tilde{M} = M/\mathcal{I}M$ is considered as module over $\tilde{\mathcal{A}} = \mathcal{A}/\mathcal{I}$, the canonical functor $M_N^k \to \tilde{M}_{N/\mathcal{I}M}^k$ is an epivalence. By induction hypothesis, $\tilde{M}$ admits a bond $\mathcal{K}$ formed by submodules $L_i/\mathcal{I}M \supset \mathcal{R}M/\mathcal{I}M$, $1 \le i \le r$, such that $\tilde{M}_{\mathcal{K}}^k \to \overline{M}_{\overline{\mathcal{K}}}^k$ is an epivalence with the notations of 2. 4 ($\overline{M} = \tilde{M}/\mathcal{R}\tilde{M} = M/\mathcal{R}M\ldots$). If we set $\mathcal{L} = \{L_1, \ldots, L_r, N\}$ and $\tilde{\mathcal{L}} = \mathcal{K} \cup \{N\}$, Lemma 4. 1 implies that the composition $M_{\mathcal{L}}^k \to \tilde{M}_{\tilde{\mathcal{L}}}^k \to \overline{M}_{\overline{\mathcal{L}}}^k$ is an epivalence.

**5. Pencils.** As in Sect. 4, $\mathcal{A}$ here denotes an aggregate with finite spectroid $\mathcal{S}$. If $M$ is a pointwise finite module on $\mathcal{A}$, we denote by $\dot{M} := \{x \in \mathcal{S} : (\mathcal{R}M)(x) \neq M(x)\}$ the *generation-indicator* of $M$. For each $p \in \dot{M}$, we write $M_p$ for the submodule of $M$ such that $M_p(p) = (\mathcal{R}M)(p)$ and $M_p(x) = M(x)$ if $x \in \mathcal{S}\setminus p$.

**5. 1. Definition.** *A pencil over $\mathcal{A}$ is a pointwise finite $\mathcal{A}$-module $P$ restrained by a proper bond $\mathcal{K}$ such that:*

a) *$P$ is not $\mathcal{K}$-wild;*

b) *there is no proper bond $\mathcal{B}$ on $P$ for which $P_{\mathcal{B}}^k$ has a finite spectroid.*

The condition b) obviously implies that $P$ admits infinitely many maximal submodules or, equivalently, that $\dim P/P_d \ge 2$ for some $d \in \dot{P}$. Proposition 4. 3 implies that such a $d$ is unique and satisfies $\dim P/P_d = 2$. We therefore call $d_p := d$ the *double-point* of $P$; any other point $s \in \dot{P}$ satisfies $\dim P/P_s = 1$ and will be called *ordinary*.

**Proposition.** *Let $(P, \mathcal{K})$ be a pencil with double-point $d$, and $(u_s)_{s\in\dot{P}\setminus d}$ a family of elements $u_s \in P(s)\setminus(\mathcal{R}P)(s)$. Let us further suppose that $\mathcal{K}$ is not empty and that $P$ is $\mathcal{K}$-semisimple. Then*

$$u + \sum_{s\in\dot{P}\setminus d} u_s \in P\left(d \oplus \bigoplus_s s\right)$$

*generates a maximal submodule of $P$ for each $u \in P(d)\setminus \bigcup_{K\in\mathcal{K}} K(d)$.*

We recall that, according to our terminology, each $K \in \mathcal{K}$ contains $\mathcal{R}P$ (2. 4).

***Proof.*** If $Q$ is the module generated by $u + \sum_s u_s =: v$, it suffices to show that $Q \supset \mathcal{R}P$ if $u \in P(d)\setminus\bigcup_K K(d)$. To this end, we set $\sum = d \oplus \bigoplus_s s$ and consider any $r \in (\mathcal{R}P)(x)$, $x \in \mathcal{S}$. The $P$-spaces $\left(k, [v' \ 0]^{\mathrm{T}}, \sum \oplus x\right)$ and $\left(k, [v' \ r']^{\mathrm{T}}, \sum \oplus x\right)$, where $v'(\lambda) = \lambda v$ and $r'(\lambda) = \lambda r$, then avoid $\mathcal{K}$ and give rise to the same $\overline{P}$-space. They are therefore connected by a morphism $\left(\mathbb{1}, \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\right)$ which is congruent to the identity modulo $\mathcal{R}_{\mathcal{A}}$ (2. 4). This means that $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}v\\ 0\end{bmatrix} = \begin{bmatrix}v\\ r\end{bmatrix}$ and implies that $\gamma v = r$ with $\gamma \in \mathcal{R}_{\mathcal{A}}(\Sigma, x)$.

**5. 2.** Proposition 5. 1 only concerned the module structure of a pencil. We now examine its bond.

**Proposition.** *For each ordinary point $s \in \dot{P}$ of a pencil $(P, \mathcal{K})$, $P_s$ belongs to $\mathcal{K}$.*

***Proof.*** Suppose that $P_s \notin \mathcal{K}$ and set $N = P_d \cap P_s, \overline{P} = P/N$ and $\overline{\mathcal{K}} = \{K/N : N \subset K \in \mathcal{K}\}$, where $d = d_P$. Then $\overline{P}$ is a semisimple pencil supported by $d$ and $s$. The functor $F : \operatorname{rep} Q^2 \to \overline{P}^k$ associated (2. 1) with the 2-parametric affine family of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & y \end{bmatrix}^{\mathrm{T}}$$

preserves indecomposability and heteromorphism. The $\overline{P}$-spaces represented by the displayed matrices avoid all proper submodules of $\overline{P}$ except $\overline{P}_s = P_s/N \notin \overline{\mathcal{K}}$. We infer that $\overline{P}$ is $\overline{\mathcal{K}}$-wild, and $P$ $\mathcal{K}$-wild (3. 7).

**5. 3.** From now on and throughout Section 5, $M$ *denotes a pointwise finite* $\mathcal{A}$-*module restrained by a bond* $\mathcal{L}$ *for which* $M$ *is not* $\mathcal{L}$-*wild.* All submodules $P$ of $M$ are implicitly supposed to be restrained by the trace $\mathcal{L} \cap P := \{L \cap P : L \in \mathcal{L}\}$ of $\mathcal{L}$. Our objective is to investigate the *pencils* of $M$, i. e. the submodules $P$ of $M$ such that $(P, \mathcal{L} \cap P)$ is a pencil. Our first result is easily derived form 5. 2.

**Corollary.** *If* $P$ *is a pencil of* $M$ *with double-point* $d$, $P/P_d$ *is the socle of* $M/P_d$. *As a consequence*, $P/\mathcal{R}P$ *is the socle of* $M/\mathcal{R}P$.

***Proof.*** Replacing $M \supset P$ by $M/P_d \supset P/P_d$ and applying 3. 7, we are reduced to the case where $P$ is semisimple and $\dot{P} = \{d\}$. Let then $Q$ denote the socle of $M$. Since $Q$ is not $\mathcal{L} \cap Q$-wild, $Q$ is a pencil of $M$ which satisfies $d_Q = d$. In case $Q \neq P$, $\dot{Q}$ has a simple point $t$ outside $\dot{P}$ and $\mathcal{L}$ contains an $L$ such that $L \cap Q = Q_t \supset P$: a contradiction to the assumption that $\mathcal{L} \cap P$ a proper bond on $P$.

**5. 4.** Our next result rests on the classical *submodule algorithm* [7]. Starting from a submodule $P$ of $M$ we consider a new aggregate $\hat{\mathcal{A}} = P^k_{\mathcal{L} \cap P}$ and modules $\hat{R}$ on $\hat{\mathcal{A}}$ associated with submodules $R$ of $M$ and defined by

$$\hat{R}(W, g, X) = (g(W) + R(X))/g(W) \subset M(X)/g(W) = \hat{M}(W, g, X).$$

With $\hat{\mathcal{L}}$ we denote the bond on $\hat{M}$ formed by $\hat{P}$ and the submodules $\hat{L}, L \in \mathcal{L}$. Thus, we obtain a functor

$$E: M^k_{\mathcal{L}} \to \hat{M}^k_{\hat{\mathcal{L}}},\ (V, f, X) \mapsto (V/V', f'', (V', f', X)),$$

where $V'$ equals $f^{-1}(P(X))$ and $f': V' \to P(X)$, $f'': V/V' \to M(X)/f(V')$ are induced by $f$. This functor is an epivalence, and even an equivalence if $\mathcal{L} \neq \emptyset$.

**Proposition.** *If* $P$ *is a pencil of* $M$, $P(X) = M(X)$ *holds for all* $x \in \dot{P}$. *Accordingly*, $M$ *contains only finitely many pencils.*

***Proof.*** Restricting $M, P$ and all $L \in \mathcal{L}$ to $\dot{P}$, we may suppose that $\dot{P} = \mathcal{L}$. Arguing by contradiction and replacing $M$ by a submodule if necessary, we may further suppose that $M/P$ is simple, i. e. that $\dim M(x) = 1 + \dim P(x)$ for some $x \in \mathfrak{A}$ and $M(y) = P(y)$ for all $y \in \mathfrak{A}/x$. Setting $N = P_d \cap P_x$ and replacing $M$ by $M/N$, we are reduced to the case where $P$ is semisimple and where $\mathfrak{A}$ consists of two points $d \neq x$ or of one point $d = x$.

a) *Case* $d \neq x$. For each submodule $R$ of $M$, we then denote by $R'$ the restriction of $\hat{R}$ to the full subaggregate $\mathcal{A}'$ of $\hat{\mathcal{A}} = P^k_{\mathcal{L} \cap P}$ whose spectroid consists of the indecomposables $(0, 0, x) \in \hat{\mathcal{A}}$ and $p = (k^3, \overline{p}, d^4 \oplus x^3)$, where

$$\overline{p} = \left[\begin{array}{cccc:cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]^{T}.$$
$$\qquad\qquad d \qquad\qquad\qquad x$$

The module $M'$ admits a submodule $Q$ such that $Q(0,0,x) = P'(0,0,x) = P(x)$ and $Q(p) = M'(p) = M(d^4 \oplus x^3)/\operatorname{Im}\overline{p} \supset P'(p)$. To prove this, it suffices to show that each morphism $(0,\mu)$: $(k^3, \overline{p}, d^4 \oplus x^3) \to (0,0,x)$ maps $M(d^4 \oplus x^3)$ into $P(x)$. For this, it is enough to show that $\mu$: $d^4 \oplus x^3 \to x$ is radical. This is due to the fact that a section $\sigma$ of $\mu$ would provide a section $(0,\sigma)$ of $(0,\mu)$.

The restriction $\mathcal{L}' = \{L' : L \in \mathcal{L}\} \cup \{P'\}$ of $\hat{\mathcal{L}}$ to $M'$ induces a proper bond $\mathcal{L}' \cap Q$ on $Q$, because $P' = P' \cap Q \neq Q$ and $L' \cap P' \neq P'$ for each $L \in \mathcal{L}$. Therefore, it suffices to show that $\dim Q(p)/(\mathcal{R}Q)(p) \geq 3$ (3.1). This follows from $\dim (M/P)(d^4 \oplus x^3) = \dim (M/P)(x^3) = 3$ and from $(\mathcal{R}Q)(p) \subset P(d^4 \oplus x^3)/\operatorname{Im}\overline{p}$. The inclusion is due to the fact that each morphism $(0,0,x) \to (k^3, \overline{p}, d^4 \oplus x^3)$ of $\mathcal{A}'$ maps $Q(0,0,x) = P(x)$ into $P(d^4 \oplus x^3)$, and that each radical endomorphism of $p$ is induced by a radical endomorphism of $d^4 \oplus x^3$ which annihilates $(M/P)(d^4 \oplus x^3)$.

b) *Case* $d = x$. Then the argument is simpler. We focus on the sole indecomposable $q = (k^2, \overline{q}, d^3)$ of $\hat{\mathcal{A}}$, where $\overline{q} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]^{T}$. Each element of $\hat{\mathcal{L}}$ induces a proper subspace of $\hat{M}(q) = M(d^3)/\operatorname{Im}\overline{q}$, and each radical endomorphism of $q$ maps $\hat{M}(q)$ into $\hat{P}(q)$. Replacing $\hat{M}$ by its restriction $M'$ to the full subaggregate $\mathcal{A}'$ of $\hat{\mathcal{A}}$ defined by $q$, we infer that $\dim M'(q)/(\mathcal{R}M)(q) \geq \dim M(d^3)/P(d^3) = 3$, and we conclude with 3. 1.

**5.5. Proposition.** *Let $K$ be maximal in $\mathcal{L}$ and not contained in the pencil $P$ of $M$. Then* $\sum_{x \in \dot{P}} \dim M(x)/K(x) = 1$.

***Proof.*** Suppose that the statement is wrong. Then we can find submodules $R_1 \subset Q_1$ of $M \mid \dot{P}$ which contain $K \mid \dot{P}$ and are of colength 2 and 1. We denote by $Q_0$ the maximal submodule of $P$ such that $Q_0 \mid \dot{P} = Q_1$, by $R$ the maximal submodule of $Q = Q_0 + K$ such that $R \mid \dot{P} = R_1$. (Of course, $R$ contains $K$.)

We set $d = d_P$ and $\Sigma = \bigoplus_s s$, where $s \in \dot{P} \setminus d$. Up to isomorphism there is a unique indecomposable $P$-space of the form $p = (k^3, \overline{p}, d^4 \oplus \Sigma^3)$ which avoids all maximal submodules of $P$. Applying the submodule-algorithm to $P \subset M$, we denote by $M'$ and $\mathcal{L}'$ the restrictions of $\hat{M}$ and $\hat{\mathcal{L}}$ to the full subaggregate $\mathcal{A}'$ of $\hat{\mathcal{A}} = P^k_{\mathcal{L} \cap P}$ whose spectroid consists of $p$ and of the $(0,0,y)$, where $y \in \dot{R}$. The wanted contradiction will follow from the fact that $M'$ is $\mathcal{L}'$-wild.

To prove this, we consider the submodule $N$ of $M'$ such that $N(p) = Q(d^4 \oplus \Sigma^3)$ mod $\operatorname{Im}\overline{p}$ and $N(0,0,y) = R(y)$ if $y \in \dot{R}$. Such a submodule exists because each morphism $(0,\mu)$: $(k^3, \overline{p}, d^4 \oplus \Sigma^3) \to (0,0,y)$ maps $Q(d^4 \oplus \Sigma^3)$ into $R(y)$. Otherwise, $\mu$ would induce an isomorphism of a summand $y'$ of $d^4 \oplus \Sigma^3$ onto $y$, and $(0,\mu)$ would admit a section.

Let $X'$ denote the submodule of $M'$ induced by a submodule $X$ of $M$. Then $N$ is not contained in $K'$, because $p$ avoids each proper submodule of $P$; hence, $R(d^4 \oplus \Sigma^3)$

and $Q(d^4 \oplus \Sigma^3)$ are identified with their images in $M(d^4 \oplus \Sigma^3)/\operatorname{Im}\overline{p}$, and we have $K'(p) \subset R(d^4 \oplus \Sigma^3) \neq Q(d^4 \oplus \Sigma^3) \tilde{\to} N(p)$. On the other hand, each $L \in (\mathcal{L} \setminus K) \cup \{P\}$ intersects $R$ properly; it follows that $L'(0, 0, y) = L(y) \neq R(y) = N(0, 0, y)$ for some $y \in \dot{R}$ and that $\mathcal{L}'$ is a proper bond on $N$. Hence, it suffices to prove that $\dim(N / \mathcal{R}N)(p) \geq 3$ which implies that $N$ is absolutely wild and $M'$ $\mathcal{L}'$-wild.

The announced inequality is due to the fact that each radical endomorphism of $p$ is induced by a radical endomorphism of $d^4 \oplus \Sigma^3$ and maps $N(p) \tilde{\to} Q(d^4 \oplus \Sigma^3)$ into $R(d^4 \oplus \Sigma^3)$. We conclude that $(\mathcal{R}N)(p) \subset R(d^4 \oplus \Sigma^3)$ and that

$$\dim(N / \mathcal{R}N)(p) \geq \dim(Q/R)(d^4 \oplus \Sigma^3) = 4 \text{ or } 3.$$

**5.6.** If $\tilde{\mathcal{L}}$ denotes the set of all maximal elements of $\mathcal{L}$, it is clear that $M^k_{\mathcal{L}} = M^k_{\tilde{\mathcal{L}}}$. Therefore we may always restrict ourselves to the case where $\mathcal{L}$ is *irredundant*, i.e. where $\mathcal{L} = \tilde{\mathcal{L}}$.

**Corollary.** *Suppose that $\mathcal{L}$ is an irredundant bond on $M$ and that $s \in \dot{P}$ is an ordinary point of a pencil $P$ of $M$. The conditions $L \in \mathcal{L}$ and $L(s) \neq M(s)$ then imply $L \cap P = P_s$.*

**5.7. Corollary.** *Let $K$ be a submodule of $M$ which is neither contained in the pencil $P$ of $M$ nor in any $L \in \mathcal{L}$. Then $\sum_{x \in \dot{P}} \dim M(x) / K(x) \leq 1$.*

***Proof.*** The corollary follows from Proposition 5.5 applied to a new bond $\mathcal{L} \cup \{K\}$.

**5.8. Corollary.** *Suppose that the $\mathcal{L}$-pencils $P$ and $Q$ of $M$ are not comparable. Then $d_P \notin \dot{Q}$ and $d_Q \notin \dot{P}$.*

***Proof.*** Suppose that $d_Q \notin \dot{P}$ and that $u \in Q(d_Q) \neq M(d_Q)$ lies outside $L(d_Q)$ whenever $L \in \mathcal{L}$ satisfies $L(d_Q) \neq M(d_Q)$. Let further $K$ denote a maximal submodule of $Q$ such that $u \in K(d_Q) \neq M(d_Q)$. Then $K$ is not contained in $P$ and $\mathcal{L} \cap K$ is a proper bond on $K$. On the other hand, we have $K(d_Q) \neq M(d_Q)$ and $K(s) = Q(s) \neq$ $\neq M(s)$ for some $s \in \dot{P}$, hence

$$\sum_{x \in \dot{P}} \dim M(x)/K(x) \geq 2,$$

in contradiction to 5.7.

**5.9. Corollary.** *If the $\mathcal{L}$-pencils $P$ and $Q$ of $M$ are not comparable, then $(\mathcal{R}P)(s) = (\mathcal{R}Q)(s)$ for all $s \in \dot{P} \cap \dot{Q}$.*

***Proof.*** Indeed, $s$ is ordinary by 5.8. If $L$ is maximal in $\mathcal{L}$ and such that $L \cap P =$ $= P_s$ (5.2), we have $L \cap Q = Q_s$ by 5.6; hence, $(\mathcal{R}P)(s) = L(s) = (\mathcal{R}Q)(s)$.

**5.10.** For each submodule $N$ of $M$, we set $\check{N} = \{x \in \mathfrak{S} : N(x) = M(x)\}$. Thus we have $\dot{P} \subset \check{P}$ if $P$ is a pencil of $M$.

**Corollary.** *If $P, Q$, and $R$ are 3 pairwise incomparable pencils of $M$, the equality $\dot{P} \setminus \check{R} = \dot{Q} \setminus \check{R}$ implies $\dot{R} \setminus \check{P} = \dot{R} \setminus \check{Q}$.*

***Proof.*** Let $s \in \dot{P} \cap \dot{Q}$ be such that $R(s) \neq M(s)$, and $L$ a maximal element of $\mathcal{L}$ such that $L \cap P = P_s$ and $L \cap Q = Q_s$ (5.6). If $t \in \dot{R}$ is such that $M(t) = R(t) \neq$ $L(t)$, we have $P(t) = P_s(t) \subset L(t)$ and $Q(t) = Q_s(t) \subset L(t)$, hence, $\dot{R} \setminus \check{P} = \{t\} = \dot{R} \setminus \check{Q}$.

**6. Proof of the second main theorem (reduction).** Our objective is to propose a general "construction" of locally finite sets $\mathcal{D} = \mathcal{D}(M, \mathcal{L})$ of $\mathcal{L}$-reliable punched lines which satisfy the conditions a) and b) of the second main theorem. Our sets $\mathcal{D}$ are the unions of subsets $\mathcal{D}_n = \mathcal{D}_n(M, \mathcal{L})$ formed by punched lines $D \setminus E \subset \subset \mathrm{Hom}_k(V, M(X))$ whose points have *space-dimension* $\dim V = n$. We construct the slices $\mathcal{D}_n(M, \mathcal{L})$ by induction on $n$ and *simultaneously* for "all" non-wild pairs $(M, \mathcal{L})$. The construction is rather precise and rather involved, as nature seems to be.

In order to classify the indecomposable $M$-spaces, we can examine the finite full subspectroids $\mathfrak{S}'$ of $\mathfrak{S}$ separately and focus on the $M$-spaces with "support" $\mathfrak{S}'$. We are thus reduced to the case examined in the present section where *the spectroid* $\mathfrak{S}$ *of* $\mathcal{A}$ *is supposed to be finite. From* 6.2 *until the end of the section, we suppose that* $M$ *is not* $\mathcal{L}$*-wild.*

**6.1.** Since our construction proceeds by induction on the space-dimension, we first example the indecomposable $M$-spaces with space-dimension 1. For this purpose, no restriction is needed on the representation type of $(M, \mathcal{L})$.

**Proposition.** *The map* $(V, f, X) \mapsto \mathcal{A}f(V)$, *which assigns to* $(V, f, X)$ *the submodule of* $M$ *generated by* $f(V)$, *induces a bijection between the set of isoclasses of indecomposables in* $M^k_{\mathcal{L}}$ *with space-dimension* 1 *and the set of submodules* $N$ *of* $M$ *for which* $\mathcal{L} \cap N$ *is a proper bond.*

***Proof.*** The inverse bijection is obtained as follows. For each $N$, we choose a projective cover $n\colon \mathcal{A}(X, ?) \to N$ and set $n' = n(X)(\mathbb{1}_X) \in N(X)$. To $N$ we then assign the isoclass of $(k, ?n', X) \in M^k_{\mathcal{L}}$.

**6.2.** Let us now return to the case where $M$ is not $\mathcal{L}$-wild. Each pencil $P$ of $(M, \mathcal{L})$ with double-point $d$ gives rise to a one-parametric family of maximal submodules $Q$ of $P$ such that $P_d \subset Q \subset P$. The other maximal submodules of $P$ have the form $P_s$, where $s$ is an ordinary point of $\dot{P}$; their number is finite, and the induced bond $\mathcal{L} \cap P_s$ is not proper (5.2).

**Proposition.** *Besides maximal submodules of pencils,* $M$ *contains only finitely many submodules* $N$ *for which* $\mathcal{L} \cap N$ *is a proper bond.*

***Proof.*** We proceed by induction on the number of pencils of $(M, \mathcal{L})$, which is finite by 5.4. If $M$ contains no pencil, we denote by $\mathcal{N}$ the set of all $N \subset M$ such that $\mathcal{L} \cap N$ is proper. Each element of $\mathcal{N}$ has finitely many (direct) predecessors. Since $\mathcal{N}$ has finite height and (at most) one maximal element, $\mathcal{N}$ is finite.

If $M$ contains pencils, we consider a minimal pencil $P$ (with double-point d) and maximal submodules $Q_1, \ldots, Q_s$ $(s \geq 1)$ of $P$ containing $P_d$ and such that each $u \in \in P(d) \setminus \bigcup_{i=1}^{s} Q_i(d)$ satisfies the statement of Proposition 5.1. Then each non-maximal submodule of $P$ is contained in some $Q_i$ or some $P_s$ with $s \in \dot{P} \setminus d$. And each non-maximal submodule $N \subset P$ for which $\mathcal{L} \cap N$ is proper is contained in some $Q_i$. Together with $Q_1, \ldots, Q_s$, these $N$ form a poset $\mathcal{N}$ which has finite height and a finite number of maximal elements. Since each element of $\mathcal{N}$ has a finite number of (direct) predecessors, $\mathcal{N}$ is finite.

On the other hand, since $(M, \mathcal{L} \cup \{P\})$ admits less pencils than $(M, \mathcal{L})$, we know by induction that there are only finitely many submodules $N'$ which are not contained in $P$, which are not maximal in a pencil of $(M, \mathcal{L})$ and for which $\mathcal{L} \cap N'$ is proper.

**6.3.** ***The construction of*** $\mathcal{D}_1$. For each pencil $P$ of $M$, we pick vectors $u_s \in P(s) \setminus (\mathcal{R}P)(s)$, $s \in \dot{P} \setminus d_P$, and a basis $(u, v)$ of a supplement of $(\mathcal{R}P)(d_P)$ in $P(d_P)$. Thus we obtain a straight line

$$D_P = \{u + \lambda v + \sum_s u_s : \lambda \in k\}$$

of $M(d_P \oplus \bigoplus_s s) \tilde{\to} \mathrm{Hom}_k(k, M(d_P \oplus \bigoplus_s s))$ whose associated functor $F: \operatorname{rep} Q^1 \to M^k$ preserves indecomposability and heteromorphism. Erasing from $D_P$ the points lying in the various subspaces $L(d_P \oplus \bigoplus_s s)$, $L \in \mathcal{L}$, we get an $\mathcal{L}$-reliable punched line, which seems to be a good applicant for a position in $\mathcal{D}_1$. Unfortunately, if the lines $D_P$ are to be retained, the present state of our technology urges us to overpunch them as will be explained below.

First we consider the *minimal pencils* of $M$, which we stack up in a finite set $\mathcal{P}$ equipped with an arbitrary *linear order*. If $\mathcal{P} \neq \emptyset$, we construct an ideal $\mathcal{J}$ of $\mathcal{A}$ and a bond $\mathcal{K}$ on $M$ which satisfy the statements of Lemma 6.4 below. Finally, for each $P \in \mathcal{P}$, we construct a proper bond $\mathcal{K}'_P$ on $P$, formed by maximal submodules $N$ such that $P$ is $\mathcal{K}'_P$-semisimple, and that $v \in N(d_P)$ for some $N$. The submodules $N$ give birth to a bond

$$\mathcal{L}'_P = (\mathcal{L} \cap P) \cup (\mathcal{K} \cap P) \cup \mathcal{K}'_P \cup \{X : P > X \in \mathcal{P}\}$$

on $P$ and to a finite subset

$$E_P = \bigcup_{L \in \mathcal{L}'_P} D_P \cap L(d_P \oplus \bigoplus_s s)$$

of the straight line $D_P$. The associated punched lines $D_P \setminus E_P$ are the first selected constituents of $\mathcal{D}_1$.

The restraint imposed by $\mathcal{K}$ will permit us to prove Lemma 6.4 below. As a result of the insertion of $\mathcal{K}'_P$ into $\mathcal{L}'_P$, *all maximal elements of* $\mathcal{L}'_P$ *and all proper submodules* $K$ *of* $P$ *for which* $\mathcal{L}'_P \cap K$ *is proper are maximal in* $P$ (5.1). Accordingly, each $u + \lambda v + \sum_s u_s \in D_P \setminus P_P$ generates a maximal submodule of $P$.

In order to puncture the lines $D_P$ *when* $P$ *is not minimal*, we now set $\mathcal{P}_1 := \mathcal{P}$ and $\mathcal{K}_1 := \mathcal{K}$. We denote by $\mathcal{P}_2$ the set of minimal pencils of $(M, \mathcal{L} \cup \mathcal{P}_1)$ or, equivalently, of $(M, \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1)$, by $\mathcal{P}_3$ the set of minimal pencils of $(M, \mathcal{L} \cup \mathcal{P}_1 \cup \mathcal{P}_2)$... . Replacing $\mathcal{L}$ by $\mathcal{L}_1 = \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1$, we construct a bond $\mathcal{K}_2$ which satisfies the statements of Lemma 6.4 for $(M, \mathcal{L}_1)$. Adapting the recipe above to the new data, we obtain a proper bond $\mathcal{L}'_P$ on each $P \in \mathcal{P}_2$ and the associated finite subset $E_P \subset D_P$. Then replacing $\mathcal{L}_1 = \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1$ by $\mathcal{L}_2 = \mathcal{L}_1 \cup \mathcal{K}_2 \cup \mathcal{P}_2$, we construct bonds $\mathcal{K}_3$ on $M$ and $\mathcal{L}'_P$ on each $P \in \mathcal{P}_3$, thus obtaining finite sets $E_P \subset D_P$ for all $P \in \mathcal{P}_3$... If $\mathcal{P}_h$ is the last non-empty set of pencils constructed in this way, we finally set

$$\mathcal{D}_1(M, \mathcal{L}) = \{D_P \setminus E_P : P \in \mathcal{P}_i, \ 1 \leq i \leq h\}.$$

If $M$ contains no pencil, $\mathcal{D}_1(M, \mathcal{L})$ is empty.

**6.4. Lemma.** *Let $M$ be a pointwise finite module over an aggregate $\mathcal{A}$ with finite spectroid $\mathfrak{S}$, $\mathcal{L}$ a bond on $M$ such that $M$ is not $\mathcal{L}$-wild, $\mathcal{P}$ a non-empty set of pairwise incomparable pencils* (5.1) *of $M$ and $R = \bigcap_{P \in \mathcal{P}} \mathcal{R}P$ the intersection of their radicals. Then there is an ideal $\mathcal{J} \subset \mathcal{R}_{\mathcal{A}}$ and a bond $\mathcal{K}$ on $M$ such that:*

a) $\mathcal{J}M \subset R \subset B \cap P \neq P$ *and* $(\mathcal{J}M)(x) = (\mathcal{R}P)(x)$ *for all* $B \in \mathcal{K}$, *all* $P \in \mathcal{P}$ *and all* $x \in \dot{P}$;

b) *if $M / \mathcal{J}M$ is considered as a module over $\mathcal{A} / \mathcal{J}$ and $\mathcal{K} / \mathcal{J}M$ denotes the set of all $B / \mathcal{J}M$, $B \in \mathcal{K}$ then the canonical functor $M^k_{\mathcal{K}} \to (M / \mathcal{J}M)^k_{\mathcal{K} / \mathcal{J}M}$ is an epivalence.*

The proof of the lemma is given in 7.1 below.

**6.5. *The construction of* $\mathcal{D}_r$, $r \geq 2$.** The construction is based on a *sequence of submodules* of $M$ which we must present beforehand. First supposing $\mathcal{P}_1 \neq \emptyset$, we consider the submodules $X$ such that: a) $\mathcal{L} \cap X$ is a proper bond on $X$; b) $X$ is contained in a module belonging to $\mathcal{K} = \mathcal{K}_1$ or to some $\mathcal{K}'_P$, where $P \in \mathcal{P} = \mathcal{P}_1$ (6.3). These submodules form a *finite* set (6.2), which we denote by $O_0 = O_0(M, \mathcal{L})$ and equip with some linear order $\leq$ such that $X \subset Y$ implies $X \leq Y$. By construction, $O_0$ contains all the non-maximal submodules $N$ of $P$, $P \in \mathcal{P}_1$, for which $\mathcal{L} \cap N$ is a proper bond.

Replacing $\mathcal{L}$ by $\mathcal{L}_1 = \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1$, then by $\mathcal{L}_2 = \mathcal{L}_1 \cup \mathcal{K}_2 \cup \mathcal{P}_2, \ldots, \mathcal{L}_h = \mathcal{L}_{h-1} \cup \mathcal{K}_h \cup \mathcal{P}_h$, we may repeat the construction of $O_0$ and obtain further linearly ordered sets $O_1 = O_0(M, \mathcal{L}_1)$, $O_2 = O_0(M, \mathcal{L}_2), \ldots, O_{h-1} = O_0(M, \mathcal{L}_{h-1})$. To these sets we add a set $O_h$, formed by the submodules $N$ of $M$ for which $\mathcal{L}_h \cap N$ is proper, and also equipped with a linear order $\leq$ such that $X \subset Y$ implies $X \leq Y$. Together with the linear orders imposed onto $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_h$, we finally obtain a *finite linearly ordered set* $Q$ which has $M$ as maximum and is formed by the disjoint intervals

$$O_0 < \mathcal{P}_1 < O_1 < \mathcal{P}_2 < O_2 < \ldots < \mathcal{P}_n < O_h.$$

If $M$ contains no pencil, $O_0$ denotes the set of all submodules $X$ of $M$ for which $\mathcal{L} \cap X$ is proper. We then set $Q = O_0$.

Our construction of $\mathcal{D}_r(M, \mathcal{L})$ now results from an application of our *main algorithm* to each submodule $N \in Q$ and to the associated bond $\mathcal{B}N = \mathcal{L} \cup \{X \in Q; X < N\}$ on $M$. For this sake, we introduce the aggregate $\mathcal{A}^N = N^k_{\mathcal{B}N \cap N}$, its spectroid $\mathfrak{S}^N$, the module $M^N$ on $\mathcal{A}^N$ defined by $M^N(W, g, X) = M(X) / g(W)$ and a bond $\hat{\mathcal{B}}N$ on $M^N$ which consists of the submodules of $M^N$ induced by $N$ and the modules $X \in \mathcal{B}N$. The resulting epivalence $M^k_{\mathcal{B}N} \to M^{Nk}_{\hat{\mathcal{B}}N}$ will allow us to lift various slices of the wanted $\mathcal{D}_r(M, \mathcal{L})$ from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{B}N)$. We distinguish two cases:

1) Case $N \in O_i$. Then $\mathcal{B}N \cap N$ contains all maximal submodules of $N$. The spectroid $\mathfrak{S}^N$ is finite and contains one point $(k, g, \bigoplus_{s \in \dot{N}}, s)$ with space-dimension 1. The remaining points of $\mathfrak{S}^N$ have the form $(0, 0, t)$, $t \in \mathfrak{S}$.

Obviously, $M^N$ is not $\hat{\mathcal{B}}N$-wild, because two-parametric families of indecomposables could be lifted from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{L})$. Proceeding by induction on $r$, we may therefore suppose that the sets $\mathcal{D}_s(M^N, \hat{\mathcal{B}}N)$ are at our disposal for all $s < r$. Here we are concerned with $\hat{\mathcal{B}}N$-reliable punched lines formed by $M^N$-spaces $(U, h, Z)$ whose bases $Z = (W, g, X) \in N^k_{\mathcal{B}N \cap N}$ have a space-dimension $\dim W =: t \geq 1$. These lines form a subset $\mathcal{D}^t_s(M^N, \hat{\mathcal{B}}N)$ of $\mathcal{D}_s(M^N, \hat{\mathcal{B}}N)$. Lifting the lines of $\mathcal{D}^t_{r-t}(M^N, \hat{\mathcal{B}}N)$ from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{B}N)$, we finally obtain a set $\tilde{\mathcal{D}}^t_{r-t}(M^N, \hat{\mathcal{B}}N)$ of $\mathcal{L}$-reliable punched lines and the requested contribution of $N$ to $\mathcal{D}_r(M, \mathcal{L})$:

$$\bigcup_{t=1}^{r-1} \tilde{\mathcal{D}}^t_{r-t}(M^N, \hat{\mathcal{B}}N).$$

2) Case $N \in \mathcal{P}_i$. We then proceed as in case 1, the difference being that $\mathfrak{S}^N$ is infinite. According to Lemma 6.6 below, $\mathfrak{S}^N$ contains a *finite* full subspectroid $\mathfrak{S}^N_r$ which supports the bases $Z = (W, g, X)$ of all indecomposables $(U, h, Z) \in M^{Nk}_{\hat{\mathcal{B}}N}$ such that $1 \leq \dim U$ and $\dim W < r$. (More precisely, $\mathfrak{S}^N_r$ is formed by the points $(0, 0, t)$, $t \in \mathfrak{S}$, and by at most $5(r-1)$ points of the form $(W', g', X')$ with $1 \leq$ $\leq \dim W' < r$.) Since $\mathfrak{S}^N_r$ is finite, our induction provides us with finite sets $\mathcal{D}_s(M^N \mid \mathfrak{S}^N_r, \hat{\mathcal{B}}N \mid \mathfrak{S}^N_r)$ for $s < r$. As in case 1, these sets are partitioned into subsets $\mathcal{D}^t_s(M^N \mid \mathfrak{S}^N_r, \hat{\mathcal{B}}N \mid \mathfrak{S}^N_r)$. Lifted from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{L})$, these subsets give rise to finite sets of $\mathcal{B}N$-reliable punched lines denoted by $\tilde{\mathcal{D}}^t_s(M^N, \hat{\mathcal{B}}N)$.

Putting together the various pieces obtained above, we finally set

$$\mathcal{D}_r(M, \mathcal{L}) = \bigcup_{N \in Q} \bigcup_{t=1}^{r-1} \tilde{\mathcal{D}}^t_{r-t}(M^N, \hat{\mathcal{B}}N). \qquad (*)$$

The fact that $\mathcal{D}(M, \mathcal{L}) = \bigcup_{r \geq 1} \mathcal{D}_r(M, \mathcal{L})$ satisfies the statements of the second main theorem is easy and will be checked in 6.7.

**6.6.** Let us provisionally consider an arbitrary pointwise finite module $M'$ over an aggregate $\mathcal{A}'$ and a bond $\mathcal{L}'$ on $M'$. We then say that an indecomposable $s \in \mathcal{A}'$ is $(M', \mathcal{L}')$-relevant if $s$ is a direct a summand of the base $X$ of some indecomposable $(V, f, X) \in M'^k_{\mathcal{L}'}$.

**Lemma.** *With the notations of 6.5, let $N$ be a pencil of $M$ and $r \geq 2$. Then there are at most $5(r-1)$ isoclasses of indecomposable $N$-spaces $(W, g, X)$ which avoid $\mathcal{B}N \cap N$, satisfy $1 \leq \dim W < r$ and are $(M^N, \hat{\mathcal{B}}N)$-relevant.*

**6.7.** ***Checking the statements of the second main theorem.*** The statements result almost immediately from the construction.

Since $\mathfrak{S}$ is supposed to be finite, the finiteness of the cardinality of $\mathcal{D}_r(M, \mathcal{L})$ follows from 6.5 (*).

In order to prove statement a), we denote by $\nu_r(M, \mathcal{L})$ the number of isoclasses of indecomposable $M$-spaces $(V, f, X) \in M^k_{\mathcal{L}}$ which have space-dimension $r$ and are not produced by punched lines of $\mathcal{D}(M, \mathcal{L})$. We shall prove that $\nu_r(M, \mathcal{L})$ is finite by induction on $r$. Clearly, $\nu_0(M, \mathcal{L})$ is equal to the number of points of $\mathfrak{S}$. So let us assume $r = 1$. By 6.1, the isoclasses of the indecomposables $(k, f, X) \in M^k_{\mathcal{L}}$ with space-dimension 1 correspond bijectively to the submodules $X = \mathcal{A}f(k)$ for which $\mathcal{L} \cap X$ is proper. In case $\mathcal{A}f(k) \notin \mathcal{Q}$, $(k, f, X)$ is produced by $\mathcal{D}(M, \mathcal{L})$ and $\mathcal{A}f(k)$ is maximal submodule of a pencil. *We infer that* $v_1(M, \mathcal{L}) = |\mathcal{Q}|$.

In the case $r \geq 2$, let $(V, f, X) \in M_{\mathcal{L}}^k$ be an indecomposable with space-dimension $r$ which is not produced by $\mathcal{D}(M, \mathcal{L})$, and let $N$ be the smallest element of $\mathcal{Q}$ such that $t = \dim f^{-1}(N(X)) \geq 1$. *If $N$ is not a pencil,* our induction-hypothesis and the finiteness of $\mathfrak{F}^N$ imply that $M_{\hat{\mathcal{B}}N}^{Nk}$ has a finite number, say, $v_{r-t}^t(M^N, \hat{\mathcal{B}}N)$, of iso-classes of indecomposables $(U, h, Z)$ not produced by $\mathcal{D}(M^N, \hat{\mathcal{B}}N)$ and such that $\dim U = r - t$ and that $Z$ has space-dimension $t \geq 1$. The contribution of $N$ to $v_r(M, \mathcal{L})$ is therefore equal to $\sum_{t=1}^r v_{r-t}^t(M^N, \hat{\mathcal{B}}N)$. (We recall that $v_0^r(M^N, \hat{\mathcal{B}}N) = 0$ in the considered case $r \geq 2$.)

*If $N$ is a pencil,* the numbers $v_{r-t}^t(M^N, \hat{\mathcal{B}}N) \in \mathbb{N} \cup \{\infty\}$ can still be defined. Now $v_0^r(M^N, \hat{\mathcal{B}}N) = 1$. In case $1 \leq t < r$, the finiteness of $v_{r-t}^t(M^N, \hat{\mathcal{B}}N)$ follows from the fact that the bases $Z$ of the indecomposables $(U, h, Z)$ considered above are supported by a *finite* subspectroid $\mathfrak{F}_r^N$ of $\mathfrak{F}^N$ (6.5, case 2, and 6.6). It follows that $N$ still has a finite contribution $\sum_{t=1}^r v_{r-t}^t(M^N, \hat{\mathcal{B}}N)$ and that

$$v_r(M, \mathcal{L}) = \sum_{N \in \mathcal{Q}} \sum_{t=1}^{r} v_{r-t}^t(M^N, \hat{\mathcal{B}}N).$$

Finally, in order to check statement b), we prove by induction on $r$ that indecomposable $M$-spaces $(V, f, X) \in M_{\mathcal{L}}^k$ and $(V', f', X') \in M_{\mathcal{L}}^k$ cannot be isomorphic if they are produced by different punched lines $D$ and $D'$ of $\mathcal{D}_{\leq r}(M, \mathcal{L}) := \bigcup_{s \leq r} \mathcal{D}_s(M, \mathcal{L})$. This is clear by construction if $D \in \mathcal{D}_1(M, \mathcal{L})$ or $D' \in \mathcal{D}_1(M, \mathcal{L})$. Otherwise, $r$ is $\geq 2$. Then we consider the smallest elements $N$ and $N'$ of $\mathcal{Q}$ which are not avoided by $(V, f, X)$ and $(V', f', X')$, respectively. Our claim is clear if $N \neq N'$. In the case $N \neq N'$, $D$ and $D'$ are obtained by lifting punched lines defined on finite spectroids $\mathfrak{F}^N$ or $\mathfrak{F}_r^N$. These punched lines consist of $M^N$-spaces with space-dimension $< r$. They produce the $M^N$-spaces associated with $(V, f, X)$ and $(V', f', X')$. Since these $M^N$-spaces are not isomorphic by induction-hypothesis, $(V, f, X)$ and $(V', f', X')$ are not isomorphic either.

## 7. Simultaneous eradication of incomparable pencils.

**7.1. Theorem.** *Let $M$ be a pointwise finite module over an aggregate $\mathcal{A}$ with finite spectroid $\mathfrak{F}$, $\mathcal{L}$ a bond on $M$ such that $M$ is not $\mathcal{L}$-wild, $\mathcal{P}$ a non-empty set of pairwise incomparable pencils of $M$, and $R = \bigcap_{P \in \mathcal{P}} \mathcal{R}P$ the intersection of their radicals. We suppose that $R(q) \neq 0$, where $q \in \mathfrak{F}$ satisfies $R(q) = M(q)$ or belongs to the generation-indicator $\dot{P} = \{x \in \mathfrak{F} : P(x) \neq (\mathcal{R}P)(x)\}$ of some $P \in \mathcal{P}$. Then $R$ contains a simple submodule $S$ such that the transporter* Transp$(M, S)$, *i. e. the ideal of $\mathcal{A}$ formed by the radical morphisms $\mu: X \to Y$ satisfying $\mu M(X) \subset S(Y)$, annihilates no $P \in \mathcal{P}$.*

*Before entering the proof of the theorem, we show that it implies Lemma 6.4 given above:*

In the notations of 6.4, we proceed by induction on $d = \sum_x \dim R(x)$, where $x \in \bigcup_{P \in \mathcal{P}} \dot{P}$. In case $d = 0$, we set $\mathcal{J} = \{0\}$ and $\mathcal{K} = \emptyset$. In case $d > 0$, we apply our theorem setting $\mathcal{I} = \operatorname{Transp}(M, S)$ and $B = N + R$, where $N$ is the annihilator of $\mathcal{I}$ in $M$. Considering $\overline{M} = M/S = M/\mathcal{I}M$ as a module over $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{I}$, we then obtain an epivalence $M_B^k \to \overline{M}_{B/S}^k$ (4.2.b). Applying the induction hypothesis to $\overline{M}$ and $\overline{\mathcal{P}} = \{P/S \colon P \in \mathcal{P}\}$, we get an ideal $\overline{\mathcal{J}}$ of $\overline{\mathcal{A}}$ and a bond $\overline{\mathcal{K}}$ on $\overline{M}$ which satisfy the statements of the lemma mutatis mutandis. For $\mathcal{J}$, it then suffices to choose the inverse image of $\overline{\mathcal{J}}$ in $\mathcal{A}$, for $\mathcal{K}$ the set formed by $B$ and by the inverse images of the submodules in $\overline{\mathcal{K}}$.

**7.2. *Beginning of the proof of Theorem* 7.1.** The proof occupies the whole Section 7. We are really interested in the case $q \in \dot{P}$; the alternative $R(q) = M(q)$ only serves our inductive argument.

If $\mathcal{P}$ has cardinality $|\mathcal{P}| = 1$, we apply Lemma 4.3 to $P$ and use the fact that $P(x) = M(x)$ for all $x \in \dot{P}$ (5.4). Hence, we may suppose that $|\mathcal{P}| \geq 2$ and proceed by induction on $|\mathcal{P}|$. We set $\dot{\mathcal{P}} = \bigcup_{P \in \mathcal{P}} \dot{P}$ and call a point $s \in \dot{P}$ *double* if $s = d_P$ for some $P \in \mathcal{P}$, otherwise, $s$ is called *ordinary*.

**Lemma.** *For each $p \in \mathcal{P}$ and each $x \in \dot{P}$, we have $R(x) = (\mathcal{R}P)(x)$. Accordingly, $R(x)$ has codimension 1 in $M(x)$ if $x$ is ordinary and codimension 2 if $x = d_P$.*

***Proof.*** Consider any $Q \in \mathcal{P} \setminus P$. If $x \in \dot{Q}$, $x$ is ordinary (5.8), and we have $(\mathcal{R}Q)(x) = (\mathcal{R}P)(x)$ by 5.9. If $x \notin \dot{Q}$, we have $(\mathcal{R}Q)(x) = Q(x)$; on the other hand, the restriction $Q \mid \dot{P}$ is a maximal submodule of $P \mid \dot{P}$ (5.7); it follows that $Q \mid \dot{P} \supset \mathcal{R}(P \mid \dot{P}) = \mathcal{R}(P) \mid \dot{P}$, hence $Q(x) \supset (\mathcal{R}P)(x)$. Accordingly, $(\mathcal{R}Q)(x)$ contains $(\mathcal{R}P)(x)$ in all cases.

**7.3. *First reduction.*** Let $\mathcal{T}$ denote the full subspectroid of $\mathfrak{A}$ formed by $\dot{\mathcal{P}}$ and by the points $x \in \mathfrak{A}$ such that $R(x) = M(x)$. Let further $n \in \mathbb{N}$ be such that $\mathcal{R}_{\mathfrak{A}}^{n+1}$ annihilates all $R(x)$, $x \in \mathcal{T}$, whereas $\mathcal{R}_{\mathfrak{A}}^{n}(t, s)R(t) \neq 0$ for some $t \in \mathcal{T}$ and some $s \in \mathfrak{A}$. Denoting by $R'$ the annihilator of $\mathcal{R}_{\mathfrak{A}}^{n}$ in $R$, we replace $M$ by $M/R'$, $\mathcal{L}$ by $\mathcal{L}/R' = \{L/R' \colon R' \subset L \in \mathcal{L}\}$ and $\mathcal{P}$ by $\mathcal{P}/R' = \{\mathcal{P}/R' \colon P \in \mathcal{P}\}$.

We claim that our theorem is true if it holds for $M/R'$, $\mathcal{L}/R'$ and $\mathcal{P}/R'$. Indeed, let $N/R'$ be a simple submodule of $R/R'$ such that the transporter $\mathcal{J}$ of $M/R'$ into $N/R'$ annihilates no $P/R'$, $P \in \mathcal{P}$. If $N/R'$ is located at $x \in \mathfrak{A}$, there is a morphism $\mu \in \mathcal{R}_{\mathfrak{A}}^{n}(x, y)$ and a simple submodule $S$ of $M$ such that $S(y) = \mu N(x) \neq 0$. Our claim then follows from the observation that the ideal $\mathcal{I}$ such that $\mathcal{I}(z, y) = \mu \mathcal{J}(z, x)$ and $\mathcal{J}(z, t) = 0$ in case $t \neq y$ is contained in $\operatorname{Transp}(M, S)$ and annihilates no $P \in \mathcal{P}$.

Thus we are reduced to the case where $\mathcal{R}_{\mathfrak{A}}$ annihilates all $R(t)$, $t \in \mathcal{T}$, and $R(q)$ is $\neq 0$ for some $q \in \mathcal{T}$. Restricting $M$ to the full subspectroid of $\mathfrak{A}$ formed by $\dot{\mathcal{P}}$ and $q$, we are further reduced to the case where $R$ is semisimple. Factoring out the submodule $R'$ of $R$ such that $R'(q) = 0$ and $R'(t) = R(t)$ if $t \neq q$, *we are finally reduced to the following situation, to which we restrict ourselves in the sequel: $R$ is a semisimple module vanishing outside some point $q \in \mathfrak{A}$; the set of points of $\mathfrak{A}$ is $\dot{\mathcal{P}} \cup \{q\}$; finally, $M(q) = (RM)(q) = R(q)$ if $q \notin \dot{\mathcal{P}}$.*

**7.4. *Second reduction and dichotomy of the proof.*** Suppose that there is an ordinary point $s \in \dot{\mathcal{P}}$ such that $P(s) = M(s)$ for all $P \in \mathcal{P}$ and $\mathcal{R}_{\mathfrak{A}}(s, q)M(s) \neq 0$. Then we have

$$\mathcal{R}_{\mathfrak{A}}(s, q)M(s) \subset \bigcap_{P \in \mathcal{P}} (\mathcal{R}P)(q) = R(q),$$

and each $\mu \in \mathcal{R}_{\mathfrak{s}}(s, q)$ satisfying $\mu M(s) \neq 0$ determines a simple submodule $S$ of $R$ such that $S(q) = \mu M(s)$ (7.2). Since Transp $(M, S)$ contains $\mu$, it annihilates no $P \in \mathcal{P}$.

*Thus, we are reduced to the case considered in the sequel where* $\mathcal{R}_{\mathfrak{s}}(s, q)M(s) = 0$ *for each ordinary* $s \in \dot{\mathcal{P}}$ *such that* $P(s) = M(s)$, $\forall\, P \in \mathcal{P}$.

*From now on, we fix a pencil* $F \in \mathcal{P}$ *subjected to the sole condition that* $q \in \dot{F}$ if $q \in \dot{\mathcal{P}}$. Since we have $M \neq F$ and $M(t) = F(t)$ for all $t \in \dot{F}$ (5.4), the generation-indicator $\dot{M}$ of $M$ is not contained in $\dot{F}$. Thus $\dot{M} \setminus \dot{F}$ contains a double or an ordinary point. The two cases are examined separately in 7.5 and 7.6 below.

**7.5. *First half:*** *Suppose that* $\dot{M} \setminus \dot{F}$ *contains the double point* $d = d_Y$ *of some* $Y \in \mathcal{P}$.

Let us then examine any $X \in \mathcal{P}$ different from $Y$. Since $d \notin \dot{X}$ (5.8), we have $X(d) = (\mathcal{R}X)(d) \subset (\mathcal{R}M)(d) \neq M(d) = Y(d)$. Since the restriction $X \cap Y | \dot{Y}$ is a maximal submodule of $Y | \dot{Y}$ (5.7), $X(d) = (\mathcal{R}M)(d)$ is a hyperplane of $M(d)$ containing $(\mathcal{R}Y)(d) = R(d)$. Thus, we can choose vectors $u \in M(d) \setminus X(d)$ and $v \in X(d) \setminus R(d)$ such that $M(d) = ku \oplus kv \oplus R(d)$ and $R(q) \subset (\mathcal{R}\,Y)(q) = \mathcal{R}_{\mathfrak{s}}(d, q)u + \sum_s \mathcal{R}_{\mathfrak{s}}(s, q)Y(s)$, where $s$ runs through the ordinary points of $\dot{Y}$ (5.1).

If $X_1 \in \mathcal{P}$ differs from $Y$ and $X$, we have $X_1(s) = M(s) = X(s)$ for all ordinary $s \in \dot{Y}$. Using 7.4, we infer that $\mathcal{R}_{\mathfrak{s}}(s, q)Y(s) = 0$ and $(\mathcal{R}Y)(q) = \mathcal{R}_{\mathfrak{s}}(d, q)u$. On the other hand, we have $\mathcal{R}_{\mathfrak{s}}(d, q)v \subset R(q)$ because $v$ belongs to $Y(d) = M(d)$ and to all $X_1(d) = (\mathcal{R}M)(d) = X(d)$.

Now set $E = \{\mu \in \mathcal{R}_{\mathfrak{s}}(d, q){:}\ \mu u \in R(q)\}$. Since $\mathcal{R}_{\mathfrak{s}}(d, q)u = (\mathcal{R}Y)(q)$ contains $R(q)$, the multiplication by $u$ provides a surjection $?u{:}\ E \to R(q)$. This implies that the representation $?u, ?v{:}\ E \rightrightarrows R(q)$ of the double-arrow is a direct sum of tubular and preinjective indecomposables. We distinguish two cases:

a) Case $?v \neq 0$. Our representation then admits an indecomposable summand which is isomorphic neither to $1, 0{:}\ k \rightrightarrows k$ nor to $0, 0, k \rightrightarrows 0$. Such a summand contains vectors $\mu, \nu \in E$ satisfying $0 \neq \mu u = \nu v =: r$ and $\mu v \in kr$. Accordingly, if $S \subset R$ is the simple module such that $S(q) = kr$, $\mu$ belongs to Transp $(M, S)$, and Transp $(M, S)$ does not annihilate $Y$. On the other hand, each $X \in \mathcal{P} \setminus Y$ satisfies some relation $v \in \varphi w + R(d)$, where $w \in X(s)$, $s \in \dot{X}$ and $\varphi \in \mathcal{R}_{\mathfrak{s}}(s, d)$. From $\nu\varphi M(s) \subset \nu X(d) = k\nu v$ and $\nu\varphi w = \nu v = r$ we infer that Transp $(M, S)$ contains $\nu\varphi$ and does not annihilate $X$.

b) Case $?v = 0$. Then we apply our induction hypothesis to $\mathcal{P} \setminus Y$. Since $q$ satisfies $R(q) = M(q)$ or $q \in \dot{F}$ where $F \in \mathcal{P} \setminus Y$, we infer that $R$ contains a simple submodule $S$ located at $q$ and such that Transp $(M, S)$ annihilates no $X \in \mathcal{P} \setminus Y$. On the other hand, since $S(q) \subset R(q) \subset \mathcal{R}_{\mathfrak{s}}(d, q)u$, there exists a $\varphi \in \mathcal{R}_{\mathfrak{s}}(d, q)$ such that $\varphi v = 0 \neq \varphi u \in S(q)$; thus, Transp $(M, S)$ also contains $\varphi$ and does not vanish on $Y$.

**7.6. *Second half:*** *Suppose that* $\dot{M} \setminus \dot{F}$ *contains an ordinary point* $y$.

Our premiss implies the existence of pencils $X, Y \in \mathcal{P}$ such that $y \notin \dot{X}$ and $y \in \dot{Y}$, hence, $X(y) = (\mathcal{R}X)(y) \subset (\mathcal{R}M)(y) \neq M(y) = Y(y)$. By 5.7 there is a unique point $x_X = x \in \dot{X}$ such that $Y(x) \neq M(x) = X(x)$; by 5.10 $x_X$ depends only on $X$ and $y$, but not on $Y$.

*Let us now examine the points* $z \in \dot{Y} \setminus y$ *such that* $\mathcal{R}_{\mathfrak{s}}(z, q)M(z) \neq 0$. By 5.7 $z$ satisfies $X(z) = M(z) = Y(z)$; by 7.4 $z$ is the double-point $d_Y$ of $Y$ or satisfies $Y_1(z) \neq Y(z)$ for some $Y_1 \in \mathcal{P}$, whose indicator $\dot{Y}_1$ runs through $y$ (5.7). In both cases,

$z \notin \dot{X}$. This follows from 5.8 if $z = d_Y$, from $Y_1(z) \neq M(z)$, $Y_1(x) \neq M(x)$ and 5.7 if not. *We conclude that*

$$M(z) \ni (\mathcal{R}X)(z) = \sum_{t \in \dot{X}} \mathcal{R}_{\mathfrak{S}}(t, z)X(t) = \mathcal{R}_{\mathfrak{S}}(x, z)X(x) = \mathcal{R}_{\mathfrak{S}}(x, z)n \qquad (*)$$

*for all* $n \in X(x) \setminus Y(x)$. The last equalities result from the fact that each $t \in \dot{X} \setminus x$ satisfies $X(t) = Y(t)$ (5.7); hence we have $\mathcal{R}_{\mathfrak{S}}(t, z)X(t) \subset \mathcal{R}(Y)(z) = R(z)$ (7.2) and $\mathcal{R}_{\mathfrak{S}}(x, z)Y(x) \subset R(z)$; but $y \notin \dot{F}$ implies $z \notin \dot{F}$ (as we have seen above in the case of $X$), hence $z \neq q$ and $R(z) = 0$.

When $Y$ varies, the points $z \in \dot{Y}$ considered above give rise to a subset of $\dot{\mathcal{P}}$ which we denote by $Z$. *The contribution*

$$R^Z = \sum_{z \in Z} \mathcal{R}_{\mathfrak{S}}(z, q)M(z)$$

*of* $Z$ *to* $M(q)$ *is contained in* $R(q)$. Indeed, this is clear if $R(q) = M(q)$ and follows from

$$R^Z = \sum_{z \in Z} \mathcal{R}_{\mathfrak{S}}(z, q)\mathcal{R}_{\mathfrak{S}}(x_F, z)F(x_F) \subset (\mathcal{R}F)(q) = R(q)$$

if $q \in \dot{F}$ (Lemma 7.3). On the other hand, we have $R(q) \subset R^Z + \mathcal{R}_{\mathfrak{S}}(y, q)M(y)$ because each $Y$ satisfies

$$R(q) \subset (\mathcal{R}Y)(q) = \sum_{z \in \dot{Y}} \mathcal{R}_{\mathfrak{S}}(s, q)M(s) = \mathcal{R}_{\mathfrak{S}}(y, q)M(y) + \sum_{z \in Z \cap \dot{Y}} \mathcal{R}_{\mathfrak{S}}(z, q)M(z).$$

Thus we are lead to distinguish the following three cases:

a) Case $R^Z + \mathcal{R}_{\mathfrak{S}}(y, q)M(y) \neq 0$. The nonzero intersection then contains some

$$r = \sum_{z \in Z} \varphi_z m_z = \varphi_y m_y \neq 0,$$

where $\varphi_s \in \mathcal{R}_{\mathfrak{S}}(s, q)$ and $m_s \in M(s)$. If $S \subset R$ denotes the simple module such that $S(q) = kr$, $\varphi_Y$ clearly belongs to Transp $(M, S)$. On the other hand, for each $X \in \mathcal{P}$ satisfying $y \notin \dot{X}$ and each $z \in Z \cap \dot{Y}$, $m_z$ can be written as $m_z = \psi_z n$ with $\psi_z \in \mathcal{R}_{\mathfrak{S}}(x_X, z)$, where $n \in M(x_X) \setminus \bigcup_Y Y(x_X)$ (see $(*)$ above). We infer that $r = \varphi_x n$, where $\varphi_x = \sum_{z \in Z} \varphi_z \psi_z$ vanishes on $Y(x_X)$ together with $\psi_z$, hence has rank 1 and belongs to Transp $(M, S)$.

b) Case $R^Z = 0$, i. e. $Z = \emptyset$. In this case, we have

$$R(q) \subset (\mathcal{R}Y)(q) = \mathcal{R}_{\mathfrak{S}}(y, q)M(y)$$

for all $Y \in \mathcal{P}$ such that $y \in \dot{Y}$. Removing these $Y$ from $\mathcal{P}$, we obtain a set $\mathcal{P}'$ of smaller cardinality which contains $F$ and satisfies the assumptions of Theorem 7.1 because $R(q) \neq M(q)$ implies $q \in \dot{F}$. The induction hypothesis then guarantees the existence of a simple submodule $S$ of $R$ such that Transp $(M, S)$ annihilates no $X \in \mathcal{P}'$, and no $Y \in \mathcal{P} \setminus \mathcal{P}'$ because of $0 \neq S(q) \subset R(q) \subset \mathcal{R}_{\mathfrak{S}}(y, q)M(y)$, $\mathcal{R}_{\mathfrak{S}}(y, q)R(y) = 0$ and $\dim M(y) / R(y) = 1$.

c) Case $R^Z \neq 0$ and $R^Z \cap \mathcal{R}_{\mathfrak{S}}(y, q)R(y) = 0$. Then we set $\mathcal{P}' = \{Y \in \mathcal{P}\colon y \in \dot{Y}\}$, and accordingly, $\dot{\mathcal{P}}' = \bigcup_{Y \in \mathcal{P}'} \dot{Y}$. We denote by $\mathfrak{S}'$ the full subspectroid of $\mathfrak{S}$ supported by $\{q\} \cup \dot{\mathcal{P}}'$, by $\mathcal{A}'$ the corresponding full subaggregate of $\mathcal{A}$. We finally set $Y' = Y \mid \mathcal{A}'$ for each $Y \in \mathcal{P}'$, $M' = \sum_{Y \in \mathcal{P}'} Y'$ and $R' = \bigcap_{Y \in \mathcal{P}'} \mathcal{R}\, Y'$. Thus we have $R'(s) = 0$ if $s \in \dot{\mathcal{P}}' \setminus q$ and

$$R'(q) = R^Z \oplus \mathcal{R}_{\mathcal{S}}(y, q)M(y) = (\mathcal{R}M')(q);$$

in particular, $R'(q) = M'(q)$ holds if $(\mathcal{R}M')(q) = M'(q)$, hence if $q \notin \dot{\mathcal{P}}'$. It follows that $M'$ and $\mathcal{P}' \mid \mathcal{A}' = \{Y' : Y \in \mathcal{P}'\}$ satisfy the assumptions of Theorem 7.1. (But we may of course have $q \notin \dot{\mathcal{P}}'$ even if $q \in \mathcal{P}'$. Here is precisely the point where the alternative $R(q) = M(q)$ of Theorem 7.1 enters the inductive argument.)

The assumptions of 7.1 pass from $M'$ and $\mathcal{P}' \mid \mathcal{A}'$ to $M'' = M'/N$ and $\mathcal{P}' = \{Y'/N : Y \in \mathcal{P}'\}$, where $N$ denotes the submodule of $R'$ such that $N(q) = \mathcal{R}_{\mathcal{S}}(y, q)M(y)$; we then have

$$R'' := \bigcap_{T \in \mathcal{P}} \mathcal{R}T = R'/N.$$

Applying our induction hypothesis to $M''$ and $\mathcal{P}''$, we find a simple submodule $S''$ of $R''$ such that Transp $(M'', S'')$ annihilates no $T = Y'/N$. Since $R^Z \xrightarrow{\sim} R''(q)$, $S''$ can be "lifted" to a simple submodule $S'$ of $R'$ such that $S'(q) \subset R^Z$. Extending $S'$ by $0$ to $\mathcal{A}$, we finally obtain the required $S \subset R$. Indeed, the construction implies that each $Y \in \mathcal{P}'$ contains a point $z \in Z \cap \dot{Y}$ such that $M(z)$ is not annihilated by Transp $(M, S)$. Since $z$ satisfies $M(z) = \mathcal{R}_{\mathcal{S}}(x_X, z)M(x_X)$ for each $X \in \mathcal{P} \setminus \mathcal{P}'$, Transp $(M, S)$ does not annihilate $X$ either.

**8. The case of a semisimple pencil.** Our main objective in this section is to prove Lemma 6.6 above.

Sticking to our previous notations and assumptions, we further suppose throughout the Sections 8.1, 8.2 and 8.4 – 8.10 that $M$ is a *faithful* module over $\mathcal{A}$ and $P$ a *semisimple* $\mathcal{L}$-pencil. This implies that $P$ *is the socle of* $M$ (5.3) and that the points $x \in \mathcal{S}$ satisfy either $0 \neq P(x) = M(x)$ or $P(x) = 0 \neq M(x)$ (5.4). In case $0 \neq P(x)$, we keep the basis chosen in 6.3, setting $M(x) = ku_x$ if $x$ is an ordinary point of $\dot{P}$ and $M(d) = ku \oplus kv$ if $d = d_P$ is the double-point. Finally, we set $\mathcal{K} = \{L \in \mathcal{L} : L(d) = M(d)\}$.

To help intuition, we may and shall choose $\mathcal{A}$ as the aggregate of all finite-dimensional projective modules over some finite-dimensional algebra. Accordingly, if $\mathcal{A}_p$ denotes the full subaggregate of $\mathcal{A}$ formed by the objects isomorphic to $p^n$, where $p \in \dot{P}$ is fixed and $n$ ranges over $\mathbb{N}$, the inclusion $\mathcal{A}_p \to \mathcal{A}$ admits a *canonical* right adjoint which maps $X \in \mathcal{A}$ onto the largest submodule $X_p$ belonging to $\mathcal{A}_p$; moreover, if $p$ is an ordinary point of $\dot{P}$ and $Y \in \mathcal{A}_p$, each vector subspace of $M(Y)$ is identified with $M(Z)$ for some submodule $Z \in \mathcal{A}_p$ of $Y$.

**8.1.** We first apply our *main algorithm* to the submodule $P$ of $M$ and to the bond $\mathcal{K}$ defined above. As usual, we set $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$, $\hat{L}(W, h, Z) = (L(Z) + h(W))/h(W)$ for all submodules $L \subset M$ and all $(W, h, Z) \in \hat{\mathcal{A}}$, and $\hat{\mathcal{K}} = \{\hat{L} : L \in \mathcal{K}\} \cup \{\hat{P}\}$. The canonical epivalence $M^k_{\mathcal{K}} \to \overline{M}^k_{\hat{\mathcal{K}}}$ (5.4) then reduces the investigation of $M^k_{\mathcal{K}}$ to $\overline{M}^k_{\hat{\mathcal{K}}}$, and we are lead to examine $\hat{\mathcal{A}}$.

The relevant part of $\mathcal{K} \cap P$ consists of the maximal submodules $P_s$, where $s \in \dot{P} \setminus d$ (5.2). In order to choose a spectroid of $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$, we consider a pair of adjoint functors

$$(P \mid \mathcal{A}_d)^k \overset{R}{\underset{S}{\rightleftarrows}} P^k.$$

The right adjoint $R$ is defined by $R(V, g, Y) = (V, g_d, Y_d)$, where $g_d$ is the $d$-component of $g: V \to P(Y) = \bigoplus_{p \in \dot{P}} P(Y_p)$. The left adjoint is such that $S(W, h, Z) = (W, \bar{h}, Z \oplus \oplus W \otimes \Sigma)$, where $\Sigma = \oplus s \in \mathcal{A}$ is the sum of all $s \in \dot{P} \setminus d$ and $\bar{h}$ maps $x \in W$ onto

$$(h(x), (x \otimes u_s)) \in P(Z) \oplus (\bigoplus_s W \otimes P(s)).$$

This left adjoint factors through $P^k_{\mathcal{K} \cap P}$ and is fully faithful and exact (for the short exact sequences considered in 2.3). Accordingly, the indecomposables $\Lambda_n, T_n^\lambda, V_n$ of $(P \mid \mathcal{A}_d)^k$ are associated with pairwise non-isomorphic indecomposables of $P^k_{\mathcal{K} \cap P}$ of the following form:

$$S\Lambda_n = (k^{n-1}, a_n, d^n \oplus \Sigma^{n-1}), \quad a_{nd} = [\mathbb{1}_{n-1} 0 \mid 0\, \mathbb{1}_{n-1}]^{\mathrm{T}},$$

$$ST_n^\lambda = (k^n, t_n^\lambda, d^n \oplus \Sigma^n), \quad t_{nd}^\lambda = [\mathbb{1}_n \mid \lambda \mathbb{1}_n + J_n]^{\mathrm{T}},$$

$$ST_n^\infty = (k^n, t_n^\infty, d^n \oplus \Sigma^n), \quad t_{nd}^\infty = [J_n \mid \mathbb{1}_n]^{\mathrm{T}},$$

$$SV_n = (k^n, z_n, d^{n-1} \oplus \Sigma^n), \quad z_{nd} = \begin{bmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & \mathbb{1}_{n-1} \end{bmatrix}.$$

The scalar $\lambda$ ranges over $k$, $n$ is $\geq 1$, $J_n$ a nilpotent Jordan-block, $a_{nd}: k^{n-1} \to P(d^n)$ the component of $a_n$ relative to $d, \ldots$ .

As a spectroid $\hat{\mathfrak{S}}$ of $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$ we choose the indecomposables $S\Lambda_n, ST_n^\lambda, SV_n$ ($n \geq 1, \lambda \in k \cup \infty$) and the $P$-spaces $(0, 0, x), x \in \mathfrak{S} \setminus d$.

**Proposition.** *There are at most* 4 *"scalars"* $\lambda \in k \cup \infty$ *such that* $ST_n^\lambda$ *is* $(\hat{M}, \hat{\mathcal{K}})$*-relevant* (6.7) *for some* $n \geq 1$.

Sections 8.4 – 8.9 are devoted to the proof of the proposition. Heretofore, we shall show that the proposition implies Lemma 6.6 above.

**8.2.** Proposition 8.1 deals with a lopped bond $\mathcal{K}$ on $M$, not with the given $\mathcal{L}$. So it remains for us to adapt the arguments of 8.1 to $\mathcal{L}$. First, we must replace $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$ by a full subaggregate $\tilde{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$. The corresponding spectroid $\tilde{\mathfrak{S}}$ is obtained from $\hat{\mathfrak{S}}$ by deletion of some $SV_n$ and some $ST_n^\lambda$. For each submodule $L$ of $M$, the $\hat{\mathcal{A}}$-module $\hat{L}$ is then replaced by its restriction $\tilde{L} = \hat{L} \mid \tilde{\mathcal{A}}$, and $\tilde{M}$ is restrained by $\tilde{\mathcal{L}} = \{\tilde{L}: L \in \mathcal{L}\} \cup \{\tilde{P}\}$. The resulting aggregate $\tilde{M}^k_{\tilde{\mathcal{L}}}$ is identified with a full subaggregate of $\hat{M}^k_{\hat{\mathcal{K}}}$. Thus we finally obtain the following corollary of Proposition 8.1.

**Proposition.** *With the preceding notations, there are at most* 4 *scalars* $\lambda \in k \cup \infty$ *such that* $ST_n^\lambda$ *is* $(\tilde{M}, \tilde{\mathcal{L}})$*-relevant for some* $n \geq 1$.

**8.3.** ***Proof of Lemma* 6.6.** The lemma follows directly from Proposition 8.2 when $M$ is faithful and $N = P$ semisimple. Our objective here is to reduce the general case to the particular one. If $N \in \mathcal{P}_e$ with $e \geq 2$, we first replace $\mathcal{L}$ by $\mathcal{L}_{e-1}$ (6.3) and are thus reduced to the case of a minimal pencil $N \in \mathcal{P}_1$. We may also replace $\mathcal{L}$ by $\mathcal{L} \cup \mathcal{K} \cup \bigcup_{P \in \mathcal{P}_1} \mathcal{K}'_P$, hence, suppose that $O_0 = \emptyset$ (6.5). Our further reduction consists of 3 steps.

***First Step.*** Here we factor out the ideal $\mathcal{J}$ of 6.4, replacing $\mathcal{A}$ by $\overline{\mathcal{A}} = \mathcal{A} / \mathcal{J}$, $M$ by $\overline{M} = M / \mathcal{J}M$ and $N$ by $\overline{N} = N / \mathcal{J}M$. The bond $\mathcal{B}N$ is replaced by the set of all $X / \mathcal{J}M$ such that $\mathcal{J}M \subset X \in \mathcal{B}N$. This set equals $\mathcal{B}\overline{N}$ if $\mathcal{L}$ is replaced by the corresponding bond on $\overline{M}$. Applying the main algorithm to the submodules $N$ and $\overline{N}$ of $M$ and $\overline{M}$, we obtain the diagram

$$\begin{array}{ccc} M^k_{\mathcal{B}N} & \xrightarrow{\ F\ } & \overline{M}^k_{\mathcal{B}\overline{N}} \\ \downarrow & & \downarrow \\ M^{Nk}_{\hat{\mathcal{B}}N} & \xrightarrow{\ G\ } & \overline{M}^{\overline{N}k}_{\hat{\mathcal{B}}\overline{N}} \end{array}$$

Since some $Y \in \mathcal{B}N$ give no contribution to $\mathcal{B}\overline{N}$, it is possible that $F$ is not an epivalence. But it is the restriction of an epivalence to a full subcategory. Hence it is surjective on the morphism-spaces and detects isomorphisms. Since the vertical arrows of the diagram are equivalences, $G$ preserves indecomposability and heteromorphism. We infer that $\mathfrak{S}^N$ (6.5) has fewer "relevant points" than $\mathfrak{S}^{\overline{N}}$, and the required statements can be lifted from $\overline{M}$ to $M$.

***Second Step.*** We suppose that $(\mathcal{R}N)(x) = 0$ for all $x \in \dot{N}$. Under this condition, we now set $\overline{M} = M / \mathcal{R}N$, $\overline{N} = N / \mathcal{R}N$ and equip $\overline{M}$ with the bond formed by all $L / \mathcal{R}N$, where $\mathcal{R}N \subset L \in \mathcal{B}N$. Applying the main algorithm to $N \subset M$ and $\overline{N} \subset \overline{M}$, we obtain modules $M^N$ and $\overline{M}^{\overline{N}}$ over some aggregates with spectroids $\mathfrak{S}^N$ and $\mathfrak{S}^{\overline{N}}$. The induced functor $\mathfrak{S}^N \to \mathfrak{S}^{\overline{N}}$ is an isomorphism because, for each $Z = (W, g, X) \in \mathfrak{S}^N$ with space-dimension $\dim W \geq 1$, $X$ is supported by $\dot{N}$ which is disjoint from the support of $\mathcal{R}N$. Accordingly, if $\mathcal{R}(N)^N$ denotes the submodule of $M^N$ associated with $\mathcal{R}N$, we have $(\mathcal{R}N)^N(Z) = 0$, and we may identify $\mathfrak{S}^N$ with $\mathfrak{S}^{\overline{N}}$ and $M^N / (\mathcal{R}N)^N$ with $\overline{M}^{\overline{N}}$. The equality $(\mathcal{R}N)^N(Z) = 0$ implies that, for any $M^N$-space $(U, h, Z')$, the canonical map

$$M^{Nk}((U, h, Z'), (0, 0, Z)) \to \overline{M}^{\overline{N}k}((U, h, Z'), (0, 0, Z))$$

is bijective. Therefore, $Z$ is relevant with respect to $(M^N, \hat{\mathcal{B}}N)$ if it is so with respect to $(\overline{M}^{\overline{N}}, \hat{\mathcal{B}}\overline{N})$. Thus we are reduced from $M$ to $\overline{M}$.

***Third Step.*** Here we may suppose that $\mathcal{R}N = 0$. But formally we still have to reduce our statement to the case where $M$ is faithful. For this sake, we denote by $\overline{\mathcal{A}}$ the residue-category of $\mathcal{A}$ modulo the annihilator of $M$. If $\overline{M}$ and $\overline{N}$ are the $\overline{\mathcal{A}}$-modules associated with $M$ and $N$, the canonical functor $M^{Nk}_{\hat{\mathcal{B}}N} \to \overline{M}^{\overline{N}k}_{\hat{\mathcal{B}}\overline{N}}$ is quasi-surjective. Therefore, the isoclasses of "relevant" points of $\mathfrak{S}^N$ correspond bijectively to those of $\mathfrak{S}^{\overline{N}}$.

**8.4.** We now return to Proposition 8.1. Before entering its proof, we examine the notion of *relevance*. Let us provisionally consider an arbitrary pointwise finite module $M'$ over an aggregate $\mathcal{A}'$ and a bond $\mathcal{L}'$ on $M'$. Equipped with the short exact sequences defined in 2.3, $M'^k_{\mathcal{L}'}$ is an exact category. Accordingly, an $M'$-space $(V, f, X) \in$

$\in M'^k_{\mathcal{L}'}$ is called $(M', \mathcal{L}')$-*injective* if, for each short exact sequence

$$0 \longrightarrow (W', g', Y') \xrightarrow[(i,j)]{} (W, g, Y) \xrightarrow[(p,q)]{} (W'', g'', Y'') \longrightarrow 0$$

formed by $M'$-spaces avoiding $\mathcal{L}'$, each morphism from $(W', g', Y')$ to $(V, f, X)$ factors through $(i, j)$. It is equivalent to say that, for each $(W, g, Y) \in M'^k_{\mathcal{L}'}$, each linear map $m: W \to M(X) / f(V)$ is a composition of the from

$$W \xrightarrow{g} M(Y) \xrightarrow{M(\eta)} M(X) \xrightarrow{\text{can.}} M(X) / f(V).$$

The indecomposable $(M', \varnothing)$-injectives are easy to describe; they have the form $(k, 0, 0)$ or $(M', (s), \mathbb{1}, s)$. The general case $\mathcal{L}' \neq \varnothing$ seems to be more intricate. In the following lemma we examine indecomposables $s \in \mathcal{A}'$ such that $(0, 0, s)$ is $(M', \mathcal{L}')$-injective; then we simply say that $s$ is $(M', \mathcal{L}')$-*injective*.

**Lemma.** *An indecomposable* $s \in \mathcal{A}'$ *is* $(M', \mathcal{L}')$-*irrelevant if and only if* $s$ *is* $(M', \mathcal{L}')$-*injective and satisfies* $L'(s) = M'(s)$ *for each maximal element* $L'$ *of* $\mathcal{L}'$.

***Proof.*** a) The condition is *sufficient*: if $(V, [f\, g]^{\mathrm{T}}, Y \oplus s)$ avoids $\mathcal{L}'$, the equalities $L'(s) = M'(s)$ considered above imply that $(V, f, X) \in M^k_{\mathcal{L}'}$. Hence, we have a short exact sequence

$$0 \longrightarrow (0, 0, s) \longrightarrow (V, [fg]^{\mathrm{T}}, Y \oplus s) \longrightarrow (V, f, Y) \longrightarrow 0$$

of $M'^k_{\mathcal{L}'}$, which splits because $s$ is $(M', \mathcal{L}')$-injective.

b) The condition is *necessary*. In order to show that $s$ is $(M', \mathcal{L}')$-injective, it suffices to prove that the exact sequence

$$0 \longrightarrow (0, 0, s) \xrightarrow[(0,\, [0\, \mathbb{1}]^{\mathrm{T}})]{} (V, [fg]^{\mathrm{T}}, Y \oplus s) \xrightarrow[(\mathbb{1},\, [\mathbb{1}\, 0]^{\mathrm{T}})]{} (V, f, Y) \longrightarrow 0$$

splits if $(V, f, Y)$ is indecomposable. But this is clear if $(V, f, Y) \xrightarrow{\sim} (0, 0, s)$. If not, $Y$ has no direct summand isomorphic to $s$. Decomposing the middle term into indecomposables, we obtain an isomorphism

$$(V, [fg]^{\mathrm{T}}, Y \oplus s) \xrightarrow[i]{\sim} (V, h, Y) \oplus (0, 0, s)$$

whose components are, say $(e, [a\, b])$ and $(0, [c\, d])$. The composition of $i$ with $(0, [0\, \mathbb{1}]^{\mathrm{T}})$ is a section with components $(0, b)$ and $(0, d)$. Since $b$ cannot be a section, $d$ is an isomorphism, and our short exact sequence splits.

Let us now turn to a maximal $L' \in \mathcal{L}'$. In case $L'(s) \neq M'(s)$, we consider the submodule $N'$ of $M'$ which is generated by $L'$ and $M'(s)$. Since the generation-indicator of $N'$ contains $s$, the indecomposable $M'$-space associated with $N'$ in 6.1 has the form $(k, f, Y \oplus s)$ and avoids $\mathcal{L}'$. This contradicts our assumptions that $s$ is $(M', \mathcal{L}')$-irrelevant.

**8.5.** We now return to the *assumptions of Proposition* 8.1 and start with the proof. By 5.6, each $L \in \mathcal{K}$ satisfies $L \cap P = P_s$ for some ordinary point $s \in \dot{P}$. It easily follows that $\hat{K}(ST_n^{\lambda}) = \hat{P}(ST_n^{\lambda}) = \hat{M}(ST_n^{\lambda})$ holds for each $\hat{K} \in \hat{\mathcal{K}}$. Hence, $ST_n^{\lambda}$ is $(\hat{M}, \hat{\mathcal{K}})$-relevant if and only if it is not $(\hat{M}, \hat{\mathcal{K}})$-injective.

*Thus, our objective is to show that* Ext $(X, (0, 0, ST_n^{\lambda})) = 0$ *for all* $X \in \hat{M}^k_{\hat{\mathcal{K}}}$ *provided* $\lambda$ *avoids some finite set* $e$. The extension-groups Ext $(X, (W, h, Z))$ considered here can be computed within the surrounding category $\hat{M}^k$ with the help of an injective resolution of $(W, h, Z)$ in $\hat{M}^k$ of the following form:

$$0 \longrightarrow (W, h, Z) \longrightarrow (\operatorname{Ker} h, 0, 0) \oplus (\hat{M}(Z), \mathbb{1}, Z) \longrightarrow (\operatorname{Coker} h, 0, 0) \longrightarrow 0.$$

The resolution shows that Ext is right exact on the short exact sequences of $\hat{M}^k$ considered here (2.3).

We display the spectroid $\hat{\mathfrak{L}}$ of $\hat{\mathcal{A}}$ (8.1) in such a way that all morphisms from the right to the left vanish ($s \in \mathfrak{L} \setminus d, \lambda \in k \cup \infty$):

$$(0, 0, s), S\Lambda_1, S\Lambda_2, S\Lambda_3, \ldots, ST_n^{\lambda}, \ldots, SV_3, SV_2, SV_1.$$

In particular, $\operatorname{Hom}(SF, (0, 0, s)) = 0$ for all $s \in \mathfrak{L} \setminus d$ and all $F \in (\dot{P} \mid \mathcal{A}_d)^k$. It follows that each $A \in \hat{\mathcal{A}}$ gives rise to a canonical split sequence

$$0 \longrightarrow A_P \xrightarrow[\iota]{} A \xrightarrow[\pi]{} A/A_P \longrightarrow 0,$$

where $A_P$ is isomorphic to some $SF$, and $A/A_P$ to some $\bigoplus_{i \in I}(0, 0, s_i)$ with $s_i \in \mathfrak{L} \setminus d$. Accordingly, each $(U, f, A) \in \hat{M}^k$ gives rise to an *exact sequence*

$$0 \longrightarrow (0, 0, A_P) \xrightarrow[(0, \iota)]{} (U, f, A) \xrightarrow[(\mathbb{1}, \pi)]{} (U, \operatorname{can} \circ f, A/A_P) \longrightarrow 0 \qquad (*)$$

of $\hat{M}^k$. In case $(U, f, A) \in \hat{M}^k_{\hat{\mathcal{K}}}$, the end terms $(0, 0, A_P)$ and $(U, \operatorname{can} \circ f, A/A_P)$ also belong to $\hat{M}^k_{\hat{\mathcal{K}}}$ because $\hat{L}(SF) = \hat{M}(SF)$, $\forall L \in \mathcal{K}$, $\forall F \in (\dot{P} \mid \mathcal{A}_d)^k$. We shall denote by $\hat{M}_1^k$ and $\hat{M}_2^k$ the *full subaggregates* of $\hat{M}^k_{\hat{\mathcal{K}}}$ formed by the $(U, f, A)$ such that $A_P = A$ and $A_P = 0$ respectively.

Now, since we have $\operatorname{Ext}((0, 0, A_P), (0, 0, ST_n^{\lambda})) = 0$ by the definition of the exact sequences of $\hat{M}^k$, we infer that the map

$$\operatorname{Ext}((U, f, A), (0, 0, ST_n^{\lambda})) \leftarrow \operatorname{Ext}((U, \operatorname{can} \circ f, A/A_P), (0, 0, ST_n^{\lambda})),$$

is surjective, and we are reduced to proving the following lemma.

**Lemma.** *If $M$ is not $\mathcal{L}$-wild, there exists a subset $e \subset k \cup \infty$ of cardinality $\leq 4$ such that* $\operatorname{Ext}(X, (0, 0, ST_n^{\lambda})) = 0$ *for all $X \in \hat{M}_2^k$, all $n \geq 1$ and all $\lambda \in (k \cup \cup \infty) \setminus e$.*

**8.6.** Lemma 8.5 concerns the aggregate $\hat{M}^k_{\hat{\mathcal{K}}}$. Our next step brings us back to $M^k_{\mathcal{K}}$ via the rum functor

$$\Phi : \hat{M}^k_{\hat{\mathcal{K}}} \to M^k_{\mathcal{K}}, \ (U, f, (W, h, Z)) \mapsto (V, g, Z) \oplus (\operatorname{Ker} h, 0, 0),$$

where $V \subset M(Z)$ is the inverse image of $f(U) \subset M(Z)/h(W)$ and $g$ the inclusion. This functor induces a bijection between the sets of isoclasses of $\hat{M}^k_{\hat{\mathcal{K}}}$ and $M^k_{\mathcal{K}}$. It is a quasi-inverse of the classical equivalence $M^k_{\mathcal{K}} \to \hat{M}^k_{\hat{\mathcal{K}}}$ if $\mathcal{K} \neq \varnothing$, i. e. if $\dot{P} \setminus d \neq \varnothing$. In general, the main virtue of $\Phi$ is to be exact, whereas $\hat{M}^k_{\hat{\mathcal{K}}} \to M^k_{\mathcal{K}}$ is not because $M^k_{\mathcal{K}}$ has "more" exact sequences than $\hat{M}^k_{\hat{\mathcal{K}}}$. In fact, for all $A_1, A_2 \in \hat{M}^k_{\hat{\mathcal{K}}}$, $\Phi$ induces an *injection*

$$\operatorname{Ext}(A_2, A_1) \to \operatorname{Ext}(\Phi A_2, \Phi A_1),$$

whose image consists of all classes of short exact sequences

$$0 \to \Phi A_1 = (V_1, g_1, Z_1) \to (V_3, g_3, Z_3) \to \Phi A_2 = (V_2, g_2, Z_2) \to 0$$

of $M^k_{\mathcal{K}}$ such that the induced sequence

$$0 \to (g_1^{-1}(PZ_1), g_1', Z_1) \to (g_3^{-1}(PZ_3), g_3', Z_3) \to (g_2^{-1}(PZ_2), g_2', Z_2) \to 0$$

is split exact in $\hat{\mathcal{A}} = P^k_{\mathcal{K}\cap P}$. Such exact sequences of $M^k_{\mathcal{K}}$ will be called *P-exact.*

In particular, if $(U, f, A)$ ranges over $\hat{M}^k_{\hat{\mathcal{K}}}$, the images of the sequences (*) under $\Phi$ are short exact sequences of $M^k_{\mathcal{K}}$. Up to isomorphism, they can be described directly as follows. Let us consider the two pairs of adjoint functors

$$(P \mid \mathcal{A}_d)^k \underset{R}{\overset{S}{\rightleftarrows}} P^k_{\mathcal{K}\cap P} \underset{R'}{\overset{S'}{\rightleftarrows}} M^k_{\mathcal{K}},$$

where $R, S$ are defined as in 8.1, $S'$ is the functor $(W, h, Z) \mapsto (W, h, Z)$ induced by the inclusion $P \to M$, and $R'$ is the *trace-functor* $(V, g, Y) \mapsto (g^{-1}(PY), g', Y)$ already considered above. With each $(V, g, Y) \in M^k_{\mathcal{K}}$, the adjoint pair $(RR', S'S)$ associates a *canonical short exact sequence*

$$0 \to (g^{-1}(PY), \bar{g}_d, Y') \xrightarrow{(\nu, \iota)} (V, g, Y) \xrightarrow{(\varphi, \pi)} (V/g^{-1}(PY), g'', Y/Y') \to 0, \qquad (**)$$

of $M^k_{\mathcal{K}}$, where $Y' = Y_d \oplus g^{-1}(PY) \otimes \Sigma$. These sequences are related to the short exact sequences (*) of 8.5 via the rum $\Phi$. If we denote by $M^k_1$ and $M^k_2$ *the full subaggregates* of $M^k_{\mathcal{K}}$ formed by the pairs $(V, g, Y)$ which induce isomorphisms $(\nu, \iota)$ and $(\varphi, \pi)$ respectively, then $S'S$ induces an equivalence $(P \mid \mathcal{A}_d)^k \xrightarrow{\sim} M^k_1$, whereas $M^k_2$ is equivalent to $M'_{\mathcal{K}'\cap P'}$ where $M'$, $\mathcal{K}'$, $P'$ denote the restrictions of $M$, $\mathcal{K}$, $P$ to $\mathfrak{L} \setminus d$. The functor $\Phi : \hat{M}^k_{\hat{\mathcal{K}}} \to M^k_{\mathcal{K}}$ maps $\hat{M}^k_1$ into $M^k_1$ and induced an equivalence $\hat{M}^k_2 \xrightarrow{\sim} M^k_2$. Moreover, in the case $A_1 \in \hat{M}^k_1$ and $A_2 \in \hat{M}^k_2$, all short exact sequences

$$0 \to \Phi A_1 \to E \to \Phi A_2 \to 0$$

of $M^k_{\mathcal{K}}$ are obviously $P$-exact. Hence, $\Phi$ induces a bijection

$$\mathrm{Ext}\,(A_2, A_1) \xrightarrow{\sim} \mathrm{Ext}\,(\Phi A_2, \Phi A_1),$$

and Lemma 8.5 is reduced to the following lemma, where we set $E^{\mathfrak{L}} = S'SE$ for all $E \in (P \mid \mathcal{A}_d)^k$.

**Lemma.** *If $M$ is not $\mathcal{L}$-wild, there exists a subset $e \subset k \cup \infty$ of cardinality $\le 4$ such that* $\mathrm{Ext}\,(H, T_n^{\lambda \mathfrak{L}}) = 0$ *for all $H \in M^k_2$, all $n \ge 1$, and all $\lambda \in (k \cup \infty) \setminus e$.*

**8.7.** In order to prove Lemma 8.6, we start with an arbitrary $H \in M^k$ and some $F = E^{\mathfrak{L}} \in M^k_1$, where $E \in (P \mid \mathcal{A}_d)^k$. For the exact structure defined in 2.3, $M^k$ admits almost split sequences [8, 9]. If $\tau H$ denotes the cotranslate of $H$, we know that

$$\mathrm{Ext}(H, F) \xrightarrow{\sim} \underline{\mathrm{Hom}}\,(F, \tau H)^{\mathrm{T}},$$

where $W^{\mathrm{T}}$ denotes the dual of a vector space $W$ and $\underline{\mathrm{Hom}}\,(F, \tau H)$ the residue-space of $\mathrm{Hom}(F, \tau H)$ obtained by annihilation of the morphisms factoring through injectives of $M^k$. Now, since $F$ admits an injective resolution whose indecomposable injective summands have the form $(k, 0, 0)$ or $(M(p), \mathbb{1}, p)$, $p \in \dot{P}$, it suffices to annihilate the morphisms factoring through these injectives. But $\tau H$ has no nonzero injective direct summand. It easily follows that all morphisms from $(k, 0, 0)$ or $(M(p), \mathbb{1}, p)$ to $\tau H$ vanish and that

$$\mathrm{Ext}(H, F) \tilde{\to} \mathrm{Hom}(E^{\mathfrak{S}}, \tau H)^{\mathrm{T}} \tilde{\to} \mathrm{Hom}(E, (\tau H)_d)^{\mathrm{T}}$$

if we set $K_d = RR'K \in (P \mid \mathcal{A}_d)^k$ for all $K \in M^k$.

Now, *in case* $H \in M_2^k$, the following lemma states that $(\tau H)_d$ is a direct sum of indecomposables $\Lambda_n$ and $T_n^{\lambda}$, where $\lambda$ belongs to some subset $e \subset k \cup \infty$ of cardinality $\leq 4$. If follows that $\mathrm{Hom}\,(E, (\tau H)_d) = 0$ if $E = V_n$ or $E = T_n^{\mu}$ with $\mu \subset \subset k \cup \infty \setminus e$. So it remains for us to prove the following lemma.

**Lemma.** *Let* $e \subset k \cup \infty$ *be the set of all* $\lambda \subset k \cup \infty$ *such that, for some* $n \geq \geq 1$ *and some* $H \in M_2^k$, $T_n^{\lambda}$ *is isomorphic to a direct summand of* $(\tau H)_d$. *Then the cardinality of* $e$ *is* $\leq 4$. *Furthermore, if* $H \in M_2^k$, $(\tau H)_d$ *has no direct summand isomorphic to* $V_n$, $n \geq 1$.

**8.8.** Lemma 8.7 will finally result from the virtues of some restriction $\overline{M}$ of the module $\hat{M}$ examined in 8.1. Let $\overline{\mathfrak{S}}$ denote the finite full subspectroid of $\tilde{\mathfrak{S}}$ formed by $S\Lambda_3$ and all $(0, 0, s)$, $s \in \mathfrak{S} \setminus d_P$. Let $\overline{\mathcal{A}}$ be the full subaggregate of $\hat{\mathcal{A}}$ formed by the points of $\overline{\mathfrak{S}}$, all isomorphic indecomposable, and their finite direct sums. The restriction $\overline{M} = \hat{M} \mid \overline{\mathcal{A}}$ and $\overline{\mathcal{K}} = \left\{ K \mid \overline{\mathcal{A}} : K \in \hat{\mathcal{K}} \right\}$ then satisfy the following lemma.

**Lemma.** $\overline{M}$ *is not* $\overline{\mathcal{K}}$*-wild.*

***Proof.*** We know that the module $\tilde{M}$ of 8.2 is not $\tilde{\mathcal{L}}$-wild. It has a submodule $N$ which vanishes at $S\Lambda_3$, $S\Lambda_2$, $S\Lambda_1$, and all $(0, 0, s)$ with $s \in \mathfrak{S} \setminus d_P$, and which takes the same values as $\tilde{M}$ at all other points of $\tilde{\mathfrak{S}}$. By 3.7 $\tilde{M}/N$ is not $(\tilde{\mathcal{L}}/N)$-wild if we set $\tilde{\mathcal{L}}/N = \{K/N : N \subset K \in \tilde{\mathcal{L}}\}$. The condition $N \subset K$ eliminates all $K$ of the form $K = \tilde{L}$ with $L(d_P) \neq M(d_P)$. Hence, only $\mathcal{K}$ contributes to $\tilde{\mathcal{L}}/N$, and $\overline{M}, \overline{\mathcal{K}}$ are identified with the restrictions of $\tilde{M}/N$, $\tilde{\mathcal{L}}/N$ to $\overline{\mathcal{A}}$.

**8.9.** ***Proof of Lemma*** **8.7.** a) Obviously, $\overline{M}_{\overline{\mathcal{K}}}^k$ can be identified with the full subcategory of $\hat{M}_{\hat{\mathcal{K}}}^k$ formed by the $\hat{M}$-spaces $(U, f, A)$ such that $A_P$ (8.5) is a direct sum of copies of $S\Lambda_3$. Setting $X = (U, \mathrm{can} \circ f, A/A_P) \in \hat{M}_2^k$ and denoting by

$$\varepsilon \in \mathrm{Ext}(X, (0, 0, A_P)) \tilde{\to} \mathrm{Hom}_k(\mathrm{Hom}(A_P, S\Lambda_3), \mathrm{Ext}(X, (0, 0, S\Lambda_3))$$

the extension associated with an $\overline{M}$-space $(U, f, A) \in \overline{M}_{\overline{\mathcal{K}}}^k$ and with the sequence

$$0 \longrightarrow (0, 0, A_P) \xrightarrow[(0,\,\iota)]{} (U, f, A) \xrightarrow[(\mathbb{1},\,\pi)]{} X = (U, \mathrm{can} \circ f, A/A_P) \longrightarrow 0$$

in 8.5, we obtain an epivalence

$$\Psi : \overline{M}_{\overline{\mathcal{K}}}^{k\,\mathrm{op}} \longrightarrow \hat{E}^k, \quad (U, f, A) \mapsto (\mathrm{Hom}(A, S\Lambda_3), \varepsilon, X),$$

where $\hat{E}$ is the module on $\hat{M}_2^{k\,\mathrm{op}}$ such that $\hat{E}(X) = \mathrm{Ext}(X, (0, 0, S\Lambda_3))$. This epivalence can be composed with an equivalence $\hat{E}^k \tilde{\to} E^k$ which results from the equivalence $\hat{M}_{\hat{\mathcal{K}}}^k \tilde{\to} M_{\mathcal{K}}^k$ and from the invariance

$$\mathrm{Ext}(A_2, A_1) \tilde{\to} \mathrm{Ext}(\Phi A_2, \Phi A_1),\ A_1 \in \hat{M}_1^k, A_2 \in \hat{M}_2^k$$

examined in 8.6. By $E$ we here denote the module

$$H \mapsto \mathrm{Ext}(H, \Lambda_3^{*}) \tilde{\to} \mathrm{Hom}(\Lambda_3, (\tau H)_d)^{\mathrm{T}}$$

which is defined on the aggregate $M_2^{k\,\mathrm{op}}$ (8.6).

b) In the epivalence $\overline{M}_{\mathcal{K}}^{k\,\mathrm{op}} \to E^k$ derived above, the point is that $E$ is free of any bond. Before exploiting this point, we must transfer "tameness" from $\overline{M}$ to $E$.

**Lemma.** *$E$ is not wild.*

***Proof.*** It suffices to prove that $\hat{E}$ is tame. If not, there is a plane coordinate system

$$e_0, e_1, e_2 \in \mathrm{Ext}((U, g, B), (0, 0, W^T \otimes S\Lambda_3)) \tilde{\to} \mathrm{Hom}_k(W, \hat{E}(U, g, B))$$

such that the induced functor $\mathrm{rep}Q^2 \to \hat{E}^k$ preserves indecomposability and heteromorhism. The extensions $e_i$ are the classes of short exact sequences which we may write as follows

$$0 \longrightarrow (0, 0, W^T \otimes S\Lambda_3) \xrightarrow[(0,\iota)]{} \left(U, \begin{bmatrix} h_i \\ g \end{bmatrix}, W^T \otimes S\Lambda_3 \oplus B\right) \xrightarrow[(\mathbb{1},\pi)]{} (U, g, B) \longrightarrow 0$$

where $\iota$ and $\pi$ are the canonical immersion and projection. Setting $f_0 = [h_0\, g]^T$ and $f_i = [h_i\ 0]^T$ for $i = 1, 2$, we obtain a plane coordinate system

$$f_0, f_1, f_2 \in \mathrm{Hom}_k(U, \overline{M}(W^T \otimes S\Lambda_3 \oplus B)).$$

The induced functor $F_f\colon \mathrm{rep}Q^2 \to \overline{M}^k$ factors through $\overline{M}_{\mathcal{K}}^k$ by construction. We claim that the composition

$$\mathrm{rep}Q^2 \xrightarrow[D]{} (\mathrm{rep}Q^2)^{\mathrm{op}} \xrightarrow[F_f]{} \overline{M}_{\mathcal{K}}^{k\,\mathrm{op}} \xrightarrow[\Psi]{} \hat{E}^k,$$

where $D$ is induced by the duality of vector spaces, is isomorphic to $F_e$. This implies that $F_f$ preserves indecomposability and heteromorphisms, a contradiction to Lemma 8.8.

Our claim follows from the observation that the map

$$\mathrm{Hom}_k(U, \overline{M}(C)) \longrightarrow \mathrm{Ext}((U, g, B), (0, 0, C)),\ h \mapsto \bar{h},$$

where $\bar{h}$ denotes the class of the short exact sequence

$$0 \longrightarrow (0, 0, C) \xrightarrow[(0,\iota)]{} \left(U, \begin{bmatrix} h_i \\ g \end{bmatrix}, C \oplus B\right) \xrightarrow[(\mathbb{1},\pi)]{} (U, g, B) \longrightarrow 0, \qquad (***)$$

is $k$-linear for all $C = W^T \otimes S\Lambda_3$. To ascertain this point, we compute the extension group using the injective resolution

$$0 \longrightarrow (0, 0, C) \xrightarrow[(0,\mathbb{1})]{} (\overline{M}(C), \mathbb{1}, C) \xrightarrow[(\mathbb{1},0)]{} (\overline{M}(C), 0, 0) \longrightarrow 0$$

of $(0, 0, C)$ in $\overline{M}^k$. The induced linear map

$$\mathrm{Hom}((U, g, B), (\overline{M}(C), 0, 0)) \longrightarrow \mathrm{Ext}((U, g, B), (0, 0, C))$$

maps $(h, 0)$ onto the induced pull-back of the chosen resolution. This pull-back is isomorphic to (***).

c) Let us now suppose that Lemma 8.7 is false, and *let* $H \in M_2^k$ *be such that* $(\tau H)_d$ *has a direct summand of the form* $V_n$. Then we may further assume that $H$ is indecomposable and denote by $\mathcal{H}$ the full subaggregate of $M_2^k$ formed by the objects isomorphic to $H^r$, $r \in \mathbb{N}$. If $m$ is the smallest number satisfying $\mathrm{Hom}(V_m, (\tau H)_d) \neq 0$, then $\mathrm{Hom}(V_m, (\tau H)_d) \otimes V_m$ is identified with a nonzero direct summand of $(\tau H)_d$, and

$$X \mapsto \mathrm{Hom}(V_m, (\tau H)_d) \otimes \mathrm{Hom}(\Lambda_3, V_m)$$

with a submodule of

$$E^{\mathrm{T}} \mid \mathcal{H}: X \mapsto \mathrm{Hom}(\Lambda_3, (\tau H)_d) \xrightarrow{\sim} \mathrm{Ext}(X, \Lambda_3^{\circ})^{\mathrm{T}}.$$

Accordingly, each simple submodule $S$ of $X \mapsto \mathrm{Hom}(V_m, (\tau H)_d)$ provides a semisimple submodule $S \otimes \mathrm{Hom}(\Lambda_3, V_m)$ of $E^{\mathrm{T}} \mid \mathcal{H}$ such that

$$\dim S(H) \otimes \mathrm{Hom}(\Lambda_3, V_m) = \dim \mathrm{Hom}(\Lambda_3, V_m) = m + 2 \geq 3.$$

We infer that $E \mid \mathcal{H}^{\mathrm{op}}$ has a semisimple residue-module whose dimension at $H$ is $\geq$ 3; and hence, that $E$ is wild in contradiction to the lemma of part b).

d) *Let us finally suppose that* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ *are distinct scalars and* $H$ *is an object of* $M_2^k$ *such that, for each* $i$, $(\tau H)_d$ *has a direct summand of the form* $T_{n_i}^{\lambda_i}$. We then denote by $\mathcal{H}$ the full subaggregate of $M_2^k$ formed by the objects isomorphic to direct summands of $H^r$, $r \in \mathbb{N}$. The restriction $E^{\mathrm{T}} \mid \mathcal{H}$ contains a direct sum of 5 nonzero submodules of the form

$$X \mapsto \mathrm{Hom}(T_1^{\lambda_i}, (\tau H)_d) \otimes \mathrm{Hom}(\Lambda_3, T_1^{\lambda_i}).$$

Accordingly, if $S_i$ is a simple submodule of $X \mapsto \mathrm{Hom}(T_1^{\lambda_i}, (\tau H)_d)$, $E^{\mathrm{T}} \mid \mathcal{H}$ has a semisimple submodule of the form

$$\bigoplus_{i=1}^{5} S_i \otimes \mathrm{Hom}(\Lambda_3, T_1^{\lambda_i}),$$

and $E \mid \mathcal{H}^{\mathrm{op}}$ has a semisimple residue-module of length 5. We infer that $E$ is wild in contradiction to the lemma of part b).

**9. From subspaces to modules.** In the present section, we apply our second main theorem (2.5) to a finite-dimensional $k$-algebra $B$. For this sake, we consider a proper quotient $\overline{\mathcal{T}}$ of a spectroid $\mathcal{T}$ of $B$ and reduce $\mathrm{mod}\, B \xrightarrow{\sim} \mathrm{mod}\, \mathcal{T}$ to a "subspace-category" $M_N^k$, where $M$ and $N$ are suitable left modules over $\mathrm{mod}\, \overline{\mathcal{T}}$.

**9.0.** Since we prefer working with finite spectroids rather than with finite-dimensional algebras, we first adapt the language introduced in 2.6 to the case of a *finite spectroid* $\mathcal{T}$.

First, we introduce the $k$-category $\otimes\mathcal{T}$ whose objects are the points of $\mathcal{T}$ and whose morphism-spaces are defined by

$$(\otimes\mathcal{T})(r, s) = \bigoplus_x \mathcal{T}(x_{n-1}, s) \otimes_k \ldots \otimes_k \mathcal{T}(x_1, x_2) \otimes_k \mathcal{T}(r, x_1),$$

where $x$ ranges over the sequences of points of $\mathcal{T}$ of length $n \geq 0$. (In case $n = 0$, the displayed tensor-product coincides with $\mathcal{T}(r, s)$.) The composition of $\otimes\mathcal{T}$ is induced by tensor-multiplication.

Let $\operatorname{mod}\otimes\mathcal{T}$ and $\operatorname{mod}\mathcal{T}$ denote the categories of all finite-dimensional right modules over $\otimes\mathcal{T}$ and $\mathcal{T}$, i. e. of all *contravariant* $k$-linear functors from $\otimes\mathcal{T}$ and $\mathcal{T}$ to $\operatorname{mod}k$. An object of $\operatorname{mod}\otimes\mathcal{T}$ is given by a family $U = (U(s))_{s\in\mathcal{T}}$ of "*stalks*" $U(s) \in \operatorname{mod} k$ and by a family of linear maps lying in

$$H_U := \prod_{r,s\in\mathcal{T}} \operatorname{Hom}_k(U(r)\otimes\mathcal{T}_k(r,s), U(s)).$$

We shall identify $\operatorname{mod}\mathcal{T}$ with a full subcategory of $\operatorname{mod}\otimes\mathcal{T}$ by the aid of the canonical functor $\otimes\mathcal{T}\to\mathcal{T}$.

Each coordinate system $e = (e_0,\dots, e_t)$ of an affine subspace $S \subset H_U$ gives rise to a functor $F_e : \operatorname{rep}Q^t \to \operatorname{mod}\otimes\mathcal{T}$ which maps a sequence $a = (a_1,\dots, a_t)$ of $t$ endomorphisms $a_i : W\to W$ onto the family $W\otimes U = (W\otimes_k U(s))_{s\in\mathcal{T}}$ equipped with the linear maps

$$\mathbb{1}_W\otimes e_0(r,s) + a_1\otimes e_1(r,s) + \dots + a_t\otimes e_t(r,s) : W\otimes U(r)\otimes\mathcal{T}(r,s) \to W\otimes U(s).$$

The space $S$ is called *$\mathcal{T}$-reliable* if $F_e$ factor through $\operatorname{mod}\mathcal{T}$ and preserves indecomposability and heteromorphism. And $\mathcal{T}$ is called *wild* if it admits a $\mathcal{T}$-reliable plane. If not, $\mathcal{T}$ is *tame.*

**Lemma.** *Let $B$ be a finite-dimensional algebra with spectroid $\mathcal{T}$. Then $B$ is wild if so is $\mathcal{T}$.*

***Proof.*** We may suppose that the points of $\mathcal{T}$ are projective $B$-modules $\varepsilon_1 B,\dots,\dots,\varepsilon_m B$, where the $\varepsilon_1$ denote primitive idempotents. Choosing an isomorphism $B \xrightarrow{\sim} \bigoplus_{i=1}^{m}(\varepsilon_i B)^{n_i}$ of $\operatorname{mod} B$, we then identify the algebra $B$ with the matrix-algebra $\bigoplus_{i,j}(\varepsilon_i B\,\varepsilon_j)^{n_i\times n_j}$.

Now let $U = (U_i)_{1\le i\le m}$ be a family of stalks and $e_0, e_1, e_2 \in \prod_{i,j}\operatorname{Hom}_k(U_i\otimes\varepsilon_i B\,\varepsilon_j, U_j)$ be a coordinate system of a $\mathcal{T}$-reliable plane. If $V$ denotes the direct sum of the spaces $U_i^{1\times n_i}$ formed by the rows with $n_i$ entries in $U_i$, we obtain a coordinate system $f_0, f_1, f_2 \in \operatorname{Hom}_k(V\otimes B, V)$ of a $B$-reliable plane by setting

$$f_p(v\otimes b) = \Big(\sum_{i=1}^{m} v^i e_p(i,j;b^{ij})\Big)_{1\le j\le m} \in \bigoplus_{j=1}^{m} U_j^{1\times n_j} = V$$

for all $v = (v^i) \in \bigoplus_i U^{1\times n_i} = V$ and all $b = (b^{ij}) = \bigoplus_{i,j}(\varepsilon_i B\,\varepsilon_j)^{n_i\times n_j} = B$. Here

$$e_p(i,j;\, b^{ij}) \in \operatorname{Hom}_k(U_i, U_j)^{n_i\times n_j}$$

denotes a matrix whose entries are defined by

$$e_p(i,j;\, b^{ij})_{rs}(x) = e_p(x\otimes b^{ij}_{rs}).$$

In the case $t = 1$, we also consider *punched lines* $S\setminus E$, where $E$ is a finite subset of $S$. Setting $C = \{\lambda \in k : e_0 + \lambda e_1 \in S\setminus E)$ as in 2.5 and 2.6, we say that $S\setminus E$ is $\mathcal{T}$-reliable if $F_e : \mathrm{rep}_C Q^1 \to \mathrm{mod} \otimes \mathcal{T}$ factors through $\mathrm{mod}\,\mathcal{T}$ and preserves indecomposability and heteromorphism. As in the case of reliable planes considered above, $\mathcal{T}$-reliable punched lines give rise to $B$-reliable punched lines whenever $B$ is a finite-dimensional algebra with spectroid $\mathcal{T}$. Thus, in order to prove our third main theorem, it suffices to construct suitable $\mathcal{T}$-reliable punched lines whenever $\mathcal{T}$ is tame and to carry them over to $B$. As a corollary, we obtain the converse of the lemma above ($B$ is tame if so is $\mathcal{T}$), which of course could also be proved directly.

**9.1.** Let $\mathcal{T}$ be an arbitrary *finite spectroid* over $k$, $\sigma \in \mathcal{R}_{\mathcal{T}}(s, t)$ a nonzero radical morphism of $\mathcal{T}$ such that $\mathcal{R}_{\mathcal{T}}(t, x)\sigma = 0 = \sigma\mathcal{R}_{\mathcal{T}}(x, s)$ for all $x \in \mathcal{T}$, and $\overline{\mathcal{T}} = \mathcal{T}/\sigma$. For each $X \in \mathrm{mod}\mathcal{T}$, we denote by $\underline{X}$ the largest submodule of $X$ annihilated by $\sigma$. Concretely, $\underline{X}$ satisfies $\underline{X}(x) = X(x)$ for all $x \in \mathcal{T}\setminus t$, whereas $\underline{X}(t)$ is the kernel of $X(\sigma) : X(t) \to X(s)$. Accordingly, $X/\underline{X}$ is semisimple and located at $t$. The obvious exact sequence

$$0 \longrightarrow \underline{X} \longrightarrow X \longrightarrow X/\underline{X} \longrightarrow 0,$$

therefore, provides a linear map

$$\varepsilon_X \in \mathrm{Hom}_k(\mathrm{Hom}_{\mathcal{T}}(t^-, X/\underline{X}), \mathrm{Ext}^1_{\mathcal{T}}(t^-, \underline{X})) \tilde{\leftarrow} \mathrm{Ext}^1_{\mathcal{T}}(X/\underline{X}, \underline{X}),$$

where $t^- \in \mathrm{mod}\mathcal{T}$ is the simple module located at $t$. Finally, we obtain an epivalence

$$G : \mathrm{mod}\,\mathcal{T} \longrightarrow M_N^k, \quad X \longrightarrow (\mathrm{Hom}_{\mathcal{T}}(t^-, X/\underline{X}), \varepsilon_X, \underline{X}),$$

where $M$ and $N$ are the left modules over $\mathcal{A} = \mathrm{mod}\overline{\mathcal{T}}$ such that $N(Z) = \mathrm{Ext}^1_{\overline{\mathcal{T}}}(t^-, Z) \subset M(Z) = \mathrm{Ext}^1_{\mathcal{T}}(t^-, Z)$ ([9], 4.2).

Our proof of the third main theorem uses the epivalence $\mathrm{mod}\,\mathcal{T} \to M_N^k$, the second main theorem and the following statement. There, $\mathrm{ind}\,\overline{\mathcal{T}}$ denotes the chosen spectroid $\mathcal{S}$ of $\mathcal{A} = \mathrm{mod}\,\overline{\mathcal{T}}$.

**Proposition.** *With the notations above, suppose that $M$ is not $N$-wild. Then, for each $d \in \mathbb{N}$, $\mathrm{ind}\,\overline{\mathcal{T}}$ contains only finitely many $(M, N)$-relevant modules of length $d$ (6.6).*

The proposition will be proved in 9.6.

**9.2. Proposition.** *$\mathcal{T}$ is wild if $M$ is $N$-wild.*

***Proof.*** Let $e = (e_0, e_1, e_2)$ be a coordinate system of an $N$-reliable plane in some $\mathrm{Hom}_k(V, M(X)) \tilde{\leftarrow} \mathrm{Ext}^1_{\mathcal{T}}(V \otimes_k t^-, X)$ $(V \in \mathrm{mod}k, X \in \mathcal{A})$. To produce a $\mathcal{T}$-reliable plane, we start from the tensor product

$$0 \longrightarrow V \otimes_k \underline{p} \longrightarrow V \otimes_k p \longrightarrow V \otimes_k t^- \longrightarrow 0 \qquad (*)$$

of $V$ with the obvious sequence (9.1) associated with $p = \mathcal{T}(?, t)$.

The induced connecting homomorphism $\mathrm{Hom}_{\mathcal{T}}(V \otimes_k \underline{p}, X) \to \mathrm{Ext}^1_{\mathcal{T}}(V \otimes_k t^-, X)$ is subjective and maps $f : V \otimes_k \underline{p} \to X$ onto the class of the pushout of (*) along $f$. Choosing the preimages $h_i$ of the given $e_i$, we construct the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & W\underset{k}{\otimes}V\underset{k}{\otimes}\underline{p} & \to & W\underset{k}{\otimes}V\underset{k}{\otimes}p & \to & W\underset{k}{\otimes}V\underset{k}{\otimes}t^- & \to & 0 \\ \mathbb{1}_W\otimes h_0+a_W\otimes h_1+b_W\otimes h_2 \downarrow & & & & c\downarrow & & \| & & \\ 0 & \to & W\otimes_k X & \xrightarrow{d} & Y_{a,b} & \to & W\underset{k}{\otimes}V\underset{k}{\otimes}t^- & \to & 0 \end{array}, (**)$$

where $a_W$ and $b_W$ map $w \in W$ onto $wa$ and $wb$.

For $Y_{a,b}$, we choose the following concrete construction. Let $Y = (Y(q))$ be a family of stalks such that $Y(t) = X(t) \oplus V$ and $Y(r) = X(r)$ if $r \neq t$. We set $Y_{a,b}(q) = W \otimes_k Y(q)$ for all $q \in \mathcal{T}$. Thus, the stalks of $W \otimes X$ are subspaces of the stalks $Y_{a,b}(q)$; on these subspaces, the structure maps

$$f_{a,b}(r, q) : Y_{a,b}(q) \otimes \mathcal{T}(r, q) \longrightarrow Y_{a,b}(r)$$

coincide with those of $W \otimes X$. Accordingly, $d$ is an inclusion, and it remains for us to describe $c$ and the restriction

$$Y_{a,b}(t) \otimes \mathcal{R}_{\mathcal{T}}(r, t) \longrightarrow Y_{a,b}(r)$$

of $f_{a,b}(r, t)$. The morphism $c$ is determined by the commutativity of the left square of (**) and by the equations $c(w \otimes v \otimes \mathbb{1}_p) = w \otimes v$. These equations imply

$$f_{a,b}(r, t)(w \otimes v) = w \otimes h_0(v \otimes \mu) + wa \otimes h_1(v \otimes \mu) + wb \otimes h_2(v \otimes \mu)$$

for all $\mu \in \mathcal{R}_{\mathcal{T}}(n, t)$. Thus, we have

$$f_{a,b}(r, q) = \mathbb{1}_W \otimes f_0(r, q) + a \otimes f_1(r, q) + b \otimes f_2(r, q),$$

where $f_1(r, q)$, $f_2(r, q)$ vanish on $X(q) \otimes \mathcal{T}(r, q)$, whereas $f_0(r, q)$ coincides there with the structure map of $X$. In other words, we have $Y_{a,b} = F_f(W, a, b)$ where $f = (f_0, f_1, f_2) \in H_Y^3$ (9.0).

Furthermore, the construction of $Y_{a,b}$ as a push-out shows that the composition

$$\operatorname{rep} Q^2 \xrightarrow[F_f]{} \operatorname{mod} \mathcal{T} \xrightarrow[G]{} M_N^k$$

of $F_f$ with the epivalence $G$ of 9.1 coincides with $F_e$. Since $F_e$ preserves indecomposability and heteromorphism, so does $F_f$.

**9.3. *Proof the third main theorem.*** Supposing that $\mathcal{T}$ is not wild, we shall construct a family of $\mathcal{T}$-reliable punched lines which (mutatis mutandis) satisfy statement b) of 2.6 (see 9.0 above).

Using induction on the dimension $\sum_{a, b \in \mathcal{T}} \dim \mathcal{T}(a, b)$ of $\mathcal{T}$, we may suppose that such a family is already available for $\overline{\mathcal{T}} = \mathcal{T}/\sigma$. Hence we restrict our attention to the "new" indecomposables which are not annihilated by $\sigma$, i. e. are transformed by $\operatorname{mod}\mathcal{T} \to M_N^k$ into $M$-spaces with nonzero first components. By 9.2, $M$ is not $N$-wild. By 9.1, the full subaggregate $\mathcal{A}_d$ of $\mathcal{A}$ "generated" by the indecomposables $X$ of dimension $\leq d$ which are $(M, N)$-relevant, has a finite spectroid for each $d \geq 1$. Denoting by $M_d$ and $N_d$ the restrictions of $M$ and $N$ to $\mathcal{A}_d$, there exists a locally finite set $\mathcal{D}^d$ of $N_d$-reliable punched lines which, for each $X \in \mathcal{A}_d$, produce almost all indecomposables of $(M_d)^k_{N_d}$ of the form $(V, f, X)$ up to isomorphism. Of course, we may and shall assume that $\mathcal{D}^1 \subset \mathcal{D}^2 \subset \dots$.

Now let $S \setminus E$ be an element of $\mathcal{D} = \bigcup_{d\geq 1}\mathcal{D}^d$, $e = (e_0, e_1)$ a coordinate system of $S$ and $C = \{\lambda \in k \mid e_0 + \lambda e_1 \in S \setminus E\}$. As in the proof of 9.2, we can construct a $\mathcal{T}$-reliable punched line with coordinate system $f = (f_0, f_1) \in H^2_Y$ such that the composition $\operatorname{rep} Q^1 \xrightarrow{F_f} \operatorname{mod} \mathcal{T} \xrightarrow{G} M^k_N$ is isomorphic to $\operatorname{rep}_C Q^1 \xrightarrow{F_e} M^k_N$. It is easy to check that the punched lines arising in this way from $\mathcal{D}$ "parametrize" the new indecomposables over $\mathcal{T}$ as wanted.

**9.4.** We now turn to the proof of Proposition 9.1. Our first objective is to shake off the bond $N = \operatorname{Ext}^1_{\overline{\mathcal{T}}}(t^-, ?)$ on $M = \operatorname{Ext}^1_{\mathcal{T}}(t^-, ?)$. For this sake, we resort to the injective $\mathcal{T}$-module $i = \mathcal{T}(s, ?)^{\mathrm{T}}$. The largest submodule $\underline{i}$ of $i$ annihilated by $\sigma$ is identified with $\overline{\mathcal{T}}(s, ?)^{\mathrm{T}}$, and $i / \underline{i}$ can be identified with $t^-$ via

$$i(t) = \mathcal{T}(s, t)^{\mathrm{T}} \to k, \quad f \mapsto f(\sigma).$$

It easily follows that $0 = N(\underline{i}) \subset M(\underline{i}) = k\,\varepsilon_i$, where $\varepsilon_i$ denotes the extension associated with the exact sequence $0 \to \underline{i} \to i \to t^- \to 0$. As a consequence, the submodule of $M$ generated by $\varepsilon_i \in M(\underline{i})$ coincides with $\mathfrak{I}M$, where $\mathfrak{I}$ is the ideal of $\mathcal{A} = \operatorname{mod}\overline{\mathcal{T}}$ generated by $\mathbb{1}_{\underline{i}}$. In the following proposition, $\overline{M} := M / \mathfrak{I}M$ is considered as a module over the aggregate $\overline{\mathcal{A}} = \mathcal{A}/\mathfrak{I}$, whose spectroid $\overline{\mathfrak{L}}$ is obtained by deleting the point $\underline{i}$ from the quotient $\mathfrak{L}/\mathbb{1}_{\underline{i}}$ of the spectroid $\mathfrak{L} = \operatorname{ind}\overline{\mathcal{T}}$ of $\mathcal{A} = \operatorname{mod}\overline{\mathcal{T}}$.

**Proposition.** *The canonical functor* $M^k_N \to \overline{M}^k$ *is quasisurjective. Up to isomorphism, it annihilates just one indecomposable* $(0, 0, \underline{i}) \in M^k_N$.

We postpone the proof to 9.7.

**9.5. Proposition.** *With the notations of* 9.4, *suppose that* $\overline{M}$ *is not wild. Then, for each* $d \in \mathbb{N}$, $\overline{M}$ *vanishes on almost all modules in* $\overline{\mathfrak{L}}$ *of length* $d$.

It seems advisable here to recall that the points of $\overline{\mathfrak{L}}$ are *genuine* modules over $\overline{\mathcal{T}}$, even though the morphisms of $\overline{\mathfrak{L}}$ are classes of morphisms of $\operatorname{mod}\overline{\mathcal{T}}$.

***Proof.*** Let us denote by $\overline{\mathfrak{L}}_d$ the full subspectroid of $\overline{\mathfrak{L}}$ formed by the modules of dimension $d$, by $\overline{M}_d$ the restriction of $\overline{M}$ to $\overline{\mathfrak{L}}_d$. By the lemma of Harada and Sai ([9], 3.2 Example 2), the radical $\mathcal{R}_d$ of $\overline{\mathfrak{L}}_d$ is nilpotent. If $\overline{M}_d(x) \neq 0$ for infinitely many $x \in \overline{\mathfrak{L}}_d$, we infer that $(\mathcal{R}^n_d \overline{M}_d / \mathcal{R}^{n+1}_d \overline{M}_d)(x) \neq 0$ for some $n \in \mathbb{N}$ and (at least!) 5 points $x \in \overline{\mathfrak{L}}_d$. This means that $\overline{M}_d$ has a subquotient which is a sum of 5 non-isomorphic simple modules. Hence, the subquotient is wild, and so are $\overline{M}_d$ and $\overline{M}$.

**9.6. *Proof of proposition* 9.1.** a) We first that *$M$ is $N$-wild if $\overline{M}$ is wild.* Indeed, let ${}_{\mathcal{A}}\overline{M}$ denote the quotient $M / \mathfrak{I}M$ considered as a module over $\mathcal{A}$. If $\overline{M}$ is wild, it is clear that ${}_{\mathcal{A}}\overline{M}$ is wild. Since ${}_{\mathcal{A}}\overline{M}$ is a quotient of $M$ and $N$ does not contain $\mathfrak{I}M$, Proposition 3.7 implies that $M$ is $N$-wild.

b) Let us now suppose that $M$ is not $N$-wild. Then $\overline{M}$ is not wild. Hence, for each $d \in \mathbb{N}$, $\mathfrak{Q}$ has a finite number $n(d)$ of points $x$ of dimension $d$ such that $\overline{M}(x) \neq 0$. Of course, all these $x \in \mathfrak{Q} \setminus \underline{i}$ are $(M, N)$-relevant. On the other hand, if $y \in \mathfrak{Q} \setminus \underline{i}$ is $(M, N)$-relevant, $M_N^k$ admits an indecomposable $(V, f, y \otimes Y)$ such that $V \neq 0$. Since this triple is also indecomposable as an object of $\overline{M}^k$ (9.4), we have $\overline{M}(y) \neq 0$. We infer that, besides $\underline{i}$, $\mathfrak{Q}$ has $n(d)$ points of dimension $d$ which are $(M, N)$-relevant.

**9.7.** *It remains for us to prove Proposition* 9.4, which follows from 4.2 b), 4.1, and the following lemma.

**Lemma.** *The annihilator of* $\mathfrak{I}$ *in* $M = \mathrm{Ext}^1_{\mathcal{T}}(t^-, ?)$ *is* $N = \mathrm{Ext}^1_{\overline{\mathcal{T}}}(t^-, ?)$.

***Proof.*** For each $Z \in \mathcal{A}$, the annihilator of $\mathfrak{I}$ in $M(Z)$ consists of the classes of short exact sequences $0 \longrightarrow Z \xrightarrow[\iota]{} Y \xrightarrow[\pi]{} t^- \longrightarrow 0$ of $\mathrm{mod}\,\mathcal{T}$ whose push-out splits for each $\mu \in \mathrm{Hom}_{\overline{\mathcal{T}}}(Z, \underline{i})$. If the class belongs to $\mathrm{mod}\,\overline{\mathcal{T}}$, $Y$ is a $\overline{\mathcal{T}}$-module and the push-out splits because $\underline{i}$ is injective in $\mathrm{mod}\,\overline{\mathcal{T}}$. Hence, $N$ is contained in the annihilator.

Conversely, suppose that the class of $(\iota, \pi)$ is annihilated by $\mathfrak{I}$. Since each $\mu \in \mathrm{Hom}_{\overline{\mathcal{T}}}(Z, \underline{i})$ factors through $Y$, the first row of

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_{\mathcal{T}}(t^-, \underline{i}) & \to & \mathrm{Hom}_{\mathcal{T}}(Y, \underline{i}) & \to & \mathrm{Hom}_{\mathcal{T}}(Z, \underline{i}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}_{\mathcal{T}}(t^-, i) & \to & \mathrm{Hom}_{\mathcal{T}}(Y, i) & \to & \mathrm{Hom}_{\mathcal{T}}(Z, i) & \to & 0 \end{array}$$

is exact. Since the first and the second vertical arrows are invertible, so is the second. Since $i$ is, up to isomorphism, the only indecomposable injective $\mathcal{T}$-module outside $\mathrm{mod}\,\overline{\mathcal{T}}$, we infer that $Y \in \mathrm{mod}\,\overline{\mathcal{T}}$.

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