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Z. Önder, İ. Çanak (Ege Univ., Izmir, Turkey)

TAUBERIAN CONDITIONS UNDER WHICH CONVERGENCE FOLLOWS FROM THE WEIGHTED MEAN SUMMABILITY AND ITS STATISTICAL EXTENSION FOR SEQUENCES OF FUZZY NUMBER

ТАУБЕРОВІ УМОВИ, ЗА ЯКИХ ЗБІЖНІСТЬ ВИПЛИВАЄ З СЕРЕДНЬОВАГОВОЇ СУМОВНОСТІ, ТА ЇХ СТАТИСТИЧНЕ РОЗШИРЕННЯ НА ПОСЛІДОВНОСТІ НЕЧІТКИХ ЧИСЕЛ

Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \qquad \text{as} \qquad n \to \infty.$$

Let (u_n) be a sequence of fuzzy numbers. The weighted mean of (u_n) is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \qquad \text{for} \qquad n = 0, 1, 2, \ldots .$$

It is known that the existence of the limit $\lim u_n = \mu_0$ implies that of $\lim t_n = \mu_0$. For the the existence of the limit $st\text{-}\lim t_n = \mu_0$, we require the boundedness of (u_n) in addition to the existence of the limit $\lim u_n = \mu_0$. But, in general, the converse of this implication is not true. In this paper, we obtain Tauberian conditions, under which the existence of the limit $\lim u_n = \mu_0$ follows from that of $\lim t_n = \mu_0$ or $st\text{-}\lim t_n = \mu_0$. These Tauberian conditions are satisfied if (u_n) satisfies the two-sided condition of Hardy type relative to (P_n).

Нехай (p_n) — послідовність невід'ємних чисел таких, що $p_0 > 0$ і

$$P_n := \sum_{k=0}^{n} p_k \to \infty \qquad \text{при} \qquad n \to \infty.$$

Нехай (u_n) — послідовність нечітких чисел. Вагове середнє для (u_n) визначається як

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \qquad \text{для} \qquad n = 0, 1, 2, \ldots .$$

Відомо, що з існування границі $\lim u_n = \mu_0$ випливає $\lim t_n = \mu_0$. Для існування границі $st\text{-}\lim t_n = \mu_0$ вимагається обмеженість (u_n) як додаткова умова до існування границі $\lim u_n = \mu_0$. Але обернена імплікація взагалі не є правильною. У цій роботі запропоновано тауберові умови, за яких існування границі $\lim u_n = \mu_0$ випливає з того, що $\lim t_n = \mu_0$ або $st\text{-}\lim t_n = \mu_0$. Ці тауберові умови виконуються, якщо (u_n) задовольняє двосторонні умови типу Гарді відносно (P_n).

1. Introduction. In this section, we begin with some remarks about history of fuzzy set theory and its applications to $(\overline{N}, p)$ summability method, that is about the history from almost fifty years ago until these days. We shortly mention emergence of the concept of statistical convergence for sequences of fuzzy numbers and advancement of that in Tauberian theory. In the sequel, bringing together the concepts of $(\overline{N}, p)$ summability and statistical convergence for sequences of fuzzy numbers under the same roof, we refer certain results obtained by several researchers concerning these concepts. After dwelling on studies that encourages us to do this research, we complete this section summarizing theorems and corollaries attained in this paper.

Improved based upon the fuzzy sets and fuzzy set operations which was introduced by Zadeh [1], fuzzy set theory has increasingly received attention from researchers in a diverse range of disciplines in the last few years. Aspiring to apply concept of fuzziness to individual works with a broad viewpoint from theoretical to practical in almost all sciences and technology, researchers have reached numerous and varied applications of its in fields such as statistics, nuclear science, biomedicine, agriculture, geography, weather prediction, finance and stock market, engineering, computer science, artificial intelligence, pattern recognition, handwriting analysis, decision theory, robotics etc. In addition to these, one of areas which the concept of fuzziness was carried out is also pure mathematics and there have been several authors discussing many important properties and applications of fuzzy sets. Dubois and Prade [2] introduced the fuzzy numbers and defined basic operations of addition, subtraction, multiplication and division. In [3], Goetschel and Voxman presented a less restrictive definition of fuzzy numbers. Matloka [4] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. Nanda [5] studied the spaces of bounded and convergent sequence of fuzzy numbers and proved that they are complete metric spaces.

In recent years, there has been an increasing interest on summability methods of sequences of fuzzy numbers. One of these summability methods which has attracted the attention of many researchers is $(\overline{N}, p)$ summability method. Tripathy and Baruah [6] introduced $(\overline{N}, p)$ method for sequences of fuzzy numbers and obtained fuzzy analogues of classical Tauberian theorems for this method. Çanak [7] investigated some conditions needed for the $(\overline{N}, p)$ summable sequences to be convergent. Later, Önder et al. [8] established the Tauberian condition controlling one-sided oscillatory behavior of a sequence of fuzzy numbers for the $(\overline{N}, p)$ summability method. Contrary to the common belief that the concept of statistical convergence, which is a natural generalization of that of ordinary convergence, was introduced by Fast [9] and Schoenberg [10], this concept was firstly came up with by Zygmund [11] who used the term *almost convergence* in place of statistical convergence and proved some theorems related to it. After the definition of statistical convergence was put into the final form by Fast [9] and Schoenberg [10], Nuray and Savaş [12] extended this concept to sequences of fuzzy numbers and discussed some properties related to its. Savaş [13] obtained some equivalent conditions for a sequence of fuzzy numbers to be statistically convergent and statistically Cauchy. Aytar and Pehlivan [14] indicated that statistical convergence of a sequence of fuzzy numbers is equivalent to uniform statistical convergence of the sequence of functions which are defined via the endpoints of a-cuts of same sequence. In [15], Başar presented some results on statistical convergence of sequences of fuzzy numbers. In the sequel, this concept was associated with Tauberian conditions given by several researchers from past to present. Kwon [16] established the Tauberian theorem and a decomposition theorem for statistical convergence of sequences of fuzzy numbers. Considering statistical convergence as a regular summability method, Talo and Başar [17] found out that necessary conditions for convergence of sequences of fuzzy numbers which are statistically convergent are slow decrease and one-sided condition $nu_n \succeq nu_{n-1} - \bar{H}$ for some $H > 0$. After the results obtained related to concept of statistical convergence were published, it was combined with $(\overline{N}, p)$ summability method. In relation to that, Talo and Bal [18] presented Tauberian conditions under which statistical convergence of sequence of fuzzy numbers follows from its statistically $(\overline{N}, p)$

summability. For some interesting results on statistical convergence of fuzzy numbers, we refer to [6, 19].

Besides the studies mentioned up to now, the studies that encourage us to do this research is in fact those including some results obtained by Móricz [20, 21] for Cesàro (or $(C, 1)$) and Harmonic (or $(H, 1)$) summability methods for sequence of real numbers. Móricz formulated these results as follows.

Theorem 1 [20]. *If the real (or complex) sequence* (s_n) *is statistically* $(C, 1)$ *summable to* μ *and slowly decreasing (or slowly oscillating), then* (s_n) *is convergent to* μ.

Theorem 2 [21]. *If the real (or complex) sequence* (s_n) *is statistically* $(H, 1)$ *summable to* μ *and slowly decreasing (or slowly oscillating) with respect to the* $(H, 1)$ *summability, then* (s_n) *is convergent to* μ.

In case that $p_n = 1$ and $p_n = \dfrac{1}{n}$ for all nonnegative integers $n,$ $(\overline{N}, p)$ summability method reduces to Cesàro and Harmonic summability methods, respectively. Here, our aim extend the theorems presented by Móricz for Cesàro and Harmonic summability methods for sequence of real (or complex) numbers to the $(\overline{N}, p)$ summability method for sequence of fuzzy numbers.

In this paper, we indicate that some conditions under which convergence follows from $(\overline{N}, p)$ summability and its statistical extension for sequences of fuzzy numbers. In Section 2, we recall some notations, basic definitions and theorems for fuzzy numbers. In Section 3, we present some lemmas which will be benefited in the proofs of main results for sequences of fuzzy numbers. In Section 4, we prove the Tauberian theorem for $(\overline{N}, p)$ summable sequences of fuzzy numbers. In the sequel, replacing $(\overline{N}, p)$ summable sequences of fuzzy numbers by statistically $(\overline{N}, p)$ summable sequences of fuzzy numbers, we establish the Tauberian theorem which convergence follows from statistically $(\overline{N}, p)$ summability under condition of slow oscillation relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. We end this section by giving an extension of the obtained theorem and corollary from statistically $(\overline{N}, p)$ summability method to statistically $(\overline{N}, p, \alpha)$ summability method.

2. Preliminaries. In this section, we present background needed to make easier readability of our study. For this, we begin with basic definitions and notations with respect to fuzzy numbers that will be used throughout this paper. In the sequel, we mention its linear structure, set operations on the space of fuzzy numbers and some algebraic properties related to its. We recall metric on the space of fuzzy numbers and exhibit a list of fundamental properties of its. We end this section by giving some definitions concerning the sequences of fuzzy numbers. For the sake of completeness of the paper, we give our study in Section 4.

In [3], Goetschel and Voxman introduced concept of fuzzy numbers as follows.

Definition 1. *Consider a fuzzy subset of real line* $u\colon \mathbb{R} \to [0, 1]$. *Then the mapping* u *is a fuzzy number if it satisfies following additional properties*:

(i) u *is normal*, *i.e.*, *there exists* $t_0 \in \mathbb{R}$ *such that* $u(t_0) = 1$;

(ii) u *is fuzzy convex*, *i.e.*, *for any* $t_0, t_1 \in \mathbb{R}$ *and for any* $\alpha \in [0, 1]$, $u(\alpha t_0 + (1 - \alpha)t_1) \geq$ $\geq \min\{u(t_0), u(t_1)\}$;

(iii) u *is upper semicontinuous on* $\mathbb{R}$;

(iv) *the support of* u, $[u]_0 := \overline{\{t \in \mathbb{R} \colon u(t) > 0\}}$ *is compact, where* $\overline{\{t \in \mathbb{R} \colon u(t) > 0\}}$ *denotes the closure of the set* $\{t \in \mathbb{R} \colon u(t) > 0\}$ *in usual topology of* $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and call it the space of fuzzy numbers.

We recall the linear structure of E^1 as follows. For $u \in E^1$, the λ-level set of u is defined by

$$[u]_\lambda := \begin{cases} \{x \in \mathbb{R} \colon u(x) \geq \lambda\}, & 0 < \lambda \leq 1, \\ \overline{\{x \in \mathbb{R} \colon u(x) > \lambda\}}, & \lambda = 0. \end{cases}$$

Then, it is easily established (see [22]) that u is a fuzzy number if and only if $[u]_\lambda$ is a closed, bounded and nonempty interval for each $\lambda \in [0,1]$ with $[u]_\beta \subseteq [u]_\lambda$ if $0 \leq \lambda \leq \beta \leq 1$. From this characterization of fuzzy numbers, it follows that a fuzzy number u is completely determined by the end points of the intervals $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ where $u^-(\lambda) \leq u^+(\lambda)$ and $u^-(\lambda), u^+(\lambda) \in \mathbb{R}$ for each $\lambda \in [0,1]$.

In the sequel, Goetschel and Voxman [3] presented another representation of a fuzzy number as a pair of functions that satisfy some properties.

Theorem 3 [3]. *Let $u \in E^1$ and $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. Then the functions u^-, $u^+ \colon [0,1] \to \to \mathbb{R}$, defining the endpoints of the λ-level sets, satisfy following conditions:*

(i) $u^-(\lambda) \in \mathbb{R}$ *is a bounded, non-decreasing and left continuous function on* $(0,1]$;

(ii) $u^+(\lambda) \in \mathbb{R}$ *is a bounded, non-increasing and left continuous function on* $(0,1]$;

(iii) *The functions* $u^-(\lambda)$ *and* $u^+(\lambda)$ *are right continuous at* $\lambda = 0$;

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions f and g satisfies the above conditions (i)–(iv), *then there exists a unique fuzzy number u such that $[u]_\lambda := [f(\lambda), g(\lambda)]$ for each $\lambda \in [0,1]$ and $u(x) := := \sup_{\lambda \in [0,1]} \{\lambda \colon f(\lambda) \leq x \leq g(\lambda)\}$.*

Suppose that $u, v \in E^1$ are represented by $[u^-(\lambda), u^+(\lambda)]$ and $[v^-(\lambda), v^+(\lambda)]$ for each $\lambda \in \in [0,1]$, respectively. Then the operations addition, subtraction and scalar multiplication on the set of fuzzy numbers are defined as follows:

$$[u+v]_\lambda := \left[u^-(\lambda) + v^-(\lambda), u^+(\lambda) + v^+(\lambda)\right],$$

$$[u-v]_\lambda := \left[u^-(\lambda) - v^+(\lambda), u^+(\lambda) - v^-(\lambda)\right],$$

$$[ku]_\lambda = k[u]_\lambda := \begin{cases} [ku^-(\lambda), ku^+(\lambda)], & k \geq 0, \\ [ku^+(\lambda), ku^-(\lambda)], & k < 0. \end{cases}$$

The set of all real numbers can be embedded in E^1. For $r \in \mathbb{R}$, $\bar{r} \in E^1$ is defined by

$$\bar{r}(x) := \begin{cases} 1, & x = r, \\ 0, & x \neq r. \end{cases}$$

The following lemma deals with the algebraic properties of fuzzy numbers.

Lemma 1 [23]. *On the set of fuzzy numbers there are two binary operations, denoted by $+, .$ and called addition, scalar multiplication, respectively. These operations satisfy following properties:*

(i) *the addition of fuzzy numbers is associative and commutative, i.e., $u + v = v + u$ and $u + (v + w) = (u + v) + w$ for any $u, v, w \in E^1$;*

(ii) $\bar{0} \in E^1$ *is neutral element with respect to $+$, i.e., $u + \bar{0} = \bar{0} + u = u$ for any $u \in E^1$;*

(iii) *with respect to $+$, none of $u \in E^1 \setminus \mathbb{R}$ has opposite in E^1;*

(iv) $\bar{1} \in E^1$ *is neutral element with respect to, i.e., $u\bar{1} = \bar{1}u = u$ for any $u \in E^1$;*

(v) *for any* $a, b \in \mathbb{R}$ *with* $ab \geq 0$ *and any* $u \in E^1$, *we have* $(a + b)u = au + bu$; *for general* $a, b \in \mathbb{R}$, *this property does not holds*;

(vi) *for any* $a \in \mathbb{R}$ *and* $u, v \in E^1$, *we have* $a(u + v) = au + av$;

(vii) *for any* $a, b \in \mathbb{R}$ *and* $u \in E^1$, *we have* $(ab)u = a(bu)$.

As a conclusion, we attain by Lemma 1 that the space of fuzzy numbers is not a linear space.

Concept of metric space may be defined as an arbitrary fuzzy set which a distance between all elements of the set are described. It is possible to define several different metrics on the space of fuzzy numbers; however, the most well-known and preferential metric among these metrics is the Hausdorff distance for fuzzy numbers based on the classical Hausdorff distance between compact convex subsets of $\mathbb{R}^n$. Let W denote the set of all closed and bounded intervals. For the case when $A = [a^-, a^+],\ B = [b^-, b^+]$ are the two intervals, the Hausdorff distance on W is defined by

$$d(A, B) := \max\left\{\left|a^- - b^-\right|, \left|a^+ - b^+\right|\right\}.$$

It can be noted that W is a complete separable metric space on the basis of the Hausdorff distance (cf. Nanda [5]). Now, we may define the metric D on the space of fuzzy numbers with the help of the Hausdorff metric d.

Definition 2 [23]. *Let* $D\colon E^1 \times E^1 \to \mathbb{R}_+$ *and let* $u, v \in E^1$ *represented*, *respectively*, *by* $[u^-(\lambda), u^+(\lambda)]$ *and* $[v^-(\lambda), v^+(\lambda)]$ *for each* $\lambda \in [0, 1]$

$$D(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda).$$

Then D *is called the Hausdorff distance between fuzzy numbers* u *and* v.

It is easy to see that

$$D(u, \bar{0}) = \sup_{\lambda \in [0,1]} \max\left\{\left|u^-(\lambda)\right|, \left|u^+(\lambda)\right|\right\} = \max\left\{\left|u^-(0)\right|, \left|u^+(0)\right|\right\}.$$

The following lemma presents some fundamental properties of the Hausdorff distance between fuzzy numbers.

Lemma 2 [23]. *Let* $u, v, w, z \in E^1$ *and* $k \in \mathbb{R}$. *Then following statements hold true*:

(i) (E^1, D) *is a complete metric space*;

(ii) $D(u + w, v + w) = D(u, v)$, *i.e.*, D *is translation invariant*;

(iii) $D(ku, kv) = |k|\, D(u, v)$;

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$;

(v) $|D(u, \bar{0}) - D(v, \bar{0})| \leq D(u, v) \leq D(u, \bar{0}) + D(v, \bar{0})$.

We now refer following definitions concerning sequences of fuzzy numbers which will be needed in the sequel.

Definition 3 [4]. *A sequence* $u = (u_n)$ *of fuzzy numbers is a function* u *from the set* $\mathbb{N}$ *of all positive integers into the set* E^1. *The fuzzy number* u_n *denotes the value of the function at a point* $n \in \mathbb{N}$ *and is called the* n*th term of the sequence.*

We denote the set of all sequences of fuzzy numbers by $\omega(F)$.

Definition 4 [4]. *A sequence* (u_n) *of fuzzy numbers is said to be convergent to the fuzzy number* μ_0, *written as* $\lim_{n\to\infty} u_n = \mu_0$, *if for every* $\epsilon > 0$ *there exists a positive integer* n_0 *such that*

$$D(u_n, \mu_0) < \epsilon \ \textit{ whenever } \ n \geq n_0. \tag{1}$$

The number μ_0 *is called the limit of* (u_n).

We denote the set of all convergent sequences of fuzzy numbers by $c(F)$.

Definition 5 [6]. *Let (u_n) be a sequence of fuzzy numbers and $p = (p_n)$ be a sequence of nonnegative numbers such that*

$$p_0 > 0 \quad \textit{and} \quad P_n := \sum_{k=0}^{n} p_k \to \infty \qquad \textit{as} \qquad n \to \infty. \tag{2}$$

Weighted means of (u_n) is defined by

$$t_n^{(1)} := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \qquad \textit{for} \qquad n \in \mathbb{N}.$$

A sequence (u_n) is said to be summable by weighted mean method determined by the sequence p to the fuzzy number μ_0 if for every $\epsilon > 0$ there exists a positive integer n_0 such that

$$D(t_n^{(1)}, \mu_0) < \epsilon \qquad \textit{for} \qquad n > n_0.$$

Weighted mean methods are also called Riesz methods or $(\overline{N}, p)$ methods in the literature. $(\overline{N}, p)$ summability method is regular if and only if condition (2) is satisfied. In other words, every convergent sequence of fuzzy numbers is also $(\overline{N}, p)$ summable to the same number under condition (2). However, converse of this statement is not true in general. Truth of that is possible under some suitable condition which is so-called the Tauberian condition on the sequence. Any theorem stating that convergence of a sequence follows from its $(\overline{N}, p)$ summability and some Tauberian condition is said to be the Tauberian theorem for $(\overline{N}, p)$ summability method.

If $p_n = 1$ for all $n \in \mathbb{N}$, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

In addition, weighted means of integer order $\alpha \geq 0$ of a sequence (u_n) of fuzzy numbers is defined by

$$t_n^{(\alpha)} := \begin{cases} \dfrac{1}{P_n} \displaystyle\sum_{k=0}^{n} p_k t_k^{(\alpha-1)}, & \text{if} \quad \alpha \geq 1, \\ u_n, & \text{if} \quad \alpha = 0. \end{cases}$$

As similar to first order, a sequence (u_n) is said to be summable by weighted mean method of integer order $\alpha \geq 0$ determined by the sequence p to the fuzzy number μ_0 if, for every $\epsilon > 0$, there exists a positive integer n_0 such that

$$D(t_n^{(\alpha)}, \mu_0) < \epsilon \quad \text{for} \quad n > n_0.$$

We present definition of natural density of $K \subset \mathbb{N}$ and generate statistically convergent sequences of fuzzy numbers by using this concept. Let $K \subset \mathbb{N}$ be a subset of positive integers and $K_n = \{k \in \in K : k \leq n\}$. Then the set K has a natural density if sequence $\left(\dfrac{|K_n|}{n+1}\right)$ has a limit. In this case, we write $\delta(K) = \lim_{n\to\infty} \dfrac{|K_n|}{n+1}$, where the vertical bar denotes the cardinality of the enclosed set.

In [12], Nuray and Savaş introduced concept of statistical convergence for sequences of fuzzy numbers as follows.

Definition 6 [12]. *A sequence (u_n) of fuzzy numbers is said to be statistically convergent to the fuzzy number μ_0 if for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} \colon D(u_k, \mu_0) \geq \epsilon,\ k \leq n\}$ has natural density zero, i.e., for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n+1}\Big|\{k \in \mathbb{N} \colon D(u_k, \mu_0) \geq \epsilon,\ k \leq n\}\Big| = 0. \tag{3}$$

We denote the set of all statistically convergent sequences of fuzzy numbers by $st(F)$. In this case, we write $st - \lim_{n\to\infty} u_n = \mu_0$ if the limit (3) exists.

We write down that every convergent sequence of fuzzy numbers is statistically convergent to same number since all finite subsets of the natural numbers have density zero. Accordingly, statistical convergence may be considered as a regular summability method. However, converse of that is not always true. For example, sequence (u_n) of fuzzy numbers defined by

$$u_n(t) = \begin{cases} \left.\begin{array}{ll} \dfrac{n+3}{8}t - \dfrac{3n+1}{8}, & \text{if } t \in \left[\dfrac{3n+1}{n+3}, 3\right] \\ 1, & \text{if } t \in [3,5] \\ \dfrac{11n+17}{6n+2} - \dfrac{n+3}{6n+2}t, & \text{if } t \in \left[5, \dfrac{11n+17}{n+3}\right] \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } n = k^3 \text{ and } k \in \mathbb{N} \\ \overline{0}, & \text{otherwise} \end{cases}$$

is statistically convergent to $\bar{0}$, since

$$\lim_{n\to\infty} \frac{1}{n+1}\Big|\{k \in \mathbb{N} \colon D(u_k, \bar{0}) \geq \epsilon,\ k \leq n\}\Big| \leq \lim_{n\to\infty} \frac{\sqrt[3]{n}+1}{n+1} = 0$$

for every $\epsilon > 0$, but not convergent in the ordinary sense.

Recall that a sequence (u_n) of fuzzy numbers is called statistically $(\overline{N}, p, \alpha)$ summable to the fuzzy number μ_0 for each nonnegative integer α if $st - \lim_{n\to\infty} t_n^{(\alpha)} = \mu_0$.

If $\alpha = 1$, then statistically $(\overline{N}, p, \alpha)$ summability method reduces to statistically $(\overline{N}, p)$ summability method.

At present, we define concepts of slow oscillation relative to (P_n) and two-sided condition of Hardy type relative to (P_n), respectively. In pursuit of defining of these concepts, we mention about how a transition exists between them.

Definition 7. *A sequence (u_n) of fuzzy numbers is said to be slowly oscillating relative to (P_n) if*

$$\limsup_{n\to\infty} D(u_m, u_n) = 0 \qquad \textit{as} \qquad m \geq n, \qquad \frac{P_m}{P_n} \to 1 \qquad (n \to \infty). \tag{4}$$

Using ϵ and δ, (4) is equivalent to following statement: for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that

$$D(u_m, u_n) \leq \epsilon \qquad \textit{whenever} \qquad m \geq n \geq n_0 \qquad \textit{and} \qquad 1 \leq \frac{P_m}{P_n} \leq 1 + \delta.$$

We emphasize that if $p_n = 1$ for all $n \in \mathbb{N}$ in (4), then concept of slow oscillation relative to (P_n) correspond to concept of slow oscillation (cf. [24]).

Definition 8. *A sequence* (u_n) *of fuzzy numbers satisfies two-sided condition of Hardy type relative to* (P_n) *if there exist positive constants* n_0 *and* C *such that*

$$D(u_n, u_{n-1}) \leq C\frac{p_n}{P_n} \qquad \textit{for} \qquad n > n_0. \tag{5}$$

We emphasize that if $p_n = 1$ for all $n \in \mathbb{N}$ in (5), then two-sided condition of Hardy type relative to (P_n) correspond to two-sided condition of Hardy type.

Additionally, it is easy to see that if the sequence (p_n) satisfies condition (2), then two-sided condition of Hardy type relative to (P_n) implies condition of slow oscillation relative to (P_n).

As a matter of fact, we assume that (p_n) satisfies condition (2). Since two-sided condition of Hardy type relative to (P_n) is satisfied, there exist positive constants n_0 and C such that $D(u_n, u_{n-1}) \leq C\dfrac{p_n}{P_n}$ for $n > n_0$.

Let $m \geq n \geq n_0$ and $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta$. Then, for a given $\epsilon > 0$, we have

$$D(u_m, u_n) = D(u_m, u_{m-1}) + D(u_{m-1}, u_{m-2}) + \ldots + D(u_{n+1}, u_n) =$$

$$= \sum_{k=n+1}^{m} D(u_k, u_{k-1}) \leq C \sum_{k=n+1}^{m} \frac{p_k}{P_k} \leq C\left(\frac{P_m}{P_n} - 1\right) < C\delta < \epsilon$$

in case we choose $0 < \delta < \dfrac{\epsilon}{C}$. Therefore, we obtain that $(u_n) \in \omega(F)$ is slowly oscillating relative to (P_n).

3. Lemmas. In this section, we express and prove following assertions which will be benefited in proofs of our main results for sequences of fuzzy numbers. The following lemma which we prove following a procedure resembling proof done for sequences of real numbers by Mikhalin [25] plays a crucial role in proofs of subsequent two lemmas which are necessary to achieve our main results for sequences of fuzzy numbers.

Lemma 3. *Let* (p_n) *satisfy conditions* (2) *and* $\dfrac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* $(u_n) \in \omega(F)$ *satisfies condition*

$$\limsup_{n\to\infty} D(u_m, u_n) \leq r \ (0 \leq r < \infty) \qquad \textit{as} \qquad m \geq n, \qquad \frac{P_m}{P_n} \to 1 \qquad (n \to \infty), \tag{6}$$

then there exist positive numbers a *and* b *such that* $D(u_m, u_n) \leq a \log \dfrac{P_m}{P_n} + b$ *for all* $m \geq n \geq 0$.

Proof. Assume that (p_n) satisfies conditions (2), $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and (u_n) satisfies condition (6). We can say from condition (6) that for every $r + 1 > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $D(u_m, u_n) < r + 1$ whenever $n > n_0$ and $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta$.

Let $n \leq n_0$ and $P_m \leq P_{n_0}(1 + \delta)$. In this case, we find that $D(u_m, u_n)$ has a maximum depending only on n_0, and so there exist $\delta > 0$ and $\ell \geq r + 1$ such that $D(u_m, u_n) < \ell$ for all $m, n \in \mathbb{N}$ related by $0 \leq P_m - P_n \leq \delta P_n$. Choose n_1 such that $1 \leq \dfrac{P_{n+1}}{P_n} \leq (1 + \delta)$ for all $n \geq n_1$. We investigate chosen n_1 in three cases such that $q \geq w \geq n_1$, $0 \leq w < n_1 \leq q$ and $0 \leq w \leq q < n_1$ for arbitrary fixed $q, w \in \mathbb{N}$.

We firstly take the case $q \geq w \geq n_1$ into consideration. For this, we define subsequence (w_{i+1})

where $w_0 = w$ and w_{i+1} is the largest natural number for which inequality $P_n \leq P_{w_i}(1+\delta)$ holds for all $i \in \mathbb{N}$. Therefore, we attain from this defining that inequalities $P_{w_{i+1}} \leq P_{w_i}(1+\delta)$ and $P_{w_{i+1}+1} > P_{w_i}(1+\delta)$ is valid. Now, let $w_k \leq q-1 < w_{k+1}$. Then we obtain that inequalities $P_q \leq P_{w_{k+1}}$ and $0 \leq P_{w_{i+1}} - P_{w_i} \leq \delta P_{w_i}$ hold for all $k \in \mathbb{N}$. So, we get

$$D(u_q, u_w) \leq D(u_q, u_{w_k}) + D(u_{w_k}, u_{w_0}) \leq$$

$$\leq D(u_q, u_{w_k}) + D\left(u_{w_k}, u_{w_{k-1}}\right) + D\left(u_{w_{k-1}}, u_{w_{k-2}}\right) + \ldots + D(u_{w_1}, u_{w_0}) =$$

$$= \sum_{j=0}^{k-1} D\left(u_{w_{j+1}}, u_{w_j}\right) + D(u_q, u_{w_k}) \leq \sum_{j=0}^{k-1} \ell + D(u_q, u_{w_k}) \leq (k+1)\ell.$$

Due to the fact that we have also inequalities

$$P_q \geq P_{w_k+1} > P_{w_{k-1}}(1+\delta) \geq P_{w_{k-2}+1}(1+\delta) >$$

$$> P_{w_{k-3}}(1+\delta)^2 \geq \ldots \geq P_{w_0}(1+\delta)^{\left[\frac{k}{2}\right]} > P_w(1+\delta)^{\frac{k}{2}-1},$$

we reach inequality $\log P_q \geq \log P_w + \dfrac{k-2}{2}\log(1+\delta)$. This implies that inequality $(1+k) \leq$ $\leq \dfrac{\log P_q - \log P_w}{\log(1+\delta)^{1/2}} + 3$ holds and, hence, we get inequality

$$D(u_q, u_w) \leq (k+1)\ell < \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell \tag{7}$$

for any $q \geq w \geq n_1$.

On the other hand, if we take the case $0 \leq w < n_1 \leq q$ into consideration, then we obtain

$$D(u_q, u_w) \leq D(u_q, u_{n_1}) + \max_{0 \leq w < n_1} D(u_{n_1}, u_w) \leq$$

$$\leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_{n_1}) + 3\ell + \max_{0 \leq w < n_1} D(u_{n_1}, u_w) \leq$$

$$\leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell + \max_{0 \leq w < n_1} D(u_{n_1}, u_w). \tag{8}$$

Finally, if we consider the case $0 \leq w \leq q < n_1$, then we have

$$D(u_q, u_w) \leq \max_{0 \leq w \leq q \leq n_1 - 1} D(u_q, u_w) \leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell + \max_{0 \leq w \leq q \leq n_1} D(u_q, u_w). \tag{9}$$

If we define positive numbers a, b as $a = \dfrac{\ell}{\log(1+\delta)^{1/2}}$ and $b = \max\{3\ell, 3\ell + \max\limits_{0 \leq w \leq q \leq n_1} D(u_q, u_w)\}$, then we conclude by (7)–(9) that

$$D(u_q, u_w) \leq a \log \frac{P_q}{P_w} + b$$

for all $q \geq w \geq 0$.

Lemma 3 is proved.

Due to the fact that condition (6) corresponds to condition of slow oscillation relative to (P_n) in case of $r = 0$ in Lemma 3, we prove in following lemma that the below-mentioned sequence is bounded under condition of slow oscillation relative to (P_n) which is restrictive in comparison with condition (6) and some additional condition on (p_n) with the help of Lemma 3.

Lemma 4. *Let (p_n) satisfies conditions* (2) *and* $\dfrac{p_n}{P_n} \to 0$ *as $n \to \infty$. If $(u_n) \in \omega(F)$ is slowly oscillating relative to (P_n), then*

$$\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n D(u_m, u_n)\right) \tag{10}$$

is bounded.

Proof. Assume that (p_n) satisfies conditions (2), $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and (u_n) is slowly oscillating relative to (P_n). Then, by taking these hypotheses into consideration, we conclude by the help of Lemma 3 that there exist positive numbers a and b such that $D(u_m, u_n) \leq a \log \dfrac{P_m}{P_n} + b$ for all $m \geq n \geq 0$. In addition to this, since (p_n) satisfies condition $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$, we have

$$\frac{P_n}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \qquad \text{as} \qquad n \to \infty. \tag{11}$$

By the fact that $t_n \to \ell$ implies $\dfrac{1}{t_n} \to \dfrac{1}{\ell}$ whenever $\ell \neq 0$ as $n \to \infty$, we find by (11) that

$$\frac{P_{n+1}}{P_n} \to 1 \qquad \text{as} \qquad n \to \infty$$

and, so,

$$1 \leq \frac{P_m}{P_n} = \frac{P_m}{P_{m-1}} \frac{P_{m-1}}{P_{m-2}} \cdots \frac{P_{n+1}}{P_n} \to 1 \qquad \text{as} \qquad m \geq n \to \infty. \tag{12}$$

This means that for every $\delta > 0$ there exists $n_0 \in \mathbb{N}^0$ such that $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta$ whenever $m \geq n \geq n_0$. Therefore, from condition of slow oscillation relative to (P_n) we declare that for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $D(u_m, u_n) \leq \epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta$. With reference to above inequalities, we obtain that for all $m \geq 0$ and given any $\epsilon > 0$

$$\frac{1}{P_m}\sum_{n=0}^{m} p_n D(u_m, u_n) = \frac{1}{P_m}\sum_{n=0}^{n_0} p_n D(u_m, u_n) + \frac{1}{P_m}\sum_{n=n_0+1}^{m} p_n D(u_m, u_n) \leq$$

$$\leq \frac{1}{P_m}\sum_{n=0}^{n_0} p_n \left(a \log \frac{P_m}{P_n} + b\right) + \frac{1}{P_m}\sum_{n=n_0+1}^{m} p_n \epsilon \leq$$

$$\leq \frac{1}{P_m}\sum_{n=0}^{n_0} p_n \left(a \log \frac{P_m}{P_0} + b\right) + \frac{1}{P_m}\sum_{n=n_0+1}^{m} p_n \epsilon =$$

$$= \frac{P_{n_0} - P_0}{P_m} \left(a \log \frac{P_m}{P_0} + b \right) + \frac{P_m - P_{n_0}}{P_m} \epsilon =$$

$$= (P_{n_0} - P_0) \left(\frac{a}{P_m} \log \frac{P_m}{P_0} \right) + \frac{P_{n_0} - P_0}{P_m} b + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon \leq$$

$$\leq (P_{n_0} - P_0) \left(\frac{a}{P_0} \right) + \frac{P_{n_0} - P_0}{P_0} b + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon =$$

$$= \left(\frac{P_{n_0}}{P_0} - 1 \right) (a + b) + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon.$$

In conjunction with the information obtained up to now if we consider that $\left(\frac{P_{n_0}}{P_m} \right)$ is convergent to 0 by condition (2) and every convergent sequence is also bounded, then there exists a positive constant H such that

$$\frac{1}{P_m} \sum_{n=0}^{m} p_n D(u_m, u_n) \leq \left(\frac{P_{n_0}}{P_0} - 1 \right) (a + b) + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon \leq \left(\frac{P_{n_0}}{P_0} - 1 \right) (a + b) + H := M$$

for all $m \geq 0$ and some constant $M > 0$. In conclusion, we reach that the sequence in (10) is bounded.

Lemma 4 is proved.

Lemma 5. *Let* (p_n) *satisfies conditions* (2) *and* $\frac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* $(u_n) \in \omega(F)$ *is slowly oscillating relative to* (P_n), *then* $(t_n^{(1)}) \in \omega(F)$ *is also slowly oscillating relative to* (P_n).

Proof. Assume that (p_n) satisfies conditions (2), $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (u_n) is slowly oscillating relative to (P_n). Given $\epsilon > 0$. By the definition of slow oscillation relative to (P_n), this means that there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $D(u_m, u_n) \leq \epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$.

Let $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta'$. By the definition of the weighted means of first order of (u_n) and Lemma 4, we obtain

$$D\left(t_m^{(1)}, t_n^{(1)}\right) = D\left(\frac{1}{P_m} \left\{ \sum_{k=0}^{n} + \sum_{k=n+1}^{m} \right\} p_k u_k, \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \right) =$$

$$= D\left(\frac{1}{P_m} \sum_{k=0}^{n} p_k u_k + \frac{1}{P_m} \sum_{k=n+1}^{m} p_k u_k + \frac{P_m - P_n}{P_m P_n} \sum_{k=0}^{n} p_k u_n, \frac{P_m - P_n}{P_m P_n} \sum_{k=0}^{n} p_k u_n + \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \right) =$$

$$= D\left(\frac{1}{P_m} \sum_{k=0}^{n} p_k u_k + \frac{1}{P_m} \sum_{k=n+1}^{m} p_k u_k + \frac{P_m - P_n}{P_m P_n} \sum_{k=0}^{n} p_k u_n, \frac{P_m - P_n}{P_m P_n} \sum_{k=0}^{n} p_k u_n + \right.$$

$$\left. + \frac{P_m - P_n}{P_m P_n} \sum_{k=0}^{n} p_k u_k + \frac{1}{P_m} \sum_{k=0}^{n} p_k u_k \right) =$$

$$= D\left(\frac{1}{P_m}\sum_{k=n+1}^{m} p_k u_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k u_n, \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k u_n + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k u_k\right) \leq$$

$$\leq D\left(\frac{1}{P_m}\sum_{k=n+1}^{m} p_k u_k, \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k u_n\right) + D\left(\frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k u_n, \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k u_k\right) \leq$$

$$\leq \frac{1}{P_m}\sum_{k=n+1}^{m} p_k D(u_k, u_n) + \frac{P_m - P_n}{P_m}\frac{1}{P_n}\sum_{k=0}^{n} p_k D(u_n, u_k) \leq$$

$$\leq \frac{1}{P_m}\sum_{k=n+1}^{m} p_k \epsilon + \left(1 - \frac{P_n}{P_m}\right) M =$$

$$= \left(1 - \frac{P_n}{P_m}\right)(M + \epsilon)$$

whenever $m \geq k > n \geq n_0$, $1 < \dfrac{P_k}{P_n} \leq \dfrac{P_m}{P_n} \leq 1 + \delta'$ and for some constant $M > 0$. Since we have that for $m \geq n \geq n_0$ and $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta'$

$$0 \leq 1 - \frac{P_n}{P_m} \leq \frac{\delta'}{1 + \delta'},$$

if we choose $0 < \delta' \leq \dfrac{\epsilon}{M}$, then we arrive

$$D\left(t_m^{(1)}, t_n^{(1)}\right) \leq \left(1 - \frac{P_n}{P_m}\right)(M + \epsilon) \leq \frac{\delta'}{1 + \delta'}(M + \epsilon) \leq \epsilon.$$

Therefore, we obtain that $(t_n^{(1)})$ is slowly oscillating relative to (P_n), as well.

Lemma 5 is proved.

The following lemma can be given as a corollary of theorem proved by Talo and Başar [17].

Lemma 6 [17]. *If $(u_n) \in \omega(F)$ is statistically convergent to $\mu_0 \in E^1$ and slowly oscillating, then (u_n) is convergent to μ_0.*

4. Main results. In this section, we prove the Tauberian theorem for $(\overline{N}, p)$ summable sequences of fuzzy numbers following a procedure resembling proof done for sequences of real numbers by Boos [26] at first. In the sequel, replacing $(\overline{N}, p)$ summable sequences of fuzzy numbers by statistically $(\overline{N}, p)$ summable sequences of fuzzy numbers, we establish some Tauberian theorems which convergence follows from statistically $(\overline{N}, p)$ summability under some Tauberian conditions and additional conditions on (p_n) and we present some corollaries related to these theorems. We end this section by giving an extension of obtained theorems from statistically $(\overline{N}, p)$ summability method to statistically $(\overline{N}, p, s)$ summability method.

Theorem 4. *Let (p_n) satisfies conditions* (2) *and*

$$\frac{P_n}{P_{n+1}} \to 1 \qquad as \qquad n \to \infty. \tag{13}$$

If $(u_n) \in \omega(F)$ is $(\overline{N}, p)$ summable to $\mu_0 \in E^1$ and slowly oscillating relative to (P_n), then (u_n) is convergent to μ_0.

Proof. Assume that (p_n) satisfies conditions (2), (13) and that (u_n) being $(\overline{N},p)$ summable to μ_0 is slowly oscillating relative to (P_n). We may suppose that $\mu_0=\bar{0}$ without loss of generality and that (u_n) does not converge. In this case, because (u_n) cannot converge to $\bar{0}$ as well, we may consider $0<\alpha:=\limsup D(u_n,\bar{0})\le\infty$. Thus, there exists an index sequence (n_i) such that $u_{n_i}\to\alpha$ as $i\to\infty$. We examine defined α in two cases such that $0<\alpha<\infty$ and $\alpha=\infty$. If we firstly take the case $0<\alpha<\infty$ into account, then for every $\epsilon_1:=\alpha/2>0$ and $i\in\mathbb{N}$ there exists $N_1\in\mathbb{N}$ such that $\alpha/2\le D(u_{n_i},\bar{0})$ whenever $n_i\ge i\ge N_1$. On the other hand, if we consider the case $\alpha=\infty$, then for every $\epsilon_2:=1>0$ and $i\in\mathbb{N}$ there exists $N_2\in\mathbb{N}$ such that $1\le D(u_{n_i},\bar{0})$ whenever $n_i\ge i\ge N_2$. Taking account of both cases, for every $i\in\mathbb{N}$ we may choose $\vartheta:=\min\{1,\alpha/2\}$ and $N\in\mathbb{N}$ such that $\vartheta\le D(u_{n_i},\bar{0})$ whenever $n_i\ge i\ge N=\max\{N_1,N_2\}$.

In addition to these, because (u_n) is slowly oscillating relative to (P_n), the subsequence $(u_{n_i})\subset$ $\subset(u_n)$ is also slowly oscillating relative to (P_n). This means that for every $\epsilon:=\vartheta/2>0$ there exist $\delta>0$ and $N\in\mathbb{N}$ such that $D(u_m,u_{n_i})\le\vartheta/2$ whenever $m\ge n_i\ge N$ and $P_{n_i}\le P_m\le(1+\delta)P_{n_i}$. At the same time, inequality

$$D(\bar{0},u_{n_i})\le D(\bar{0},u_m)+D(u_m,u_{n_i})$$

is verified, we obtain by taking slow oscillation relative to (P_n) of (u_{n_i}) and assumption into account inequality

$$D(\bar{0},u_m)\ge D(\bar{0},u_{n_i})-D(u_m,u_{n_i})\ge\vartheta-\frac{\vartheta}{2}=\frac{\vartheta}{2}$$

for all such m. We now define index sequence

$$m_i:=\min\left\{n\in\mathbb{N}\colon P_n\ge P_{n_i}\left(1+\frac{\delta}{2}\right)\right\}\qquad\text{for all}\quad i\in\mathbb{N}.$$

It follows from the definition of the index sequence (m_i) that $m_i\ge n_i\ge N$ and $P_{m_i-1}<$ $<P_{n_i}\left(1+\dfrac{\delta}{2}\right)$. On the other hand, by the fact that $t_n\to\ell$ implies $\dfrac{1}{t_n}\to\dfrac{1}{\ell}$ whenever $\ell\ne0$ as $n\to\infty$, we find by (13) that

$$\frac{P_{n+1}}{P_n}\to1\qquad\text{as}\qquad n\to\infty$$

and, so, $\dfrac{P_n}{P_{n-1}}\le1+\dfrac{\delta}{4}$ for sufficiently large n. Thus, we get

$$P_{m_i}=\frac{P_{m_i}}{P_{m_i-1}}P_{m_i-1}\le\left(1+\frac{\delta}{4}\right)P_{n_i}\left(1+\frac{\delta}{2}\right)\le P_{n_i}(1+\delta)$$

for sufficiently large i. From this point of view, we can say that (m_i) satisfies condition $P_{n_i}\left(1+\dfrac{\delta}{2}\right)\le$ $\le P_{m_i}\le P_{n_i}(1+\delta)$ for sufficiently large i. By the definition of the weighted means of first order of (u_n), the assumption and Lemma 2, we obtain

$$D\left(t^{(1)}_{m_i},t^{(1)}_{n_i}\right)+D\left(t^{(1)}_{n_i},\frac{P_{n_i}}{P_{m_i}}t^{(1)}_{n_i}\right)\ge D\left(t^{(1)}_{m_i},\frac{P_{n_i}}{P_{m_i}}t^{(1)}_{n_i}\right)=$$

$$=D\left(\frac{1}{P_{m_i}}\left(\sum_{k=0}^{n_i}p_ku_k+\sum_{k=n_i+1}^{m_i}p_ku_k\right),\frac{1}{P_{m_i}}\sum_{k=0}^{n_i}p_ku_k\right)=$$

$$= D\left(\frac{1}{P_{m_i}}\sum_{k=n_i+1}^{m_i} p_k u_k, \bar{0}\right) \geq$$

$$\geq D\left(\frac{1}{P_{m_i}}\sum_{k=n_i+1}^{m_i} p_k u_{n_i}, \bar{0}\right) - D\left(\frac{1}{P_{m_i}}\sum_{k=n_i+1}^{m_i} p_k u_k, \frac{1}{P_{m_i}}\sum_{k=n_i+1}^{m_i} p_k u_{n_i}\right) \geq$$

$$\geq D\left(\frac{P_{m_i}-P_{n_i}}{P_{m_i}} u_{n_i}, \bar{0}\right) - \frac{1}{P_{m_i}}\sum_{k=n_i+1}^{m_i} p_k D(u_k, u_{n_i}) \geq$$

$$\geq \frac{P_{m_i}-P_{n_i}}{P_{m_i}} D(u_{n_i}, \bar{0}) - \frac{1}{P_{m_i}}\sum_{k=n_i+1}^{m_i} p_k \frac{\vartheta}{2} \geq$$

$$\geq \left(\frac{P_{m_i}-P_{n_i}}{P_{m_i}}\right)\vartheta - \left(\frac{P_{m_i}-P_{n_i}}{P_{m_i}}\right)\frac{\vartheta}{2} =$$

$$= \left(1 - \frac{P_{n_i}}{P_{m_i}}\right)\frac{\vartheta}{2} \geq \left(\frac{\delta}{2+\delta}\right)\frac{\vartheta}{2} > 0 \tag{14}$$

whenever $m_i \geq k \geq n_i \geq i \geq N$ and $P_{n_i}\left(1+\dfrac{\delta}{2}\right) \leq P_{m_i} \leq P_{n_i}(1+\delta)$. Additionally, we get inequality

$$D\left(t_{m_i}^{(1)}, \frac{P_{n_i}}{P_{m_i}} t_{n_i}^{(1)}\right) \leq D\left(t_{m_i}^{(1)}, t_{n_i}^{(1)}\right) + D\left(t_{n_i}^{(1)}, \frac{P_{n_i}}{P_{m_i}} t_{n_i}^{(1)}\right) =$$

$$= D\left(t_{m_i}^{(1)}, t_{n_i}^{(1)}\right) + D\left(\frac{P_{m_i}}{P_{n_i}P_{m_i}}\sum_{k=0}^{n_i} p_k u_k, \frac{P_{n_i}}{P_{n_i}P_{m_i}}\sum_{k=0}^{n_i} p_k u_k\right) =$$

$$= D\left(t_{m_i}^{(1)}, t_{n_i}^{(1)}\right) + \frac{1}{P_{n_i}P_{m_i}} D\left(P_{n_i} + (P_{m_i}-P_{n_i})\sum_{k=0}^{n_i} p_k u_k, P_{n_i}\sum_{k=0}^{n_i} p_k u_k\right) \leq$$

$$\leq D\left(t_{m_i}^{(1)}, t_{n_i}^{(1)}\right) + \frac{P_{m_i}-P_{n_i}}{P_{m_i}} D\left(t_{n_i}^{(1)}, \bar{0}\right) \leq$$

$$\leq D\left(t_{m_i}^{(1)}, \bar{0}\right) + D\left(t_{n_i}^{(1)}, \bar{0}\right) + \frac{\delta}{2+\delta} D\left(t_{n_i}^{(1)}, \bar{0}\right) \tag{15}$$

by the help of Lemma 2. If we take limit of both sides of inequality (15) as $i \to \infty$, because (u_n) is $(\overline{N}, p)$ summable to $\bar{0}$, we reach inequality

$$\lim_{i\to\infty} D\left(t_{m_i}^{(1)}, \frac{P_{n_i}}{P_{m_i}} t_{n_i}^{(1)}\right) \leq \lim_{i\to\infty} D(t_{m_i}^{(1)}, \bar{0}) + \lim_{i\to\infty} D(t_{n_i}^{(1)}, \bar{0}) + \left(\frac{\delta}{2+\delta}\right) D\left(t_{n_i}^{(1)}, \bar{0}\right) = 0. \tag{16}$$

Therefore, from inequalities (14) and (16) we attain

$$0 < \left(\frac{\delta}{2+\delta}\right)\frac{\vartheta}{2} \leq \lim_{i\to\infty} D\left(t_{m_i}^{(1)}, \frac{P_{n_i}}{P_{m_i}} t_{n_i}^{(1)}\right) \leq 0.$$

Because this contradicts that $(t_n^{(1)})$ converges to $\bar{0}$, we conclude that (u_n) converges to $\bar{0}$.

Theorem 4 is proved.

Theorem 5. *Let* (p_n) *satisfies conditions* (2), $\dfrac{p_n}{P_n} \to 0$ *as* $n \to \infty$ *and*

$$1 \le \frac{P_m}{P_n} \to 1 \qquad \textit{whenever} \qquad 1 < \frac{m}{n} \to 1 \quad \textit{as} \quad n \to \infty. \tag{17}$$

If $(u_n) \in \omega(F)$ *is statistically* $(\overline{N}, p)$ *summable to* $\mu_0 \in E^1$ *and slowly oscillating relative to* (P_n), *then* (u_n) *is convergent to* μ_0.

Proof. Assume that (p_n) satisfies conditions (2), (17) and $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and (u_n) being statistically $(\overline{N}, p)$ summable to μ_0 is slowly oscillating relative to (P_n). In the circumstances, we arrive by the help of Lemma 5 that $(t_n^{(1)})$ is also slowly oscillating relative to (P_n). In other words, we can say by the definition of slow oscillation relative to (P_n) that condition

$$\limsup_{n\to\infty} D(t_m^{(1)}, t_n^{(1)}) = 0 \qquad \text{as} \qquad m \ge n, \qquad \frac{P_m}{P_n} \to 1 \qquad (n \to \infty),$$

holds and, so, by condition (17) we obtain

$$\limsup_{n\to\infty} D(t_m^{(1)}, t_n^{(1)}) = 0 \qquad \text{as} \qquad m \ge n, \qquad \frac{m}{n} \to 1 \qquad (n \to \infty).$$

This statement implies slow oscillation of $(t_n^{(1)})$. Since $(t_n^{(1)})$ is slowly oscillating and statistically convergent to μ_0, we reach by the help of Lemma 6 that $(t_n^{(1)})$ is convergent to μ_0 which means that (u_n) is $(\overline{N}, p)$ summable to μ_0. In addition to this, since (p_n) satisfies condition $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$, we have

$$\frac{P_n}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \qquad \text{as} \qquad n \to \infty.$$

If we consider that condition of slowly oscillating relative to (P_n) is the Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n) as a result of Theorem 4, then we conclude that (u_n) is convergent to μ_0.

Theorem 5 is proved.

***Corollary* 1.** *Let* (p_n) *satisfies conditions* (2), (17) *and* $\dfrac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* $(u_n) \in \omega(F)$ *is statistically* $(\overline{N}, p)$ *summable to* $\mu_0 \in E^1$ *and satisfies two-sided condition of Hardy type relative to* (P_n), *then* (u_n) *is convergent to* μ_0.

As a generalization of Theorem 5 and Corollary 1, we can present Theorem 6 and Corollary 2, respectively.

Theorem 6. *Let* (p_n) *satisfies conditions* (2), (17) *and* $\dfrac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* $(u_n) \in \omega(F)$ *is statistically* $(\overline{N}, p, \alpha)$ *summable to* $\mu_0 \in E^1$ *for some integer* $\alpha \ge 0$ *and slowly oscillating relative to* (P_n), *then* (u_n) *is convergent to* μ_0.

Proof. Assume that (p_n) satisfies conditions (2), (17) and $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and (u_n) being statistically $(\overline{N}, p, \alpha)$ summable to μ_0 for some integer $\alpha \ge 0$ is slowly oscillating relative to (P_n). In the circumstances, we arrive by the help of Lemma 5 that $(t_n^{(\beta)})$ is also slowly oscillating relative to (P_n) for each integer $\beta \geq 1$. In other words, we can say by the definition of slow oscillation relative to (P_n) that condition

$$\limsup_{n\to\infty} D(t_m^{(\beta)}, t_n^{(\beta)}) = 0 \qquad \text{as} \qquad m \geq n, \qquad \frac{P_m}{P_n} \to 1 \qquad (n \to \infty),$$

holds and, so, by condition (17) we obtain

$$\limsup_{n\to\infty} D(t_m^{(\beta)}, t_n^{(\beta)}) = 0 \qquad \text{as} \qquad m \geq n, \qquad \frac{m}{n} \to 1 \qquad (n \to \infty).$$

This statement implies slow oscillation of $(t_n^{(\beta)})$ for each integer $\beta \geq 1$. Since $(t_n^{(\alpha)})$ is statistically convergent to μ_0 from the assumption, we reach by taking $\beta = \alpha$ with the help of Lemma 6 that $(t_n^{(\alpha)})$ is convergent to μ_0 which means that $(t_n^{(\alpha-1)})$ is $(\overline{N}, p)$ summable to μ_0. In the case when $\beta = \alpha - 1$, we have that $(t_n^{(\alpha-1)})$ is slowly oscillating relative to (P_n). In addition to this, since (p_n) satisfies condition $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$, we find

$$\frac{P_n}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \qquad \text{as} \qquad n \to \infty.$$

If we consider that condition of slowly oscillating relative to (P_n) is the Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n) as a result of Theorem 4, then we conclude that $(t_n^{(\alpha-1)})$ is convergent to μ_0. Continuing in a similar way, it follows that $(t_n^{(1)})$ is convergent to μ_0 which means that (u_n) is $(\overline{N}, p)$ summable to μ_0. Therefore, we accomplish by the help of Theorem 4 that (u_n) is convergent to μ_0.

Theorem 6 is proved.

***Corollary* 2.** *Let (p_n) satisfies conditions* (2), (17) *and* $\dfrac{p_n}{P_n} \to 0$ *as $n \to \infty$. If $(u_n) \in \omega(F)$ is statistically $(\overline{N}, p, \alpha)$ summable to $\mu_0 \in E^1$ for some integer $\alpha \geq 0$ and satisfies two-sided condition of Hardy type relative to (P_n), then (u_n) is convergent to μ_0.*

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