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HANDLE DECOMPOSITIONS OF SIMPLY-CONNECTED FIVE-MANIFOLDS. II

РОЗКЛАД НА РУЧКИ ОДНОЗВ'ЯЗНИХ П'ЯТИВИМІРНИХ МНОГОВИДІВ. II

The handle decompositions of simply-connected smooth or piecewise-linear five-manifolds are considered. The basic notions and constructions necessary for proving further results are introduced.

Розглядається розклад на ручки однозв'язних гладких або кусково-лінійних п'ятивимірних многовидів. Наведені основні поняття і конструкції, необхідні для одержання подальших результатів.

The main result of this paper is Theorem 3 asserting that the D. Barden's handle decomposition of a closed 1-connected smooth or PL 5-manifold is geometrically diagonal. It is obtained as a consequence of Theorem 2 apparently describing the construction of the C. T. C. Wall's diffeomorphisms for each of 1-connected 4-manifolds $S^2 \times S^2 \# S^2 \times S^2$ and $S^2 \underset{\sim}{\times} S^2 \# S^2 \times S^2$. The basic notions and tools necessary to prove these theorems were presented in [1].

4. D. Barden's constructions. As was proved by D. Barden in [2], any closed 1-connected 5-manifold is diffeomorphic to the finite connected sum of 5-manifolds of certain types. These manifolds are constructed as follows.

Consider standard 5-manifolds $M = A \natural A$ and $X = B \natural A$, where A and B are the elementary 5-manifolds designed above. Let V be either M or X; then V admits an exact handle decomposition $V = h^0 \cup h_1^2 \cup h_2^2$, which induces the canonical handle decomposition of $\partial V = h^0 \cup h_{11}^2 \cup h_{12}^2 \cup h_{21}^2 \cup h_{22}^2 \cup h^4$ with the canonical basis $\{a_1, b_1, a_2, b_2\}$ of $H_2(\partial V)$. All cycles of this basis can be realized by 2-spheres embedded in ∂V. The spheres $\tilde{a}_1$, and $\tilde{a}_2$ are determined by the cores of 5-dimensional 2-handles h_1^2, and h_2^2 of V, the spheres $\tilde{b}_1$, and $\tilde{b}_2$ are the b-spheres of these handles. The intersection form $Q(\partial V)$ in the canonical basis is

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \oplus \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ if } \partial V = \partial M \text{ or } \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \oplus \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ if } \partial V = \partial X.$$

Consider the following nondegenerate matrices with integer coefficients:

$$A(k) = \begin{pmatrix}1 & 0 & 0 & -k\\ 0 & 1 & 0 & 0\\ 0 & k & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad B(k) = \begin{pmatrix}1 & 0 & 0 & -2k\\ 0 & 1 & 0 & k\\ k & 2k & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$C(k) = \begin{pmatrix}1-2k & 2(1-2k) & -4k & 0\\ 0 & 2k-1 & 2k & k-1\\ 2k & 0 & 0 & 1-2k\\ 1-k & -2(k-1) & 1-2k & 0\end{pmatrix}. \tag{1}$$

For any integer $k \geq 1$, specify automorphisms f_{k*} of the group $H_2(\partial M)$ and automorphisms g_{k*} and h_{k*} of $H_2(\partial X)$ as follows: $f_{k*}\{a_1, b_1, a_2, b_2\} = \{a_1, b_1, a_2,$

$b_2\} A(k)$, $g_{k*}\{a_1, b_1, a_2, b_2\} = \{a_1, b_1, a_2, b_2\} B(k)$, $h_{k*}\{a_1, b_1, a_2, b_2\} = \{a_1, b_1, a_2, b_2\} C(k)$.

One can easily calculate that all f_{k*} preserve the intersection form $Q(\partial M)$, whereas g_{k*} and h_{k*} preserve $Q(\partial X)$. By the Wall's theorem [3], there exist diffeomorphisms f_k of ∂M, and g_k and h_k of ∂X, which induce the diffeomorphisms f_{k*}, g_{k*}, and h_{k*} on $H_2(\partial M)$ and $H_2(\partial X)$. For $k > 1$ introduce closed 1-connected 5-manifolds $M_k = = M \cup_{f_k} (-M)$, $X[B(k)] = X \cup_{g_k} (-X)$, and $X[C(k)] = X \cup_{h_k} (-X)$ for $k \geq 1$. Introduce also $M_1 = X_0 = S^5$, $M_\infty = S^2 \times S^3$, and $X_\infty = B \cup_{g_\infty} (-B)$, where $g_\infty = \mathrm{id}$. Since $\partial B = S^2 \tilde{\times} S^2 \approx \mathbb{CP}^2 \# (-\mathbb{CP}^2)$, the $H_2(\partial B)$ admits also a basis $\{p, q\}$ such that each of p and q corresponds to the summand $\mathbb{CP}^2$. One can easily specify the diffeomorphism g_{-1} of ∂B, which induces the following automorphism g_{-1*} of $H_2(\partial B)$: g_{-1*}: $\{p, q\} \to \{p, -q\}$. In the canonical basis $\{a = p, b = p - q\}$ of ∂B, the automorphism g_{-1*} is represented by the matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. Put $X_{-1} = = X \cup_{g-1} (-B)$. By definition, all 5-manifolds constructed above admit exact handle decompositions.

The matrices $B(k)$ and $C(k)$ differ from those considered in [2] because instead of the canonical basis for $H_2(\partial B)$ and $\partial X = \partial A \# \partial B \approx \partial B \neq \partial B$ as in [2], the corresponding bases $\{p, q\}$ and $\{p_1, q_1, p_2, q_2\}$ are used. When fixing the canonical basis, the matrices $B(k)$ and $C(k)$ change to (1).

Lemma 5 [2].

1) $H_2(M_k) = \mathbb{Z}_k \oplus \mathbb{Z}$, $k \neq 1, \infty$;

2) $H_2(X_{-1}) = \mathbb{Z}_2$, $H_2(X_\infty) = H_2(M_\infty) = \mathbb{Z}$;

3) $H_2(X[B(k)]) = \mathbb{Z}_{2k} \oplus \mathbb{Z}_{2k}$, $H_2(X[(k)]) = \mathbb{Z}_{2k-1} \oplus \mathbb{Z}_{4k-2}$, $0 < k < \infty$.

Any 1-connected closed 5-manifold W admits the linking form $b(x, y) = x \circ y \in \in \mathbb{Q} / \mathbb{Z}$ on $\mathrm{tors}(H_2(W))$. This is a nonsingular nondegenerate skew-symmetric integer bilinear form. In [2], a b-basis $\{z_1, z_2, x_1, y_1, \dots, x_m, y_m\}$ was constructed, i.e., the basis in which z_1 has an odd order φ, z_2 has the order 2φ, and $b(z_1, z_2) = 1 / \varphi$; both x_i and y_i have an odd order θ_i and $b(x_i, y_i) = 1 / \theta_i$; on the other pairs (u, v) of the basis elements except, possibly, (z_2, z_2) and (y_i, y_i), $i = 1, \dots, m$, the value of $b(u, v)$ is 0. The elements z_1 or both z_1 and z_2 may be missed from the b-basis. In this case, we include z_1 and z_2 into the basis assuming them to be equal to zero. A basis of the entire $H_2(W)$ is called a b-basis if it contains a b-basis of $\mathrm{tors}(H_2(W))$. It is shown in [2] that a b-basis may be chosen to be minimal, i.e., such that it contains a minimal number of elements. Since for each $x \in \mathrm{tors}(H_2(W))$, we have $b(x, x) = 0$ or $b(x, x) = 1 / 2$, the minimal b-basis of $\mathrm{tors}(H_2(W))$ may be modified so that $b(x, x) = 0$ for each element X of the b-basis except, possibly, for one element. For any $x \in \mathrm{tors}(H_2(W))$ we have $b(x, x) \neq 0$ if $w^2(x) \neq 0$ ([2]). If $w^2(e) \neq 0$ for each $e \in \mathrm{Fr}(H_2(W))$, then we can modify also a basis of $\mathrm{Fr}(H_2(W))$ so that $w^2(e) = 0$ for each element e of the basis except, possibly, for one element.

Thus we have constructed the basis of $H_2(W)$, which we call the minimal w^2 - b-

basis.

Theorem 1 (the Barden decomposition theorem, [2]). *For any b-basis* $\{z_1, z_2, x_1, y_1, \ldots, x_m, y_m, l_1, \ldots, l_s\}$ *of* $H_2(W)$, *there exists a diffeomorphism* ψ *of* W *into the manifold*

$$V = M_{z_1, z_2} \# M_{x_1, y_1} \# \ldots \# M_{x_r, y_r} \# M_{e_1} \# \ldots \# M_{e_s}, \tag{2}$$

where $M_{z_1, z_2} = X_{-1}$ *if the order* φ *of* z_1 *is* 1, *i. e.* $z_1 = 0$, *and* $M_{z_1, z_2} = X[C((\varphi-1)/2)]$ *if* $\varphi > 1$; $H_{x_i, y_i} = M_{\theta_i}$ *if* $b(y_i, y_i) = 0$ *and* $M_{x_i, y_i} = X[B(\theta_i/2)]$ *if* $b(y_i, y_i) \neq 0$, *where* θ_i *is the order of* x_i *and* y_i; $M_{e_i} = M_\infty$ *if* $w_2(e_i) = 0$ *and* $M_{e_i} = X_\infty$ *if* $w^2(e_i) \neq 0$. *For each pair* $(u, v) = (z_1, z_2)$ *or* (x_i, y_i), *the diffeomorphism* ψ *induces the isomorphism between* gp(u, v) *and* $H_2(M_{u,v})$, *which preserves the linking numbers. For each generator* e_i *of* Fr$(H_2(W))$, *we have* $H_2(M_{e_i}) = \mathbb{Z}$ *and* $w^2(M_{e_i}) = 0$ *iff* $w^2(e_i) = 0$.

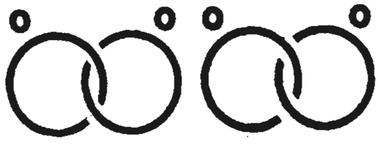
It follows from Theorem 1 that any b-basis of $H_2(W)$ determines a handle decomposition of W which contains one 0-handle, one 5-handle, and a pair of 2-handle and 3-handle for each element of this basis. The minimal w^2 - b-basis determines an exact handle decomposition of W which contains at most one summand of type X for each of tors $(H_2(W))$ and Fr $(H_2(G))$, all other summand being of type M. Since the basis is minimal, the handle decomposition is exact. In what follows, we will consider only such decompositions and call them the Barden handle decompositions.

5. Diffeomorphisms of manifolds $S^2 \tilde{\times} S^2 \# S^2 \times S^2$ **and** $S^2 \times S^2 \# S^2 \times S^2$. Let V denote either $S^2 \times S^2 \# S^2 \times S^2$ or $\partial X = S^2 \tilde{\times} S^2 \# S^2 \times S^2$. V admits an induced canonical handle decomposition with the canonical basis $\{a_1, b_1, a_2, b_2\}$ and 2-spheres $\{\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2\}$, which realize this basis. $\{a_2, b_2\}$ will always be considered as a canonical basis of the second summand, i.e., of $S^2 \times S^2$. We prove here the theorem which provides a geometric description of the Wall's diffeomorphisms of v.

Theorem 2. *For* $V = \partial M$ *or* $V = \partial X$, *let* φ_* *be an automorphism of* $H_2(V)$, *which preserves the intersection form* $Q(V)$. *Let* C *be a matrix, which represents* φ_* *in the canonical basis* $\{a_1, b_1, a_2, b_2\}$ *of an induced canonical handle decomposition of* V. *Then there exists a diffeomorphism* φ *of* V, *which induces the automorphism* φ_* *on* $H_2(V)$ *and maps each sphere of* $\{\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2\}$ *into the corresponding sphere of* $\{\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2\}C$, *where the addition operation means the connected summing and the minus sign means the altering of the orientation.*

Fix the above-mentioned induced canonical handle decomposition of V. By rearranging the handles, we can construct the proper handle decomposition of V. Let η be the corresponding diffeomorphism of V. The a-spheres of the proper handle decomposition $V = h^0 \cup h^2_{11} \cup h^2_{12} \cup h^2_{21} \cup h^2_{22} \cup h^4$ are in ∂h^0 and the cores of these 2-handles determine the 2-spheres $\{\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}\} = \eta\{\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2\}$ which realize the basis $\{a, b, x, y\} = \eta_*\{a_1, b_1, a_2, b_2\}$ with geometric intersections and $\{x, y\}$ corresponds to the second summand $S^2 \times S^2$. Consider a new proper handle decomposition $V = h^0 \cup \bar{h}^2_{11} \cup \bar{h}^2_{12} \cup \bar{h}^2_{21} \cup \bar{h}^2_{22} \cup h^4$, where $\{\bar{h}^2_{11}, \bar{h}^2_{12}, \bar{h}^2_{21}, \bar{h}^2_{22}\} = \{h^2_{11}, h^2_{12}, h^2_{21}, h^2_{22}\}C$, the addition operation means the handle summing, and the minus sign means the altering of orientation for the core of a handle. If we construct a diffeomorphism θ of V, such that $\theta(h_{ij}^2) = \bar{h}_{ij}^2$, $\forall i, j = 1, 2$, and then turn back to the induced canonical handle decomposition, we obtain the diffeomorphism $\varphi = \eta^{-1}\theta\eta$ we are searching for. Thus, our nearest aim is to construct a diffeomorphism θ.

The 2-handles of the proper handle decomposition of V are glued along a framed link in $S^3 = \partial h^0$ of type (3) for $V = \partial M$ or type (4) for $V = \partial X$.



(3)

(4)

Since any two links of type 3 are ambiently isotopic in S^3 and the same holds also for any two links of type 4, Theorem 2 will be proved if we show that the link for attaching 2-handles $\bar{h}_{ij}^2$, $i, j = 1, 2$, to $S^3 = \partial h^0$ is the same as that for attaching h_{ij}^2. Denote this property by Γ. The property Γ is equivalent to all mutual intersection indices of $\{\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}\}C$ being geometric (algebraic indices of $\{\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}\}C$ are equal to those of $\{\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}\}$ because C preserves the intersection form).

Let Y be an arbitrary closed 1-connected 4-manifold with the indefinite intersection form. Consider $V = Y \# S^2 \times S^2$. In the proof of the Wall's Theorem [3], all the generators of the group of automorphisms of $H_2(V)$ preserving the intersection form are presented. Let $\{x, y\}$ be a canonical basis of $H_2(S^2 \times S^2)$ and z be an arbitrary element of $H_2(Y)$. Consider the following automorphisms of $H_2(V)$:

$$E_\omega^y: \quad \begin{aligned} &z \to z - (z \cdot \omega) y \\ &x \to x - Ny + \omega \\ &y \to y \end{aligned} \qquad\qquad E_\omega^x: \quad \begin{aligned} &z \to z - (z \cdot \omega) y \\ &x \to x \\ &y \to y - Nx + \omega, \end{aligned}$$

where ω is the element of $H_2(Y)$ such that $\omega \cdot \omega = 2N \in \mathbb{Z}$. For $\omega \in H_2(V)$ such that $|\omega \cdot \omega| = 1$, if it exists, consider the automorphism $S(\omega)$

$$z \to z - \frac{2}{\omega \cdot \omega}(z \cdot \omega)\omega, \quad x \to x, \quad y \to y.$$

Consider also the following automorphisms

$$R_0:\ z\to -z;\ x\to x,\ y\to y \qquad R_1:\ z\to z;\ x\to -x,\ y\to -y \qquad R_2:\ z\to z;\ x\to y,\ y\to x.$$

As was shown by Wall [3], in the case where $Q(V)$ is even, the group of automorphisms of $H_2(V)$ preserving the intersection form $Q(V)$ admits the following generators:

1) E^y_ω, E^x_ω for all $\omega\in H_2(Y)$ with even $\omega\cdot\omega$;

2) R_0, R_1, R_2.

In the case where $Q(V)$ is odd, the generators are the same as specified in 1) and 2) and also $S(u)$ for a fixed $u\in H_2(V)$ such that $|u\cdot u|=1$. By applying this result to $V=\partial X$ with the basis $\{a, b, x, y\}$, we obtain $E^y_{\alpha,\beta}$: $a\to a-(2\alpha+\beta)y$, $b\to b-2\alpha y$, $x\to x-2\alpha(\alpha+\beta)y+2\alpha a+\beta b$, $y\to y$ for any $\omega=2\alpha a+\beta b\in H_2(S^2\tilde{\times} S^2)$. $E^x_{\alpha,\beta}$ can be obtained as a result of permuting x and y in $E^y_{\alpha,\beta}$. Fixing $u=a$, we obtain $S(u)=g_{-1}\oplus E$.

For $V=\partial M$, we have

$$E^y_{\alpha,\beta}:\ a\to a-\beta y,\ b\to b-\alpha y,\ x\to x-\alpha\beta y+\alpha a+\beta b,\ y\to y$$

for any $\omega=\alpha a+\beta b$, because $\omega\cdot\omega$ is always even, and $E^x_{\alpha,\beta}$ as a result permuting x and y in $E^y_{\alpha,\beta}$.

It sufficies to prove property Γ only for these generators, since the property is obvious for R_0, R_1, R_2.

To prove the property Γ for $g_{-1}\oplus E$, consider g_{-1} in the basis $\{p, q\}$ of $H_2(S^2\tilde{\times} S^2)$. This basis is realized by the embedded 2-spheres $\{\tilde{p}, \tilde{q}\}$ and determined by the handle decomposition with the 2-handles attached along the obvious framed link. This link consists of two circles in S^3 having framings 1 and -1. The first sphere corresponds to p and the second to q. Since, by definition, $g_{-1}(p)=p$ and $g_{-1}(q)=-q$, the link is not changed and property Γ is obvious. Since the canonical basis $\{a, b\}$ of $H_2(S^2\tilde{\times} S^2)$ is obtained from $\{p, q\}$ with $a=p$ and $b=p-q$, g_{-1} can be performed with one Kirby move, hence, g_{-1} has property Γ in the canonical basis of $H_2(S^2\tilde{\times} S^2)$. The same is, certainly, true for $g_{-1}\oplus E$ in the canonical basis of ∂X.

If we prove the property Γ for $E^x_{\alpha,\beta}$ and $E^y_{\alpha,\beta}$, the proof of Theorem 2 will be completed because R_0, R_1, R_2, and $g_{-1}\oplus E$ are of order 2 and the diffeomorphisms opposite to $E^x_{\alpha,\beta}$ and $E^y_{\alpha,\beta}$ are the same as $E^x_{\alpha,\beta}$ and $E^y_{\alpha,\beta}$, but with different α and β. Since x and y in $E^x_{\alpha,\beta}$ and $E^y_{\alpha,\beta}$ are symmetric, it suffices to prove property Γ only for $E^y_{\alpha,\beta}$ for $V=\partial X$ or $V=\partial M$. For $V=\partial X$, consider $E^y_{\alpha,\beta}$ as the product C_2C_1, where the automorphisms C_1 and C_2 act as follows

$$C_1:\ \begin{aligned} &a\to a'=a\\ &b\to b'=b\\ &x\to x'=x-2\alpha(\alpha+\beta)y+2\alpha a+\beta b\\ &y\to y'=y\end{aligned} \qquad C_2:\ \begin{aligned} &a'\to a''=a'-(2\alpha+\beta)y'\\ &b'\to b''=b'-2\alpha y'\\ &x'\to x''=x'\\ &y'\to y''=y'.\end{aligned}$$

Note that C_1 and C_2 do not preserve the intersection form $Q(\partial X)$. Having performed C_1 for a given proper handle decomposition h_{ij}, $i, j = 1, 2$ of ∂X, we obtain a handle decomposition attached along the framed link on the left-hand side of the picture.

In Fig. 1 we show the attaching circles of 2-handles. Near each circle, we show the framing and the cycle in $H_2(\partial X)$ determined by the core of the 2-handle attached to this circle. Denote these circles by $\gamma_{a'}, \gamma_{b'}, \gamma_{x'}, \gamma_{y'}$. Since $\gamma_{y'}$ has a trivial framing and links $\gamma_{x'}$, geometrically one time, we can apply the Kirby moves [4] to free $\gamma_{x'}$ of $\gamma_{a'}$ and $\gamma_{b'}$. It readily follows from the definition of the Kirby move that the composition of the Kirby moves we have just performed determines the automorphism C_2 of $H_2(\partial X)$ applied to the link on the left-hand side. Thus, after performing C_1 and C_2, we have a framed link on the right-hand side of the picture with all linking numbers being geometric. The property Γ for $V = \partial X$ is proved.

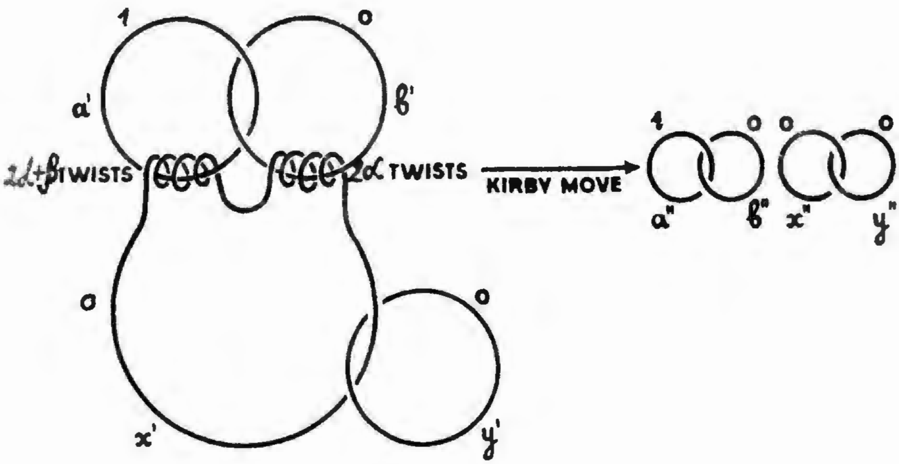


Fig. 1

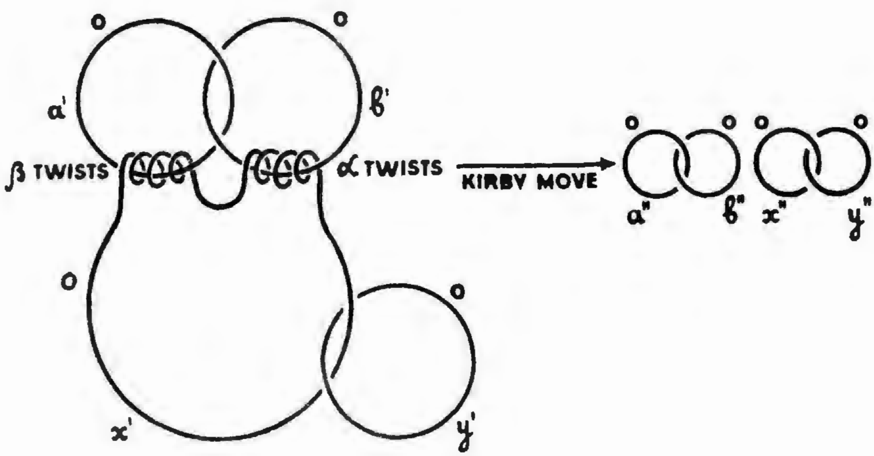


Fig. 2

For $V = \partial M$, we have $E^y_{\alpha,\beta} = C_2 C_1$ with

$$C_1:\quad \begin{aligned} a &\to a' = a \\ b &\to b' = b \\ x &\to x' = x - \alpha\beta y + \alpha a + \beta b \\ y &\to y' = y \end{aligned} \qquad C_2:\quad \begin{aligned} a' &\to a'' = a' - \beta y' \\ b' &\to b'' = b' - \alpha y' \\ x' &\to x'' = x' \\ y' &\to y'' = y'. \end{aligned}$$

The application of C_2 of $H_2(\partial M)$ to the link on the left-hand side of Fig. 2 is equivalent to performing a series of Kirby moves with it to obtain a link with geometric linking numbers on the right-hand side. This proves property Γ for $E^y_{\alpha,\beta}$ and completes the proof of Theorem 2.

6. Applications to the Barden handle decomposition. Here we use Theorem 2 to prove the following theorem.

Theorem 3. *All incidence indices of* 3*-handles and* 2*-handles in the Barden handle decomposition of a closed* 1*-connected* 5*-manifold are geometrically diagonal.*

It suffices to prove this theorem for M_∞, X_∞, X_{-1}, M_k, $X[B(k)]$ and $X[C(k)]$. For M_∞, X_∞, and X_{-1}, the theorem is obvious. To prove it for other manifolds, consider an exact handle decomposition of the standard 5-manifold $W = M$ or $W = X$. It induces the canonical handle decomposition of the standard 4-manifold ∂W with the canonical basis $\{a_1, b_1, a_2, b_2\}$ realized by the 2-spheres $\{\tilde a_1, \tilde b_1, \tilde a_2, \tilde b_2\}$ embedded into ∂W ($\tilde b_1$ and $\tilde b_2$ are the b-spheres of 5-dimensional 2-handles of W). Each of closed 5-manifolds M_k, $X[B(k)]$, and $X[C(k)]$ can be obtained as a double of M, X, and X, respectively, along the corresponding boundary diffeomorphisms f_k, g_k and h_k. By Statement 3, the homomorphism $\partial_3\colon C_3 \to C_2$ can be represented in the canonical basis of the boundary by the matrix $a^k_{ij} = f_k(\tilde b_i)\cdot \tilde b_j$ for M_k, $a^k_{ij} = g_k(\tilde b_i)\cdot \tilde b_j$ for $X[B(k)]$, and $a^k_{ij} = h_k(\tilde b_i)\cdot \tilde b_j$ for $X[C(k)]$. It is easy to calculate these matrices for M_k, $X[B(k)]$, and $X[C(k)]$ to obtain

$$\begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -2k \\ 2k & 0 \end{pmatrix}, \quad \begin{pmatrix} 2(1-2k) & 0 \\ 0 & 1-2k \end{pmatrix},$$

respectively. By Theorem 2, all the coefficients of these matrices are geometric. Thus, Theorem 3 is proved.

This theorem can be applied also to construct round Morse functions [5]. Combining it with the technique of A. T. Fomenko and V. V. Sharko [6], we obtain the following theorem.

Theorem 4. *Any closed* 1*-connected* 5*-manifold admits an exact round Morse function.*

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