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ON THE CARDINALITY OF A REDUCED UNIQUE RANGE SET*

ПРО ПОТУЖНІСТЬ РЕДУКОВАНОЇ МНОЖИНИ УНІКАЛЬНОСТІ

Two meromorphic functions are said to share a set $S \subset \mathbb{C} \cup \{\infty\}$ ignoring multiplicities (IM) if S has the same pre-images under both functions. If any two nonconstant meromorphic functions, sharing a set IM, are identical, then the set is called a "reduced unique range set for meromorphic functions" (in short, RURSM or URSM-IM).

From the existing literature, it is known that there exists a RURSM with seventeen elements. In this article, we reduced the cardinality of an existing RURSM and established that there exists a RURSM with fifteen elements. Our result gives an affirmative answer to the question of L. Z. Yang (Int. Soc. Anal., Appl., and Comput., **7**, 551 – 564 (2000)).

Дві мероморфні функції поділяють між собою множину $S \subset \mathbb{C} \cup \{\infty\}$, не враховуючи кратність, якщо S має однакові прообрази відносно обох цих функцій. Якщо для деякої множини будь-які дві мероморфні функції, що не є сталими та поділяють між собою цю множину, не враховуючи кратність, обов'язково є тотожними, то така множина називається редукованою множиною унікальності для мероморфних функцій.

З наявних робіт відомо, що існує редукована множина унікальності для мероморфних функцій, яка складається з 17 елементів. У цій роботі ми скорочуємо вказане число та доводимо, що існує редукована множина унікальності для мероморфних функцій, що складається з 15 елементів. Наш результат дає ствердну відповідь на питання, поставлене L. Z. Yang (Int. Soc. Anal., Appl., and Comput., **7**, 551 – 564 (2000)).

1. Introduction. Suppose that f and g are two nonconstant meromorphic functions and $a \in \mathbb{C}$. We say that f and g share the value a-CM (counting multiplicities), if $f-a$ and $g-a$ have the same set of zeros with the same multiplicities. Similarly, we say that f and g share the value a-IM (ignoring multiplicities), provided that $f-a$ and $g-a$ have the same set of zeros, where the multiplicities are not taken into account.

Moreover, we say that f and g share ∞-CM (resp., IM), if $1/f$ and $1/g$ share 0-CM (resp., IM).

In course of studying the factorization of meromorphic functions, in 1976, F. Gross [6] first generalized the idea of value sharing by introducing the concept of set sharing. Before going to the details of this paper, we first recall the definition of set sharing:

Definition 1.1. *Let f be a nonconstant meromorphic function and, let $S \subset \mathbb{C} \cup \{\infty\}$. The set*

$$E_f(S) = \bigcup_{a \in S} \{(z,m) \in \mathbb{C} \times \mathbb{N} \mid f(z) - a = 0\},$$

where a zero of $f(z)-a$ with multiplicity m counts m times in $E_f(S)$, is called the pre-image of S under f, which is also denoted by $f^{-1}(S)$. Also, we define

$$\overline{E}_f(S) = \bigcup_{a \in S} \{(z,1) \in \mathbb{C} \times \mathbb{N} \mid f(z) - a = 0\},$$

i.e., $\overline{E}_f(S)$ denotes the set of distinct elements in $E_f(S)$.

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Definition 1.2. *Two meromorphic functions f and g are said to share a set S CM (resp., IM), if $E_f(S) = E_g(S)$ (resp., $\overline{E}_f(S) = \overline{E}_g(S)$).*

Thus, if S is a singleton set, then it coincides with the usual definition of the value sharing notation.

In 1976, F. Gross [6] proposed the following question which has later became popular as "*Gross' question*". The question was as follows:

***Question* 1.1.** Does there exist a finite set S such that any two nonconstant entire functions f and g sharing the set S must be $f \equiv g$?

In 1982, to give an affirmative answer to the above question, F. Gross and C. C. Yang [7] introduced the terminology of unique range set for entire function (in short, URSE) as follows:

Definition 1.3. *A set $S \subset \mathbb{C}$ is said to be a unique range set for entire functions (in short, URSE), if for any two nonconstant entire functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$.*

In the same paper [7], they proved the following result.

Theorem A [7]. *Let $S = \{z \in \mathbb{C} \colon e^z + z = 0\}$. If two entire functions f, g satisfy $E_f(S) = E_g(S)$, then $f \equiv g$.*

It is to be observed that the set S given in Theorem A is an unique range set but contains infinitely many elements. Thus it can not answer to Question 1.1.

Analogue to Definition 1.3, the definition of unique range sets for meromorphic functions was also introduced in the literature.

Definition 1.4. *A set $S \subset \mathbb{C}$ is called a unique range set for meromorphic functions (in short, URSM), if for any two nonconstant meromorphic functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$.*

Later on, many authors (see, e.g., [3, 4, 10, 11, 15]) gave the existence of such finite sets for entire functions as well as meromorphic functions to confirm Question 1.1.

The prime concern of the researchers is *to find new unique range sets* or *to make the cardinalities of the existing range sets as small as possible*. To see the remarkable progress in this regard, one can go through the research monograph of C. C. Yang and H. X. Yi [13].

To carry on the research on unique range sets, in 1997, H. X. Yi [16] introduced the concept of reduced unique range sets.

Definition 1.5 [16]. *A set $S \subset \mathbb{C} \cup \{\infty\}$ is said to be a unique range set for meromorphic (resp., entire) functions in ignoring multiplicity, in short URSM-IM (resp., URSE-IM) or a reduced unique range set for meromorphic (resp., entire) functions, in short RURSM (resp., RURSE) if $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$ for any pair of nonconstant meromorphic (resp., entire) functions.*

So, the following question is natural.

***Question* 1.2** [16]. Is there any finite set S such that for any two nonconstant meromorphic (resp., entire) functions f and g, the condition $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$?

In 1997, H. X. Yi [16] gave an answer to the above question.

Theorem B [16]. *Let n and m be two integers with $n > 2m + 14$ and $m \geq 2$, and let a and b be two non-zero constants such that the algebraic equation $z^n + az^m + b = 0$ has no multiple roots. If n and m are co prime, then*

$$S = \{z \mid z^n + az^m + b = 0\}$$

is a URSM-IM.

The above theorem gives the existence of a URSM-IM with 19 elements. In 1998, H. X. Yi [17] further improved the above result as:

Theorem C [17]. *Let $n(\geq 17)$ be an integer. Let*

$$S = \{z \mid az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2 = 0\},$$

where a and b be two non-zero constants such that $ab^{n-2} \neq 2$. Then the set S is a URSM-IM.

Thus Theorem C gives the existence of a URSM-IM with 17 elements. In this direction, in 1997, M. Reinders [12] has shown that there exist URSM-IM with 16 elements. But unfortunately, the proof of a lemma which is necessary in the proof of Reinders' proof [12] has some gaps [9, p. 204].

In 1998, M. L. Fang and H. Guo [2] gave another example of URSM-IM with 17 elements using the technique of G. Frank and M. Reinders [3].

For a positive integers $n(\geq 3)$ and a complex number $c(\neq 0, 1)$, we shall denote by $P(z)$ [3] the following polynomial:

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c. \tag{1.1}$$

Clearly, the restrictions on c implies that $P(z)$ has only simple zeros.

Theorem D [2]. *Let $S = \{z \mid P(z) = 0\}$, where $P(z)$ is defined by* (1.1). *If $n \geq 17$, then the set S is a URSM-IM.*

In 1999, S. Bartels [1] gave another proof of Theorem D. Thus it is observed from the existing literature that the smallest available reduced unique range set must contains at least 17 elements (see, e.g., [1, 2, 17]). Let

$$\overline{\lambda}_M = \inf\{\sharp(S) \colon S \text{ is a URSM-IM}\},$$

where $\sharp(S)$ is the cardinality of S. It is clear from the above discussion that $\sharp(S) \leq 17$. Also, examples show that $\sharp(S) \geq 6$ [13, p. 527]. Combining the above results, L.-Z. Yang pose the following open question [14, p. 557].

***Question* 1.3** [14]. What exactly the number $\overline{\lambda}_M$ is?

The main purpose of this paper is to reduce the cardinality of the URSM-IM in Theorem D. As a result, our paper partially answers to Question 1.3.

2. Main result. The following theorem is the main result of this paper.

Theorem 2.1. *Let $S = \{z \colon P(z) = 0\}$, where $P(z)$ is the polynomial of degree n, defined in* (1.1). *If $n \geq 15$, then S is a URSM-IM.*

***Remark* 2.1.** In Theorem 2.1, if $n \geq 9$, then S is a URSE-IM.

3. Notations. We assumed that the readers are familiar with the classical Nevanlinna theory [8, 13]. Before going to the proof of the main theorem, we explain some well known definitions and notations.

Definition 3.1. *Let f be a meromorphic function. Also, let $a \in \mathbb{C} \cup \{\infty\}$ and $m \in \mathbb{N}$.*

(i) *We denote by $N(r, a; f \mid= 1)$, the counting function of simple a-points of f.*

(ii) *We denote by* $N(r,a;f\mid\leq m)$ (*resp.,* $N(r,a;f\mid\geq m)$), *the counting function of those* a*-points of* f *whose multiplicities are not greater (resp., not less) than* m *where each* a*-point is counted according to its multiplicity.*

Let $\overline{N}(r,a;f\mid\leq m)$ *and* $\overline{N}(r,a;f\mid\geq m)$ *denote the reduced counting function of* $N(r,a;f\mid\leq m)$ *and* $N(r,a;f\mid\geq m)$, *respectively.*

Similar to the above counting functions, $N(r,a;f\mid< m)$, $N(r,a;f\mid> m)$, $\overline{N}(r,a;f\mid< m)$ *and* $\overline{N}(r,a;f\mid> m)$ *are defined.*

Definition 3.2. *For* $a\in\mathbb{C}\cup\{\infty\}$ *and* $p\in\mathbb{N}$, *we denote, by* $N_p(r,a;f)$, *the sum*

$$\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\ldots+\overline{N}(r,a;f\mid\geq p).$$

Thus, clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 3.3. *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share* a *IM, where* $a\in\mathbb{C}\cup\{\infty\}$. *Let* z_0 *be an* a*-point of* f *with multiplicity* p *and* z_0 *be an* a*-point of* g *with multiplicity* q.

(i) *We denote by* $N_E^{1)}(r,a;f)$, *the counting function of those* a*-points of* f *and* g *where* $p=q=1$. *Thus* $N_E^{1)}(r,a;f)=N_E^{1)}(r,a;g)$.

(ii) *We denote by* $\overline{N}_E^{(2}(r,a;f)$, *the reduced counting function of those* a*-points of* f *and* g *where* $p=q\geq 2$. *So,* $\overline{N}_E^{(2}(r,a;f)=\overline{N}_E^{(2}(r,a;g)$.

(iii) *We denote by* $\overline{N}_L(r,a;f)$, *the reduced counting function of those* a*-points of* f *and* g *where* $p>q$ *and by* $\overline{N}_L(r,a;g)$, *we denote the reduced counting function of those* a*-points of* f *and* g *where* $q>p$. *Thus* $\overline{N}_L(r,a;f)\neq\overline{N}_L(r,a;g)$.

We denote by $\overline{N}_*(r,a;f,g)$, *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g. *Thus,*

$$\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)\qquad\text{and}\qquad\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g).$$

Definition 3.4. *Let* $a,b\in\mathbb{C}\cup\{\infty\}$. *We denote by* $N(r,a;f\mid\ g\neq b)$, *the counting function of those* a*-points of* f, *counted according to multiplicity, which are not the* b*-points of* g.

Definition 3.5. *A polynomial* $\wp(z)$ *over* $\mathbb{C}$, *is called a uniqueness polynomial for meromorphic (resp., entire) functions, if for any two nonconstant meromorphic (resp., entire) functions* f *and* g, $\wp(f)\equiv\wp(g)$ *implies* $f\equiv g$.

In 2000, H. Fujimoto [4] first discovered a special property of a polynomial, which was later termed as critical injection property.

Definition 3.6 [4]**.** *A polynomial* $\wp(z)$ *is said to satisfy critical injection property if* $\wp(\alpha)\neq$ $\neq\wp(\beta)$, *where* α *and* β *are any two distinct zeros of* $\wp'(z)$.

4. Auxiliary lemmas.

Lemma 4.1 (First fundamental theorem of Nevanlinna, [13])**.** *For a nonconstant meromorphic function* f *and for a complex number* $a\in\mathbb{C}\cup\{\infty\}$,

$$T\left(r,\frac{1}{f-a}\right)=T(r,f)+O(1),$$

where $O(1)$ *is a bounded quantity depending on* a.

Lemma 4.2 (Second fundamental theorem of Nevanlinna, [13, p. 15]). *Suppose that f is a nonconstant meromorphic function in the complex plane and $a_1, a_2, \ldots, a_q$ are $q(\geq 2)$ distinct values in $\mathbb{C}$. Then*

$$(q-1)T(r,f) \leq N(r,\infty;f) + \sum_{j=1}^{q} N(r,a_j;f) - N_{\mathrm{ram}}(r,f) + S(r,f), \tag{4.1}$$

where

$$N_{\mathrm{ram}}(r,f) = 2N(r,\infty;f) - N(r,\infty;f') + N(r,0;f')$$

and $S(r,f)$ is a quantity such that $\dfrac{S(r,f)}{T(r,f)} \to 0$ as $r \to +\infty$ outside of a set $E(\subset (0,\infty))$ with finite linear measure.

***Remark* 4.1.** Clearly, (4.1) can be written as

$$(q-1)T(r,f) \leq \overline{N}(r,\infty;f) + \sum_{j=1}^{q} \overline{N}(r,a_j;f) - N_{\circ}(r,0;f') + S(r,f),$$

where $N_{\circ}(r,0;f')$ is the counting function of those zeros of f' which is not zeros of $\prod_{j=1}^{q}(f-a_j)$.

Lemma 4.3 [13, p. 28]. *Let f be a nonconstant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f)) = d \cdot T(r,f) + S(r,f),$$

where $d = \max\{n,m\}$.

Lemma 4.4 [5]. *Let $\wp(z)$ be a monic polynomial without multiple zero whose derivative has mutually k-distinct zeros, given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively.*

Suppose that $\wp(z)$ satisfy the "critical injection property". Then $\wp(z)$ will be a uniqueness polynomial if and only if

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1,q_2,q_3\} \geq 2$ or $k = 2$, $\min\{q_1,q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

Lemma 4.5 [13, p. 376]. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non constant meromorphic functions sharing 1 CM. If*

$$N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + N_2(r,\infty;\mathcal{F}) + N_2(r,\infty;\mathcal{G}) < (\mu + o(1))T(r),$$

where $\mu < 1$, $r \in I$, $T(r) = \max\{T(r,\mathcal{F}), T(r,\mathcal{G})\}$, I is a set of infinite linear measure of $r \in (0,\infty)$. Then one of the following holds:

i) $\mathcal{F} \equiv \mathcal{G}$,

ii) $\mathcal{F}\mathcal{G} \equiv 1$.

Lemma 4.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non constant meromorphic functions sharing* 1 *IM. Then*

$$\overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G})-N_E^{1)}(r,1;\mathcal{F})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq$$

$$\leq\frac{1}{2}\{N(r,1;\mathcal{F})+N(r,1;\mathcal{G})\}+N(r,1;\mathcal{F}|\geq 2)+N(r,1;\mathcal{G}|\geq 2).$$

Proof. Given $\mathcal{F}$ and $\mathcal{G}$ share 1 IM. Let z_0 be an 1-point of $\mathcal{F}$ of multiplicity p and let z_0 be an 1-point of $\mathcal{G}$ of multiplicity q. Now, we consider following cases:

***Case* 1.** Assume $p=q$.

If $p=q=1$, then z_0 is counted $(1+1-1+0)=1$ times in the left-hand side of the above inequality whereas it is counted $\frac{1}{2}(1+1)+0+0=1$ times in the right-hand side of the same.

If $p=q\geq 2$, then z_0 is counted $(1+1-0+0)=2$ times in the left-hand side of the above inequality whereas it is counted $\frac{1}{2}(p+p)+p+p=3p$ times in the right-hand side of the same.

***Case* 2.** Assume $p>q$.

If $p>q$ and $q=1$, then $p\geq 2$ and z_0 is counted $(1+1-0+1)=3$ times in the left-hand side of the above inequality whereas it is counted $\frac{1}{2}(p+1)+p+0=\frac{3p}{2}+\frac{1}{2}\left(\geq 3+\frac{1}{2}\right)$ times in the right-hand side of the same.

If $p>q$ and $q\geq 2$, then z_0 is counted $(1+1-0+1)=3$ times in the left-hand side of the above inequality whereas it is counted $\frac{1}{2}(p+q)+p+q=\frac{3}{2}(p+q)(>3q)$ times in the right-hand side of the same.

***Case* 3.** Assume $q>p$.

The explanations are similar to Case 2. Hence, the proof is completed.

5. Proof of Theorem 2.1. Given that f and g share the set S IM. Now, we define

$$F:=Q(f),\qquad G:=Q(g),$$

where

$$Q(z):=\frac{P(z)+c}{c}=$$

$$=\frac{(n-1)(n-2)}{2c}z^{n-2}\left(z^2-\frac{2n}{n-1}z+\frac{n}{n-2}\right)$$

and $P(z)$ is defined in (1.1), $c\in\mathbb{C}\setminus\{0,1\}$.

Thus F and G share the value 1 IM, and, hence, $\overline{E}_F(\{1\})=\overline{E}_G(\{1\})$. Now we consider two cases:

***Case* 1.** First, we assume that F and G are linearly dependent. Then there exist a non zero constant k such that

$$F\equiv k\,G.$$

Thus, using Lemma 4.3, we obtain

$$T(r,f)=T(r,g)+S(r,g).$$

***Subcase* 1.1.** If $\overline{E}_F(\{1\}) \cap \overline{E}_G(\{1\}) \neq \phi$, then there exist a $z_0 \in \mathbb{C}$ such that $F(z_0) = G(z_0) = 1$. Thus, $k = 1$, i.e.,

$$F \equiv G, \text{ i.e., } P(f) \equiv P(g).$$

Since $P'(z) = \dfrac{n(n-1)(n-2)}{2} z^{n-3}(z-1)^2$ and $P(0) \neq P(1)$. So, $P(z)$ satisfies "critical injection property". Thus, in view of Lemma 4.4, $P(z)$ is a uniqueness polynomial, i.e., $f \equiv g$.

***Subcase* 1.2.** If $\overline{E}_F(\{1\}) \cap \overline{E}_G(\{1\}) = \phi$, then $\overline{E}_F(\{1\}) = \overline{E}_G(\{1\}) = \phi$. Thus, we can assume that F and G share 1 CM.

First, we show that under the given conditions,

$$FG \not\equiv 1,$$

because, otherwise if $FG \equiv 1$, then

$$f^{n-2} \prod_{i=1}^{2} (f - \gamma_i)\; g^{n-2} \prod_{i=1}^{2} (g - \gamma_i) \equiv \frac{4c^2}{(n-1)^2(n-2)^2},$$

where $\gamma_i,\ i = 1, 2$, are the roots of the equation $z^2 - \dfrac{2n}{n-1} z + \dfrac{n}{n-2} = 0$.

Let z_0 be a γ_i-point of f of order p. Then z_0 must be a pole of g (say, of order q). Then $p = nq \geq n$. So,

$$\overline{N}(r, \gamma_i; f) \leq \frac{1}{n} N(r, \gamma_i; f) \leq \frac{1}{n} T(r, f) + O(1).$$

Again, let z_0 be a zero of f of order t. Then z_0 must be a pole of g (say, of order s). Then $(n-2)t = ns$. Thus $t > s$. Now, $2s = (n-2)(t-s) \geq (n-2)$. Thus $(n-2)t = ns$ gives $t \geq \dfrac{n}{2}$. So,

$$\overline{N}(r, 0; f) \leq \frac{2}{n} N(r, 0; f) \leq \frac{2}{n} T(r, f) + O(1).$$

Similar calculations are valid for g also. Thus, applying the second fundamental theorem, we get

$$T(r, f) \leq \overline{N}(r, 0; f) + \sum_{i=1}^{2} \overline{N}(r, \gamma_i; f) + S(r, f) \leq$$

$$\leq \frac{2}{n} T(r, f) + \frac{2}{n} T(r, f) + S(r, f),$$

which is impossible as $n \geq 5$. Thus, $FG \not\equiv 1$.

Now, our claim is $F \equiv G$. Since

$$N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \leq$$

$$\leq 2\overline{N}(r, 0; f) + 2T(r, f) + 2\overline{N}(r, 0; g) + 2T(r, g) + 2\overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) +$$

$$+ S(r, f) + S(r, g) \leq$$

$$\leq 4T(r, f) + 4T(r, g) + 2\overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + S(r, f) + S(r, g) <$$

$$< (1 + o(1))T(r) \ \ (\text{as } n \geq 9 \text{ (resp., } n \geq 15) \text{ for URSE-IM (resp., URSM-IM)}),$$

where $T(r)=\max\{T(r,F),T(r,G)\}$ and $S(r)=o(T(r))$, so, in view of Lemma 4.5, we obtain

$$F\equiv G. \tag{5.1}$$

Thus, from (5.1), we have

$$P(f)\equiv P(g).$$

Since $P'(z)=\dfrac{n(n-1)(n-2)}{2}z^{n-3}(z-1)^2$ and $P(0)\neq P(1)$. So, $P(z)$ satisfies "critical injection property". Thus, in view of Lemma 4.4, $P(z)$ is a uniqueness polynomial, i.e.,

$$f\equiv g.$$

***Case* 2.** Assume that F and G are linearly independent. Then $F\not\equiv G$. Henceforth we shall denote by H the following function:

$$H:=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{5.2}$$

Again we consider two subcases:

***Subcase* 2.1.** Assume $H\equiv 0$. Then on integration, we get from (5.2) that

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B,$$

where $A(\neq 0)$, B are constants. Thus F and G share the value 1 CM. Since,

$$N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\leq$$

$$\leq 2\overline{N}(r,0;f)+2T(r,f)+2\overline{N}(r,0;g)+2T(r,g)+2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+$$

$$+S(r,f)+S(r,g)\leq$$

$$\leq\frac{\lambda}{n}T(r)+S(r)\quad(\lambda=8,\ \text{or 12 according to } f \text{ and } g \text{ both are entire,}$$

$$\text{or meromorphic functions, respectively})<$$

$$<(1+o(1))T(r)\ \ (\text{as } n\geq 9 \text{ (resp., } n\geq 15) \text{ for URSE-IM (resp., URSM-IM)}),$$

where $T(r)=\max\{T(r,F),T(r,G)\}$ and $S(r)=o(T(r))$, thus, in view of Lemma 4.5, we obtain

$$FG\equiv 1\ \text{ or }\ F\equiv G.$$

But, already we have seen that, if $n\geq 5$, then $FG\not\equiv 1$. Thus, $F\equiv G$, which contradicts the fact that F and G are linearly independent.

***Subcase* 2.2.** We assume that $H\not\equiv 0$. Then by simple calculations, we have

$$N(r,\infty;H)\leq\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+$$

$$+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not zeros of $F(F-1)$, similarly, $\overline{N}_0(r,0;G')$ is defined.

Since

$$cF = \frac{(n-1)(n-2)}{2} f^{n-2} \prod\nolimits_{i=1}^{2} (f-\gamma_i)$$

and

$$cF' = \frac{n(n-1)(n-2)}{2} f^{n-3} (f-1)^2 f',$$

where $\gamma_i,\ i=1,2$, are the roots of the equation $z^2 - \dfrac{2n}{n-1}z + \dfrac{n}{n-2} = 0$. Thus,

$$N(r,\infty;H) \le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}_*(r,1;F,G) +$$
$$+\overline{N}(r,1;f) + \overline{N}(r,1;g) + \overline{N}_\star(r,0;f') + \overline{N}_\star(r,0;g') \le$$
$$\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + 2\{T(r,f) + T(r,g)\} + \overline{N}_*(r,1;F,G) +$$
$$+\overline{N}_\star(r,0;f') + \overline{N}_\star(r,0;g'), \tag{5.3}$$

where $\overline{N}_\star(r,0;f')$ is the reduced counting function of those zeros of f' which are not zeros of $f(f-1)$ and $(F-1)$, $\overline{N}_\star(r,0;g')$ denotes similarly according to g. Again

$$N_E^{1)}(r,1;F) = N_E^{1)}(r,1;G) \le N(r,\infty;H) + S(r,f) + S(r,g), \tag{5.4}$$

where $N_E^{1)}(r,1;F)$ is the counting function of those simple 1-points of F which are also simple 1-points of G.

Thus, using (5.3), (5.4) and Lemma 4.6 and first fundamental theorem, we have

$$\overline{N}(r,1;F) + \overline{N}(r,1;G) \le$$
$$\le \overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1)}(r,1;F) + N(r,\infty;H) + S(r,f) + S(r,g) \le$$
$$\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + 2\{T(r,f) + T(r,g)\} + \overline{N}_*(r,1;F,G) +$$
$$+\overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1)}(r,1;F) + \overline{N}_\star(r,0;f') + \overline{N}_\star(r,0;g') +$$
$$+S(r,f) + S(r,g) \le$$
$$\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + 2\{T(r,f) + T(r,g)\} +$$
$$+\frac{1}{2}\{N(r,1;F) + N(r,1;G)\} + N(r,1;F \mid\ge 2) + N(r,1;G \mid\ge 2) +$$
$$+\overline{N}_\star(r,0;f') + \overline{N}_\star(r,0;g') + S(r,f) + S(r,g) \le$$
$$\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \left(2 + \frac{n}{2}\right)\{T(r,f) + T(r,g)\} +$$
$$+N(r,1;F \mid\ge 2) + N(r,1;G \mid\ge 2) + N_\star(r,0;f') + N_\star(r,0;g') +$$
$$+S(r,f) + S(r,g), \tag{5.5}$$

where $N_\star(r,0;f')$ is the counting function of those zeros of f' which are not zeros of $f(f-1)$ and $(F-1)$, $N_\star(r,0;g')$ denotes similarly according to g.

If $\alpha_1,\alpha_2,\ldots,\alpha_n$ be the n-distinct zeros of $P(z)=0$, then

$$c(F-1)=P(f)=\frac{(n-1)(n-2)}{2}\prod_{i=1}^{n}(f-\alpha_i)$$

and

$$c(G-1)=P(g)=\frac{(n-1)(n-2)}{2}\prod_{i=1}^{n}(g-\alpha_i).$$

Thus, applying the second fundamental theorem for $n+2$ distinct values $0,\ 1,\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$, we obtain

$$(n+1)\left(T(r,f)+T(r,g)\right)\le$$

$$\le\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,1;f)+\sum_{i=1}^{n}\overline{N}(r,\alpha_i;f)-N_\star(r,0,f')+$$

$$+\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,1;g)+\sum_{i=1}^{n}\overline{N}(r,\alpha_i;g)-N_\star(r,0,g')+$$

$$+S(r,f)+S(r,g)\le$$

$$\le\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+2(T(r,f)+T(r,g))+\overline{N}(r,1;F)+\overline{N}(r,1;G)-$$

$$-N_\star(r,0,f')-N_\star(r,0,g')+S(r,f)+S(r,g). \tag{5.6}$$

By using inequalities (5.5) and (5.6), we get

$$\left(\frac{n}{2}-3\right)\left(T(r,f)+T(r,g)\right)\le$$

$$\le 2(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+\{N(r,1;F|\ge 2)+N(r,1;G|\ge 2)\}+$$

$$+S(r,f)+S(r,g). \tag{5.7}$$

Also, using first fundamental theorem and elementary calculations, we have

$$N\left(r,0;\frac{f'}{f}\right)\le T\left(r,\frac{f'}{f}\right)+O(1)=N\left(r,\infty;\frac{f'}{f}\right)+S(r,f)\le$$

$$\le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\le T(r,f)+\overline{N}(r,\infty;f)+S(r,f).$$

Thus on simplifying (5.7), we obtain

$$\left(\frac{n}{2}-3\right)\left(T(r,f)+T(r,g)\right)\le$$

$$\le 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N(r,1;F|\ge 2)+N(r,1;G|\ge 2)+$$

$$+S(r,f)+S(r,g)\le$$

$$\leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N(r,0;f'|f\neq 0)+N(r,0;g'|g\neq 0)+$$

$$+S(r,f)+S(r,g)\leq$$

$$\leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N\left(r,0;\frac{f'}{f}\right)+N\left(r,0;\frac{g'}{g}\right)+$$

$$+S(r,f)+S(r,g)\leq$$

$$\leq 3\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+T(r,f)+T(r,g)+S(r,f)+S(r,g).$$

That is,

$$(n-8)(T(r,f)+T(r,g))\leq$$

$$\leq 6\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,f)+S(r,g)$$

which is impossible as $n\geq 15$ (resp., 9) for URSM-IM (resp., URSE-IM) case.

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