UDC 517.53

V. V. Andrievskii, doct. phys.-math. sci.
(Inst. Appl. Math. and Mech. Ukrainian Acad. Sci., Donetsk)

APPROXIMATION OF HARMONIC FUNCTIONS ON COMPACT SETS IN $\mathbb{C}$

НАБЛИЖЕННЯ ГАРМОНІЧНИХ ФУНКЦІЙ НА КОМПАКТАХ В $\mathbb{C}$

The direct theorem of the approximation theory of harmonic functions is given for the case where the functions are defined on a compact set, the complement of which with respect to $\mathbb{C}$ is a John domain.

Встановлюється пряма теорема теорії наближення гармонічних функцій на компакті, який є доповненням області Джона до площини.

1. Introduction. This paper deals with the qualitative theory of uniform approximation by harmonic polynomials.

To be more precise, let M be an arbitrary compact set in the complex plane $\mathbb{C}$ with the connected complement $\Omega := \overline{\mathbb{C}}\setminus M$, where $\overline{\mathbb{C}}$ is the extended complex plane.

Denote by Har (M) the set of all functions which are continuous on M and harmonic at its interior points. The best polynomial approximation is given by

$$E_{n,\Delta}(u, M) := \inf (\| u - t_n \|_M : \; t_n \in \mathbf{T}_n), \quad u \in \text{Har}\,(M),$$

where n is a positive integer, $\| \cdot \|_M$ denotes the supremum norm over M, and

$$\mathbf{P}_n := \left\{ p_n(z) = \sum_{j=0}^{n} a_j z^j : \; a_j \in \mathbb{C} \right\}, \quad \mathbf{T}_n := \{ t_n(z) = \operatorname{Re} p_n(z\;) : \; p_j \in \mathbf{P}_n \}.$$

In the monographs [1–3], one can find a survey of the papers investigating the rate of convergence $E_{n,\Delta}(u, M) \to 0$ as $n \to \infty$. In the overwhelming majority of them, M is a continuum (not a single point).

In this paper, the estimates of $E_{n,\Delta}(u, M)$ are established for the case of a compact set M whose complement Ω is a John domain [4, 5] (not necessarily simply connected). The form of the result is similar to the analogous assertions in [6].

Note that some results of this paper can be extended to $\mathbb{R}^n$, $n > 2$, instead of $\mathbb{C}$ (see [7]). However, the proofs need to be essentially modified in the case of $\mathbb{R}^n$, since we use here the conformal mappings.

2. Main definitions and results. We shall use $c, c_1, \ldots$ to denote constants and ε, $\varepsilon_1, \ldots$ to denote sufficiently small constants (in general, different in different relations) which depend only on quantities that are not important for a particular problem.

By L, l, and γ, we denote closed Jordan curves or arcs (i.e., curves with different endpoints) and by $|L|, |l|$, and $|\gamma|$ we denote their length.

For points z and $\zeta \in L$, denote by $L(z, \zeta)$ the subarc of L lying between them.

The open disk with the center at z and the radius δ is denoted by $D(z, \delta)$, $D := D(0, 1)$. Let $d(A, B)$ be the distance between $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$. Further, for $A \subset \mathbb{C}$ and $\delta > 0$, we set

$$A_\delta := \{ \zeta : \; d(\zeta, A) < \delta \}.$$

Let $\omega(\delta)$, $\delta > 0$, be a function of the type of a modulus of continuity, i.e., a positive nondecreasing function (with $\omega(+0) = 0$) which satisfies the inequality

$$\omega(t\delta) \le ct\,\omega(\delta) \quad \text{for} \quad \delta > 0, \quad t > 1.$$

We denote by $\mathrm{Har}_\omega(M)$ the class of all functions $u \in \mathrm{Har}\,(M)$ such that the inequality

$$|u(z_1) - u(z_2)| \le c\omega\,(|z_1 - z_2|) \tag{1}$$

holds for each pair of points $z_1, z_2 \in M$.

A domain Ω is called a John domain [4, 5] if any point $\zeta \in \Omega \setminus \{\infty\}$ can be joined to ∞ by an arc $\gamma = \gamma(\zeta, \infty) \subset \Omega$ such that

$$d(z, \partial\Omega) \le c\,|\gamma(\zeta, z)| \tag{2}$$

for each point $z \in \gamma$, where ∂A is the boundary with respect to $\mathbb{C}$.

It is more convenient to use another definition of the John domain (see Theorem 1 below) that can be given from the point of view of the theory of quasiconformal mappings.

A bounded Jordan domain G is called a k-quasidisk and its boundary $L := \partial G$ is called a k-quasicircle (or, briefly, a quasidisk and a quasicircle), $0 \le k < 1$, if any conformal mapping ψ of the unit disk D onto G can be extended to a K-quasiconformal homeomorphism of $\overline{\mathbb{C}}$ onto itself, $K := (1+k)/(1-k)$.

It is easy to verify (see, for example, [6]) that the domain $G = G(k, \delta)$, $0 \le k < 1$, $\delta > 0$, which is symmetric with respect to the real and imaginary axes and bounded by two circle arcs which meet in an inner angle of $\pi(1-k)$ at the vertices $\pm\delta$, is a k-quasidisk (see [8, 9]).

We say that the domain Ω satisfies a k-quasidisk condition, $0 \le k < 1$, if, for each point $\zeta \in \Omega$, there exists a k-quasidisk $D_\zeta \in \Omega$ such that $\zeta \in \partial D_\zeta$, $\operatorname{diam} D_\zeta \ge c$.

Theorem 1. *The domain Ω is a John domain if and only if it satisfies a k-quasidisk condition, $0 \le k < 1$.*

Theorem 2. *Let $M \subset \mathbb{C}$ be a compact set whose complement $\Omega = \overline{\mathbb{C}} \setminus M$ satisfies a k-quasidisk condition, $0 \le k < 1$. Then, for any $u \in \mathrm{Har}_\omega(M)$ and a positive integer n,*

$$E_{n,\Delta}(u, M) \le c\omega(n^{k-1}), \tag{3}$$

where c is independent of n.

In the case where M is a closed quasidisk, this result is presented implicitly in [10]. Hence, the thrust of our work is that we can now approximate harmonic functions on compact sets by a complement with infinite connectivity.

Note that if M is a closed quasidisk, the statement of Theorem 2 is a direct consequence of the holomorphic analog [11] obtained by passing to the holomorphic completion of a harmonic function. However, if Ω is not simply connected or M is not a quasidisk, we cannot find any way of deducing the harmonic results from the holomorphic analogs. It turns out that for each Hölder class

$$\mathrm{Har}^\alpha(M) := \mathrm{Har}_\omega(M), \quad \text{where} \quad \omega(\delta) = \delta^\alpha, \quad 0 < \alpha < 1,$$

and $0 \le k < 1$, it is not possible to improve estimate (3).

Theorem 3. *For any $0 \le k < 1$ and $0 < \alpha < 1$, there exist a closed Jordan domain $G = G(k)$, whose complement satisfies a k-quasidisk condition, and a function $u \in \mathrm{Har}^\alpha(\overline{G})$ such that*

$$E_{n,\Delta}(u, \overline{G}) \ge cn^{\alpha(k-1)}, \quad n = 1, 2, \ldots, \tag{4}$$

where c is independent of n.

In the sequel, we use the symbols $a \preccurlyeq b$, denoting that $a \leq cb$, and $a \asymp b$ if $a \preccurlyeq b$ and $b \preccurlyeq a$ simultaneously.

3. Quasidisks. Before proving our theorems, we discuss some facts of the theory of quasiconformal mappings necessary in what follows. They are consequences of the appropriate results in [9, 12, 13].

Lemma 1. *Let L be a bounded Jordan arc. The following two conditions are equivalent:*

1) *there exists a constant $c \geq 1$ such that* $\operatorname{diam} L(z_1, z_2) \leq c|z_1 - z_2|$ *for each pair of points $z_1, z_2 \in L$;*

2) *there exists a K-quasiconformal mapping $F: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $F(L) = [-1, 1]$, $F(\infty) = \infty$.*

Moreover, if K is given, c depends only on K, and if c is given, then $K = K(c)$.

Lemma 2. *Let $F: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a K-quasiconformal mapping, $F(\infty) = \infty$, $\zeta_j \in \mathbb{C}$, $w_j := F(\zeta_j)$, $j = 1, 2, 3$, $|w_1 - w_2| \leq c_1 |w_1 - w_3|$. Then $|\zeta_1 - \zeta_2| \leq c_2 |\zeta_1 - \zeta_3|$, and moreover,*

$$\left| \frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2} \right| \leq c_3 \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^K,$$

where $c_i = c_i(c_1, K)$, $i = 2, 3$.

Let G be a k-quasidisk, $L := \partial G$. Further, let $\psi: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiconformal mapping, conformal in D and such that $\psi(\partial D) = L$, $\psi(\infty) = \infty$. We set $\zeta_0 := \psi(0)$, $\varphi := \psi^{-1}$. By applying Lemma 2 to the mapping $F := \varphi$, we obtain

$$d(\zeta_0, L) \asymp \operatorname{diam} L. \tag{5}$$

Lemma 3. *For $\zeta \in L$, we set $\gamma = \gamma(\zeta, \zeta_0) := \psi([0, \varphi(\zeta)])$. Then*

$$|\gamma(\zeta, z)| \leq c_1 |\zeta - z| \leq c_2 d(z, L),$$

for each point $z \in \gamma$, where $c_i = c_i(K)$, $i = 1, 2$.

Lemma 4. *For all w_1 and $w_2 \in \overline{D}$,*

$$|\psi(w_1) - \psi(w_2)| \leq c \operatorname{diam} L\, |w_1 - w_2|^{1-k},$$

where $c = c(K)$.

The last assertion is a simple consequence of an appropriate result for the mapping $f \in \Sigma(k)$ [14, p. 347].

4. Proof of Theorem 1. To begin with, suppose that Ω is a John domain, and let $\zeta \in \Omega$ be an arbitrary point. For simplicity, we carry out the proof only for $\zeta \in M_\varepsilon$; the case where $\zeta \in \Omega \setminus M_\varepsilon$ is obvious.

Let $\zeta_0 \in \Omega$ be a fixed point and let $\gamma = \gamma(\zeta, \zeta_0)$ be a joining arc satisfying (2). Without loss of generality, we can assume that, in some neighborhood of the point ζ, the arc γ coincides with a straight line.

We replace the arc γ with the polygon $l = l(\zeta, \zeta_0)$, by joining the points ζ and ζ_0 so that

$$d(z, M) \geq c_1 |z - \zeta|, \quad z \in l, \tag{6}$$

$$|l(z_1, z_2)| \preccurlyeq |z_1 - z_2|, \quad z_1, z_2 \in l. \tag{7}$$

Let us describe an algorithm of constructing of this arc. Fix a number $0 < q < 1$ such that, for any point $z \in \gamma$, we have $D(z, 2q\,|\gamma(\zeta, z)|) \subset \Omega$.

First, we construct a sequence of points $\zeta_1, \zeta_2, \ldots \in \gamma$ in the following way. For ζ_1, we take the point of the intersection $\gamma \cap \partial D(\zeta_0, q\,|\gamma(\zeta, \zeta_0)|)$ which is the first one as we move along the arc γ from ζ to ζ_0, and so forth; for ζ_{k+1}, we take the point of the intersection $\gamma(\zeta, \zeta_k) \cap \partial D(\zeta_k, q\,|\gamma(\zeta, \zeta_k)|)$ which is the first one as we move along the arc $\gamma(\zeta, \zeta_k)$ from ζ to ζ_k (if this intersection is empty, then we set $\zeta_{k+1} := \zeta$ and terminate our construction).

If we successively join the points $\zeta_0, \zeta_1, \ldots$ by the intervals, we obtain the polygon l with properties (6) and (7).

Note that (7) and Lemma 1 imply the existence of a K-quasiconformal mapping $F: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $1 \le K \preccurlyeq 1$, which satisfies the relations

$$F(l) = [-1, 1], \quad F(\zeta) = 1, \quad F(\zeta_0) = -1, \quad F(\infty) = \infty.$$

Consider a convex lens domain $E = E(\varepsilon)$ which is symmetric with respect to the real and imaginary axes and bounded by two arcs which meet in an interior angle of $2\pi\varepsilon$ at the vertices ± 1.

Set $V := F^{-1}(E)$. We claim that, for sufficiently small ε, $V \subset \Omega$, and consequently, V can be taken as a domain D_ζ in the definition of a quasidisk condition.

Indeed, let $z \in \partial V$, $z \neq \zeta$, $z \neq \zeta_0$, be an arbitrary point. We set $w := F(z)$, $w_1 := \mathrm{Re}\, w$, $z_1 := F^{-1}(w_1)$. Consider triplets of points z_1, z, ζ and $w_1, w, 1$, respectively. If $\varepsilon < 1/4$, then $|w_1 - w| < |w_1 - 1|$ and, by using Lemma 2, we can conclude that

$$\left| \frac{z_1 - \zeta}{z_1 - z} \right| \ge c_2 \left| \frac{w_1 - 1}{w_1 - w} \right|^{1/K} \ge c_2 (\mathrm{ctg}\, \varepsilon)^{1/K}. \tag{8}$$

Therefore, if $\varepsilon < \arctan (c_2 / c_1)^K$, where c_1 and c_2 are the constants from (6) and (8), then $z \in \Omega$.

Thus, since the point z is arbitrary, we have $V \subset \Omega$.

Suppose now that Ω satisfies the k-quasidisk condition for some $0 \le k < 1$ and let $\zeta \in \Omega$ be an arbitrary point. By applying Lemma 3 to the quasidisk D_ζ, we can conclude that there exists an arc $\gamma(\zeta, \zeta_1) \subset D_\zeta \subset \Omega$ joining ζ with some point $\zeta_1 \in \mathbb{C} \setminus M_\varepsilon$ and possessing property (2). An elementary argument involving the technique of construction of the polygon l shows that $\gamma(\zeta, \zeta_1)$ can be extended to an arc $\gamma(\zeta, \infty)$ satisfying relation (2). This completes the proof.

5. Approximation of the Cauchy kernel. Let Ω satisfy the k-quasidisk condition, $0 \le k < 1$. The problem of approximating the Cauchy kernel $1/(\zeta - z)$ by a polynomial kernel of the form

$$P_n(\zeta, z) := \sum_{j=0}^{n} a_j(\zeta)\, z^j \tag{9}$$

arises in connection with several problems in approximation theory. In this and next sections, we develop a harmonic analog of the stated problem.

We begin with a description of one construction that will be useful below.

Suppose that $\zeta \in \Omega \cap M_\varepsilon$ is an arbitrary point, and let D_ζ be an appropriate k-quasidisk, i.e., $\zeta \in \partial D_\zeta$, $D_\zeta \subset \Omega$, and $\mathrm{diam}\, D_\zeta \succcurlyeq 1$. For convenience, we always assume that diam $D_\zeta \asymp 1$. Our nearest purpose is to construct an unbounded domain $B_\zeta \supset D_\zeta$ such that

$$M \subset \mathbb{C}\setminus B_\zeta \quad \text{and} \quad D_\zeta \cap D(\zeta, \varepsilon) = B_\zeta \cap D(\zeta, \varepsilon). \tag{10}$$

For this purpose, we denote by $\varphi = \varphi_\zeta := \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ an appropriate quasiconformal mapping, conformal in D_ζ and satisfying $\varphi(D_\zeta) = D$, $\varphi(\zeta) = 1$, $\varphi(\infty) = \infty$. We set $\psi := \varphi^{-1}$, $\zeta_0 := \psi(0)$. According to (5), we have $d(\zeta_0, M) \asymp 1$. Let $M \subset D(0, c_1)$. By using the procedure from the proof of Theorem 1, we join the point ζ_0 to $\partial D(0, 2c_1)$ by the polygon $l \subset \mathbb{C}\setminus M_{\varepsilon_1}$ satisfying

$$|l(\zeta_1, \zeta_2)| \preccurlyeq |\zeta_1 - \zeta_2|,$$

for any points ζ_1 and $\zeta_2 \in l$. It seems interesting to note that l consists of a finite number $m \preccurlyeq 1$ of segments, each of the length $\succcurlyeq 1$. Therefore, $l_{\varepsilon_2} := \{z\colon d(z, l) < \varepsilon_2\}$ is a simply connected domain.

From parts of the curves $\partial l_{\varepsilon_2}, \partial D_\zeta$, and $\partial D(0, 2c_1)$, we construct the Jordan curve $L = L(\zeta)$ such that the unbounded component B_ζ of $\overline{\mathbb{C}}\setminus L$ satisfies (10).

Let $\Phi(z)$ denote the function that maps B_ζ onto the exterior Δ of the unit disk conformally and univalently and is normalized by the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. This function can be naturally extended to a homeomorphism between the closed domains $\overline{B}_\zeta$ and $\overline{\Delta}$, and we retain the previous notation for the extension. We set $\Psi := \Phi^{-1}$.

Lemma 5. *For any point* w *such that* $1 < |w| \preccurlyeq 1$, *we have*

$$|\Psi(w) - \zeta| \preccurlyeq |w - \Phi(\zeta)|^{1-k}.$$

Proof. We set $z := \Psi(w)$, $\tau := \Phi(\zeta)$. We carry out the proof only for the case $|\tau - w| < \varepsilon_1$, and consequently, $|z - \zeta| < \varepsilon_2$; the case, where $|\tau - w| \le \varepsilon_1$, is trivial.

Let Γ_1 be a family of all arcs $\gamma \subset B_\zeta$ with the endpoints on ∂B_ζ separating, in B_ζ, the points ζ and z from ∞. Further, let Γ_2 be a similar family for the domain D_ζ and the points ζ, z, and ζ_0, respectively.

Since, for $\gamma \in \Gamma_1\setminus\Gamma_2$, we have $|\gamma| \ge \varepsilon_3$, by recalling the definition and elementary properties of a module of a family of arcs or curves [8], we obtain $m(\Gamma_1) \le m(\Gamma_2) + c_1$. By a slightly modified version of a result by Belyi (see [11], Theorem 1), we find that

$$|w - \tau| \asymp \exp\{-\pi m(\Gamma_1)\} \quad \text{and} \quad |\varphi(z) - \varphi(\zeta)| \asymp \exp\{-\pi m(\Gamma_2)\}.$$

Hence, $|\varphi(z) - \varphi(\zeta)| \preccurlyeq |w - \tau|$, and, by virtue of Lemma 4,

$$|z - \zeta| = |\psi[\varphi(z)] - \psi[\varphi(\zeta)]| \preccurlyeq |\varphi(z) - \varphi(\zeta)|^{1-k} \preccurlyeq |w - \tau|^{1-k}.$$

This proves Lemma 5.

Further, we apply the technique of [2] to prove the main assertion of this section.

Lemma 6. *For each positive integer* n, $\delta := n^{k-1}$, and $\zeta \in M_\varepsilon \cap \Omega$, *there exists a polynomial kernel of the form* (9) *satisfying*

$$\left| \frac{1}{\zeta - z} - P_n(\zeta, z) \right| \preccurlyeq \frac{1}{|\zeta - z|}\left(\frac{\delta}{|\zeta - z| + \delta} \right)^3 \tag{11}$$

for all $z \in M$.

Proof. Let n be sufficiently large. Starting from the domain B_ζ, we consider the Dzyadyk polynomial kernel $K_{1,1,3,n}(\zeta, z)$ (see, e.g., [2, p. 429]). In [11], it is shown that

$$\left| \frac{1}{\zeta - z} - K_{1,1,3,n}(\zeta, z) \right| \preccurlyeq \frac{|\tilde{\zeta} - \zeta|^3}{|\zeta - z||\tilde{\zeta} - z|^3}, \tag{12}$$

where we have set $\tilde{\zeta} := \Psi[(1 + 1/n)\Phi(\zeta)]$.

It follows from Lemma 5 that $|\zeta - \tilde{\zeta}| \preccurlyeq \delta$. Therefore, by Lemma 2, we have

$$\left| \frac{\tilde{\zeta} - z}{\tilde{\zeta} - \zeta} \right| \asymp 1 + \left| \frac{\zeta - z}{\tilde{\zeta} - \zeta} \right| \succcurlyeq \frac{\delta + |\zeta - z|}{\delta}. \tag{13}$$

Thus, the desired inequality (11) follows from (12) and (13) if we set $P_n(\zeta, z) :=$ $:= K_{1,1,3,[\varepsilon n]}(\zeta, z)$, where a sufficiently small constant ε is chosen so that $\deg P_n \le n$.

By virtue of standard argument (see, e.g., [2, p. 345]), we can assume that the coefficients $a_j(\zeta)$ in representation (9) are integrable with respect to ζ over $\Omega \cap M_\varepsilon$.

6. Proof of Theorem 2. The idea of the following discussion goes back to [15]. We give a sketch of the proof to show how Mergelyan's argument can be modified to obtain the required result.

Let n be sufficiently large and $\delta := n^{k-1}$. We extend the function u continuously to the whole of the complex plane $\mathbb{C}$ so that the extended function, also denoted by u, satisfies condition (1) for any $z_1, z_2 \in \mathbb{C}$ (see, for example, [16]). Without loss of generality, we can assume that $u(z) = 0$ for $z \in \mathbb{C} \setminus M_\varepsilon$.

For $z \in \mathbb{C}$, we set

$$u_n(z) := \int\int_{\mathbb{C}} u(z + \delta\zeta)\, K(\zeta)\, d\sigma_\zeta,$$

where

$$K(\zeta) := \begin{cases} c \exp\{|\zeta|^2 / (|\zeta|^2 - 1)\}, & 0 \le |\zeta| < 1; \\ 0 & , \ |\zeta| \ge 1; \end{cases}$$

and the constant c is chosen so that $\int\int_{\mathbb{C}} K(\zeta) d\sigma_\zeta = 1$.

An elementary computation shows that $u_n(z)$ is an infinitely differentiable function $\mathbb{C}$ satisfying

$$u_n(z) = 0 \quad \text{if} \quad z \in \mathbb{C} \setminus M_{\varepsilon+\delta}, \quad u_n(z) = u(z) \quad \text{if} \quad z \in M \setminus \Omega_\delta,$$

$$|u(z) - u_n(z)| \preccurlyeq \omega(\delta) \quad \text{if} \quad z \in \mathbb{C}, \quad \text{and} \quad |\Delta u_n(z)| \preccurlyeq \omega(\delta)/\delta^2, \quad \text{if} \quad z \in \mathbb{C}.$$

It remains to show that

$$E_{n,\Delta}(u_n, M) \preccurlyeq \omega(\delta). \tag{14}$$

By the Green formula, we can write

$$u_n(z) = \frac{1}{2\pi} \iint_{\Omega_\delta \cap M_\varepsilon} \Delta u_n(\zeta) \log|\zeta - z|\, d\sigma_\zeta.$$

We now describe the method for approximating the function $\log|\zeta - z|$ by harmonic polynomial kernels of the form

$$T_n(\zeta, z) := \operatorname{Re} \sum_{j=1}^{n} a_j(\zeta) z^j. \tag{15}$$

Lemma 7. *Let $\zeta_0 \in \Omega \setminus \{\infty\}$ be a fixed point. There exists a harmonic polynomial kernel of the form* (15) *satisfying, for $z \in M$ and $\zeta \in \Omega_\delta \cap M_\varepsilon$, the inequality*

$$\left| \log \left| \frac{\zeta - z}{\zeta_0 - z} \right| - T_n(\zeta, z) \right| \preccurlyeq \left(\frac{\delta}{|\zeta - z| + \delta} \right)^3 \left(1 + \log \frac{|\zeta - z| + \delta}{|\zeta - z|} \right). \tag{16}$$

Proof. Suppose first that $\zeta \in \Omega \cap M_\varepsilon$. Since Ω is the John domain, we can join ζ to ζ_0 by an arc $\gamma(\zeta, \zeta_0) \subset \Omega$ with property (2).

Next, we note that

$$\log \left| \frac{\zeta_0 - z}{\zeta - z} \right| = \operatorname{Re} \int_{\gamma(\zeta, \zeta_0)} \frac{d\xi}{\xi - z}$$

for $z \in M$. Thus, it is reasonable to consider the expression

$$T_n(\zeta, z) := -\operatorname{Re} \int_{\gamma(\zeta, \zeta_0)} P_n(\xi, z)\, d\xi,$$

where $P_n(\xi, z)$ is a polynomial function from Lemma 6, as the desired harmonic polynomial kernel.

Indeed, by setting $\mu(t) := |\gamma(\zeta, \zeta_0) \cap D(\zeta, t)|$, $t > 0$, and using Lemma 6 and the obvious relations $\mu(t) \preccurlyeq t$ and $d(z, \gamma(\zeta, \zeta_0)) \asymp |z - \zeta|$, we obtain

$$\left| \log \left| \frac{\zeta - z}{\zeta_0 - z} \right| - T_n(\zeta, z) \right| \le \int_{\gamma(\zeta, \zeta_0)} \left| \frac{1}{\xi - z} - P_n(\xi, z) \right| |d\xi| \preccurlyeq$$

$$\preccurlyeq \delta^3 \left[\int_0^{|\zeta - z|} \frac{d\mu(t)}{|\xi - z| (|\zeta - z| + \delta)^3} + \int_{|\zeta - z|}^{|\zeta - z| + \delta} \frac{d\mu(t)}{t (|\zeta - z| + \delta)^3} + \int_{|\zeta - z| + \delta}^{\infty} \frac{d\mu(t)}{t^4} \right] \preccurlyeq$$

$$\preccurlyeq \left(\frac{\delta}{|\zeta - z| + \delta} \right)^3 \left(1 + \int_{|\zeta - z|}^{|\zeta - z| + \delta} \frac{\mu(t)}{t^2} dt \right) + \delta^3 \int_{|\zeta - z| + \delta}^{\infty} \frac{\mu(t)}{t^5} dt \preccurlyeq$$

$$\preccurlyeq \left(\frac{\delta}{|\zeta - z| + \delta} \right)^3 \left(1 + \log \frac{|\zeta - z| + \delta}{|\zeta - z|} \right).$$

Now let $\zeta \in M \cap \Omega_\delta$. Consider any point $\zeta_1 \in \Omega$ with the properties $|\zeta_1 - \zeta| \le 2\delta$ and $d(\zeta_1, M) \succcurlyeq \delta$. Denote by $\gamma(\zeta_1, \zeta)$ the arc of the circle

$$\partial D \left(\frac{\zeta_1 + \zeta}{2}, \frac{|\zeta_1 - \zeta|}{2} \right),$$

which joins ζ and ζ_1 and is farthest from the point z.

By integrating the identity

$$(\xi - z)^{-1} = \sum_{j=1}^{2} \frac{(\zeta - \xi)^j}{(\zeta_1 - z)^{j+1}} + \left(\frac{\zeta_1 - \xi}{\zeta_1 - z} \right)^3 \frac{1}{\xi - z},$$

we get

$$\log\left|\frac{\zeta-z}{\zeta_1-z}\right| = \mathrm{Re}\int_{\gamma(\zeta_1,\zeta)}\frac{d\xi}{\xi-z} =$$

$$= -\sum_{j=0}^{2}\frac{1}{j+1}\mathrm{Re}\left(\frac{\zeta_1-\zeta}{\zeta_1-z}\right)^{j+1} + \mathrm{Re}\int_{\gamma(\zeta_1,\zeta)}\left(\frac{\zeta_1-\xi}{\zeta_1-z}\right)^3\frac{d\xi}{\xi-z}.$$

This relation suggests the following definition of the harmonic polynomial kernel:

$$T_n(\zeta, z) := T_n(\zeta_1, z) - \sum_{j=1}^{3}\frac{1}{j}\mathrm{Re}(\zeta_1-\zeta)^j\,[P_{[n/j]}(\zeta_1, z)]^j,$$

where $P_k(\zeta_1, z)$ are polynomial kernels from Lemma 6.

The proof of inequality (16) in this case is identical to that in the previous case. This completes the proof of Lemma 7.

Consider the function

$$g_n(z) := \frac{1}{2\pi}\iint_{\Omega_\delta\cap M_\varepsilon}\Delta u_n(\zeta)\log\left|\frac{\zeta-z}{\zeta_0-z}\right|d\sigma_\zeta, \quad z\ne\zeta_0.$$

Since $u_n(z)-g_n(z)$ is harmonic in some fixed neighborhood of M, we need, for the check-up of estimate (14), to prove a similar inequality for the function g_n.

In order to do this, we set $A := D(z,\delta/2)\cap\Omega_\delta\cap M_\varepsilon$, $B := (\Omega\cap M_\varepsilon)\setminus A$, and

$$t_n(z) := \frac{1}{2\pi}\iint_{\Omega_\delta\cap M_\varepsilon}\Delta u_n(\zeta)\,T_n(\zeta, z)\,d\sigma_\zeta, \quad z\in M,$$

where $T_n(\zeta, z)$ satisfies (16). Finally, the condition

$$\int_0^\delta x\log\frac{\delta}{x}\,dx \le \delta^2/e$$

implies, for $z\in M$, that

$$|g_n(z)-t_n(z)| \preccurlyeq \frac{\omega(\delta)}{\delta^2}\left[\iint_A\log\frac{d}{|\zeta-z|}\,d\sigma_\zeta + \delta^3\iint_B\frac{d\sigma_\zeta}{|\zeta-z|^3}\right] \le$$

$$\le 2\pi\frac{\omega(\delta)}{\delta^2}\left[\int_0^{\delta/2} r\log\frac{\delta}{r}\,dr + \delta^3\int_{\delta/2}^{\infty}\frac{dr}{r^2}\right] \preccurlyeq \omega(\delta),$$

which yields $E_{n,\Delta}(g_n, M)\preccurlyeq\omega(\delta)$. Hence, our proof is completed.

7. Proof of Theorem 3. We begin with the examination of the domain

$$G = G(k) := \{re^{i\theta}:\ 0<r<1;\ |\theta-\pi| < \pi(1+k)/2\}$$

and the functions $f(z) = f(z,\alpha) := z^\alpha$ and $u(z) := \mathrm{Re}\,f(z)$ given in G.

We only need to check the following inequality:

$$E_{n,\Delta}(u,\overline{G}) \succcurlyeq \delta^\alpha \tag{17}$$

for sufficiently large n and $\delta := n^{k-1}$.

To do this, we assume that $E_n := E_{n,\Delta}(u,\overline{G}) \le \delta^\alpha$. Let $t_n\in T_n$ be such that

$$\|u-t_n\|_{\overline{G}} = E_n,$$

and let $p_n \in P_n$ be a holomorphic completion of t_n, i.e., $t_n(z) = \operatorname{Re} p_n(z)$.

Consider the domain

$$G^n := \{z = re^{i\theta} - \delta:\ 0 < r < 1/2,\ |\theta - \pi| < \pi(1+k)/2\}.$$

By defining $d := d(\partial G, \partial G^n) \asymp \delta$, we get, by the Schwarz formula for $z \in \partial G^n$,

$$|p_n'(z)| \le |f'(z)| + |p_n'(z) - f'(z)| = \alpha |z|^{\alpha-1} + \frac{1}{\pi}\left| \int_{\partial D(z,d)} \frac{t_n(\zeta) - u_n(\zeta)}{(\zeta - z)^2} d\zeta \right| \le$$

$$\le \alpha |z|^{\alpha-1} + \frac{2E_n}{d} \preccurlyeq \delta^{\alpha-1}.$$

By applying the Bernstein–Walsh theorem, we find that the relation

$$|\operatorname{grad} t_n(z)| = |p_n'(z)| \le c_1 \delta^{\alpha-1}$$

holds if $-\delta \le z \le 0$.

We set $z_\varepsilon := -\varepsilon\delta$, where $\varepsilon = (\alpha/(2c_1))^{1/(1-\alpha)}$. Hence, we get

$$|\operatorname{grad}(u(z_\varepsilon) - t_n(z_\varepsilon))| \ge |\operatorname{grad} u(z_\varepsilon)| - |\operatorname{grad} t_n(z_\varepsilon)| \ge$$

$$\ge \alpha |z_\varepsilon|^{\alpha-1} - c_1 \delta^{\alpha-1} \ge \frac{\alpha}{2} |z_\varepsilon|^{\alpha-1}.$$

A glance at the already estimated functions indicates that we can also take

$$|\operatorname{grad}(u(z_\varepsilon) - t_n(z_\varepsilon))| = |f'(z_\varepsilon) - p_n'(z_\varepsilon)| =$$

$$= \frac{1}{\pi}\left| \int_{\partial D(z_\varepsilon, |z_\varepsilon|)} \frac{u(\zeta) - t_n(\zeta)}{(\zeta - z_\varepsilon)^2} d\zeta \right| \le 2E_n / |z_\varepsilon|,$$

which, in view of the previous inequality, yields (17) and, consequently, (4).

1. *Walsh J. L.* Interpolation and approximation by rational functions in the complex domain. – Providence, R. I.: American Math. Soc., 1965. – 508 p.
2. *Dzyadyk V. K.* Introduction to the theory of uniform approximation of functions by polynomials. – Moscow: Nauka, 1977. – 512 p. (in Russian).
3. *Gaier D.* Vorlesungen über Approximation im Komplexen. - Basel: Birkhäuser, 1980. – 132 p.
4. *Martio O., Sarvas J.* Injectivity theorems in plane and space // Ann. Acad. Sci. Fenn. Ser. AI Math. – 1978 – 1979. – **4**. – P. 383 – 401.
5. *Pommerenke Ch.* Boundary behavior of conformal maps. – Berlin etc.: Springer, 1992. – 189 p.
6. *Anderson J. M., Gehring F. W., Hinkkanen A.* Polynomial approximation on quasidisks // Differential geometry and complex analysis, Vol. dedic. H. E. Rauch. – 1985. – P. 75 – 86.
7. *Andrievskii V. V.* Uniform harmonic approximation on compact sets in R^k, $k \ge 3$ // SIAM J. Math. Anal. – 1993. – **24**, № 1. – P. 1 – 7.
8. *Ahlfors L. V.* Lectures on quasiconformal mappings. – Princeton: Van Nostrand Mathematical Studies, 10, 1966. – 134 p.
9. *Gehring F. W.* Characteristic properties of quasidisks. – Montreal: Les Presses de l'Universite de Montreal, 1982. – 102 p.
10. *Dveirin M. Z.* The Hardy–Littlewood theorem and approximation of harmonic functions in domains with a quasiconformal boundary // Dokl. Akad. Nauk Ukr. SSR. Ser. A. – 1982. – № 1. – P. 11 – 14 (in Russian).
11. *Belyi V. I.* Conformal mappings and the approximation of analytic functions in domains with quasiconformal boundary // Mat. Sb. – 1977. – **102 (144)**. – P. 331 – 361 (in Russian).
12. *Rickman S.* Characterization of quasiconformal arcs // Ann. Acad. Sci. Fenn. Ser. AI Math. – 1966. – P. 1 – 30.
13. *Andrievskii V. V.* Some properties of continua with a piecewise quasiconformal boundary // Ukr. Mat. Zh. – 1980. – **32**, № 4. – P. 435 – 440 (in Russian).
14. *Pommerenke Ch.* Univalent Functions. – Göttingen: Vandenhoek and Ruprecht, 1975. – 376 p.
15. *Mergelyan S. N.* Uniform approximation of functions of a complex variable // Uspekhi Mat. Nauk. – 1952. – **7**, № 2 (48). – P. 31 – 122 (in Russian).
16. *Stein E. M.* Singular integrals and differentiability properties of functions. – Princeton, N.J.: Princeton Univ. Press, 1970. – 376 p.

Received 22.10.93