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SHEN'S L-PROCESS ON BERWALD CONNECTION

L-ПРОЦЕС ШЕНА НА ЗВ'ЯЗНОСТІ БЕРВАЛЬДА

The Shen connection cannot be obtained by using Matsumoto's processes from the other well-known connections. Hence Tayebi – Najafi introduced two new processes called Shen's C and L-processes and showed that the Shen connection is obtained from the Chern connection by Shen's C-process. In this paper, we study the Shen's C- and L-process on Berwald connection and introduce two new torsion-free connections in Finsler geometry. Then, we obtain all of Riemannian and non-Riemannian curvatures of these connections. Using it, we find the explicit form of hv-curvatures of these connections and prove that hv-curvatures of these connections are vanishing if and only if the Finsler structures reduce to Berwaldian or Riemannian structures. As an application, we consider compact Finsler manifolds and obtain ODEs.

Зв'язність Шена неможливо отримати за допомогою процесу Мацумото з інших відомих процесів. Тому Тайєбі та Наджафі запропонували два нових процеси, названі C- та L-процесами Шена, і показали, що за допомогою C-процесу Шена із зв'язності Черна можна отримати зв'язність Шена. Ми вивчаємо C- та L-процеси Шена на зв'язності Бервальда і пропонуємо дві нові безторсіонні зв'язності у геометрії Фінслера. Далі отримуємо всі ріманові та неріманові кривини для цих зв'язностей. За допомогою цього знаходимо точну форму hv-кривини для цих зв'язностей і доводимо, що hv-кривини для цих зв'язностей є нульовими тоді й тільки тоді, коли структури Фінслера зводяться до структур Бервальда чи Рімана. Як застосування розглядаємо компактні фінслерові многовиди та отримуємо звичайні диференціальні рівняння.

1. Introduction. In [8], Matsumoto introduced a satisfactory and truly aesthetical axiomatic description of Cartan's connection in the sixties. After the Cartan connection has been constructed, easy processes, baptized by Matsumoto "L-process" and "C-process" (or briefly "ML-process" and "MC-process"), yield the Chern, the Hashiguchi and the Berwald connections. For other Finslerian connections, see [3 – 8] and [14]. The space of all connections makes an affine space modeled on the space of $(1,2)$-tensors over pulled-back bundle π^*TM. It means that adding a $(1,2)$-tensor to a connection makes a new connection. A Finsler metric F gives us two natural $(1,2)$-tensors with components $C^i{}_{jk}$ and $L^i{}_{jk}$. These two $(1,2)$-tensors play key role in Matsumoto's processes. The C-processes use Cartan tensor, and the L-processes use Landsberg tensor:

$$\begin{tikzcd}
\text{Cartan connection} \arrow[r, "MC\text{-process}"] \arrow[d, "ML\text{-process}"'] & \text{Chern connection} \arrow[d, "ML\text{-process}"] \\
\text{Hashiguchi connection} \arrow[r, "MC\text{-process}"] & \text{Berwald connection.}
\end{tikzcd}$$

It is well-known that vanishing hv-curvatures of Cartan and Berwald connections characterizes Landsberg metrics and Berwald metrics, respectively.

In [11], Shen introduced a new connection in Finsler geometry, which vanishing hv-curvature of this connection characterizes Riemannian metrics. In [9], Muzsnay and Nagy gave an invariant treatment of Shen connection. The Shen connection can not be constructed by Matsumoto's processes from these known connections. Therefore, Tayebi and Najafi introduced two new processes on connections, called Shen's C and L-processes [17]. For the sake of simplicity, we use "SC-process" and "SL-process" instead of Shen's C-process and Shen's L-process, respectively. Let (M, F) be a Finsler manifold. Suppose that ∇ is a connection with connection forms ω^i_j. Define

$$\tilde{\omega}^i_j := \omega^i_j - C^i{}_{jk}\omega^k.$$

Then $\tilde{\omega}^i_j$ are connection forms of a connection $\widetilde{\nabla}$, that is called the connection obtained from ∇ by Shen's C-process. Similarly, one can define

$$\tilde{\omega}^i_j := \omega^i_j - L^i{}_{jk}\omega^{n+k}.$$

Then $\tilde{\omega}^i_j$ are connection forms of a connection $\widetilde{\nabla}$, that is called the connection obtained from ∇ by Shen's L-process. Tayebi and Najafi showed that the Shen connection is obtained from the Chern connection by Shen's C-process.

In this paper, we are going to study the connections which obtain by Shen's C- and L-process on Berwald connection. In Section 3, we study the connection obtained by Shen's L-process on the Berwald connection, call it by D, and prove the existence and uniqueness of this connection. In Section 4, we show that the hv-curvature of D vanishes if and only if F is a Berwald metric. Let $P^j_{i\ kl} = P^j_{i\ kl}(x, y)$ and $P^j_{n\ kl} = P^j_{i\ kl}y^i$ denote the hv-curvature and contracted hv-curvature of D, respectively. In Section 5, we prove that on a compact Finsler manifold the contracted hv-curvature of D is vanishing if and only if F is a Landsberg metric. In Section 6, we study the connection obtained by Shen's C-process on the Berwald connection, call it by ∇, and prove the existence and uniqueness of this connection. Finally, in Section 7, we show that the hv-curvature of ∇ vanishes if and only if F reduce to a Riemannian metric.

2. Preliminaries. Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM := \bigcup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi\colon TM \to M$ is given by $\pi(x, y) := x$.

The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π^*_vTM at $v \in TM_0$ is T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \left\{(x, y, v)\Big|\ y \in T_xM_0,\ v \in T_xM\right\}.$$

Some authors prefer to define connections in the pull-back tangent bundle π^*TM. From geometrical point of view, the construction of these connections on π^*TM seems to be simple because here the fibers are n-dimensional $\big($i.e., $\pi^*(TM)_u = T_{\pi(u)}M \quad \forall u \in TM\big)$ thus torsions and curvatures are obtained quickly from the structure equations. When the construction is done on $T(TM)$ many geometrical objects appear twice and one needs to split $T(TM)$ in the vertical and horizontal parts where the latter is called horizontal distribution or nonlinear connection. Nevertheless we do not need to split π^*TM. Indeed the connection on $\pi^*(TM)$ is the most natural connection for physicists. In order to define curvatures, it is more convenient to consider the pull-back tangent bundle than the tangent bundle, because our geometric quantities depend on directions.

For the sake of simplicity, we denote by

$$\left\{\partial_i|_v := \left(v, \frac{\partial}{\partial x^i}\bigg|_x\right)\right\}_{i=1}^n$$

the natural basis for $\pi_v^* TM$. In Finsler geometry, we study connections and curvatures in $(\pi^* TM, \mathbf{g})$, rather than in (TM, F). The pull-back tangent bundle $\pi^* TM$ is very special tangent bundle.

A (globally defined) Finsler structure on a manifold M is a function $F\colon TM \to [0, \infty)$, with the following properties:

(i) F is a differentiable function on the manifold TM_0 and is continuous on the null section of the projection $\pi : TM \to M$;

(ii) $F : TM \to [0, \infty)$ is a positive scalar function;

(iii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iv) the Hessian of F^2 with elements

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positively defined on TM_0. Given a manifold M and a Finsler structure F on M, the pair (M, F) is called a Finsler manifold. F is called Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

The Finsler structure F defines a fundamental tensor $\mathbf{g}\colon \pi^* TM \otimes \pi^* TM \to [0, \infty)$ by the formula $\mathbf{g}(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $v = y^i \dfrac{\partial}{\partial x^i}\Big|_x$. Let

$$g_{ij}(x, y) := F F_{y^i y^j} + F_{y^i} F_{y^j},$$

where $F_{y^i} = \dfrac{\partial F}{\partial y^i}$. Then $(\pi^* TM, \mathbf{g})$ becomes a Riemannian vector bundle over TM_0.

Put

$$A_{ijk}(x, y) = \frac{1}{2} F(x, y) \frac{\partial g_{ij}}{\partial y^k}(x, y).$$

Clearly, A_{ijk} is symmetric with respect to i, j, k. The Cartan tensor

$$A\colon \pi^* TM \otimes \pi^* TM \otimes \pi^* TM \to R$$

is defined by

$$A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y),$$

where $v = y^i \dfrac{\partial}{\partial x^i}\Big|_x$ (see [15, 19]). In some literature $C_{ijk} = \dfrac{A_{ijk}}{F}$ is called Cartan tensor. Riemannian manifolds are characterized by $A \equiv 0$. The homogeneity condition (iii) holds in particular for positive λ. Therefore, by Euler's theorem we see that

$$y^i \frac{\partial g_{ij}}{\partial y^k}(x, y) = y^j \frac{\partial g_{ij}}{\partial y^k}(x, y) = y^k \frac{\partial g_{ij}}{\partial y^k}(x, y) = 0.$$

We recall that the canonical section ℓ is defined by

$$\ell = \ell(x, y) = \frac{y^i}{F(x, y)} \frac{\partial}{\partial x^i} = \frac{y^i}{F} \frac{\partial}{\partial x^i} := \ell^i \frac{\partial}{\partial x^i}.$$

Put $\ell_i := g_{ij} \ell^j = F_{y^i}$. Thus the canonical section ℓ satisfies

$$g(\ell, \ell) = g_{ij} \frac{y^i}{F} \frac{y^j}{F} = 1$$

and

$$\ell^i A_{ijk} = \ell^j A_{ijk} = \ell^k A_{ijk} = 0.$$

Thus, $A(X, Y, \ell) = 0$.

Given an n-dimensional Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are called spray coefficients and given by the following:

$$G^i = \frac{1}{4} g^{il} \left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\right].$$

$\mathbf{G}$ is called the spray associated to F.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}\Big|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = \frac{\partial^2 N^i_j}{\partial y^k \partial y^l}.$$

$\mathbf{B}_y(u, v, w)$ is symmetric in u, v and w. From the homogeneity of spray coefficients, we have $\mathbf{B}_y(y, v, w) = 0$. $\mathbf{B}$ is called the Berwald curvature. Indeed, L. Berwald first discovered that the third order derivatives of spray coefficients give rise to an invariant for Finsler metrics. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$ [16]. In this case, G^i are quadratic in $y \in T_xM$ for all $x \in M$, i.e., there exists $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ such that

$$G^i = \Gamma^i{}_{jk} y^j y^k.$$

There is another equal definition for a Berwald metric as follows. A Finsler metric F is called a Berwald metric if the Cartan torsion of F satisfies the following:

$$A_{ijk|l} = 0,$$

where the "|" and "," denote the horizontal and vertical covariant derivatives with respect to the Berwald connection.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y\left(\mathbf{B}_y(u, v, w), y\right).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L{}_{ijk}(y) u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2} y_l B^l{}_{ijk}.$$

$\mathbf{L}_y(u, v, w)$ is symmetric in u, v and w and $\mathbf{L}_y(y, v, w) = 0$. $\mathbf{L}$ is called the Landsberg curvature. A Finsler metric F is called a Landsberg metric if $\mathbf{L}_y = 0$ [12]. Equivalently, a Finsler metric F is called a Landsberg metric if the Cartan torsion of F satisfies the following:

$$A_{ijk|m} y^m = 0.$$

It is easy to see that, every Berwald metric is a Landsberg metric.

2.1. The bundle maps. In [1], Akbar-Zadeh developed the modern theory of global Finsler geometry by establishing a global definition of Cartan connection. For this aim, he introduced two bundle maps ρ and μ. Here, we give a short introduction of these bundle maps. Let TTM be the tangent bundle of TM and ρ the canonical linear mapping

$$\rho\colon TTM_0 \quad \to \quad \pi^*TM,$$
$$\hat{X} \quad \longmapsto \quad \Big(z, \pi_*(\hat{X})\Big),$$

where $\hat{X} \in T_zTM_0$ and $z \in TM_0$. The bundle map ρ satisfies

$$\rho\left(\frac{\partial}{\partial x^i}\right) = \partial_i, \qquad \rho\left(\frac{\partial}{\partial y^i}\right) = 0.$$

Let V_zTM be the set of vertical vectors at z, that is, the set of vectors tangent to the fiber through z, or equivalently $V_zTM = \ker\rho$, called the *vertical space*.

By means of these considerations, one can see that the following sequence is exact:

$$0 \to VTM \stackrel{i}{\longrightarrow} TTM \stackrel{\rho}{\longrightarrow} \pi^*TM \longrightarrow 0,$$

where i is the natural inclusion map.

Let ∇ be a linear connection on π^*TM, that is $\nabla\colon T_zTM_0 \times \pi^*TM \to \pi^*TM$ such that $\nabla\colon (\hat{X}, Y) \to \nabla_{\hat{X}}Y$. Let us define the linear mapping

$$\mu_z\colon T_zTM_0 \quad \to \quad T_{\pi z}M,$$
$$\hat{X} \quad \longmapsto \quad \nabla_{\hat{X}}F\ell,$$

where $\hat{X} \in T_zTM_0$. For a torsion-free connection ∇ the bundle map μ satisfies

$$\mu\left(\frac{\partial}{\partial x^i}\right) = N_i^k\partial_k, \qquad \mu\left(\frac{\partial}{\partial y^i}\right) = \nabla_{\frac{\partial}{\partial y^i}}F\ell = \rho\left(\left[\frac{\partial}{\partial y^i}, y^k\frac{\partial}{\partial x^k}\right]\right) = \partial_i,$$

where $N_i^k = F\Gamma_{ij}^k\ell^j$ and Γ_{ij}^k are Christoffel symbols of ∇.

Let us put

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^k\frac{\partial}{\partial y^k}.$$

Then

$$\mu\left(\frac{\delta}{\delta x^i}\right) = 0.$$

The connection ∇ is called a *Finsler connection* if for every $z \in TM_0$, μ_z defines an isomorphism of V_zTM_0 onto $T_{\pi z}M$. Therefore, the tangent space TTM_0 in z is decomposed as

$$T_zTM_0 = H_zTM \oplus V_zTM,$$

where $H_zTM = \ker\mu_z$ is called the *horizontal space* defined by ∇. Indeed any tangent vector $\hat{X} \in T_zTM_0$ in z decomposes to

$$\hat{X} = H\hat{X} + V\hat{X},$$

where $H\hat{X} \in H_zTM$ and $V\hat{X} \in V_zTM$. Thus ρ restricted to HTM is an isomorphism onto π^*TM, and μ restricted to VTM is the bundle isomorphism onto π^*TM.

The structural equations of the Finsler connection ∇ are

$$\mathcal{T}_\nabla(\hat{X}, \hat{Y}) := \nabla_{\hat{X}}Y - \nabla_{\hat{Y}}X - \rho[\hat{X}, \hat{Y}],$$

$$\Omega(\hat{X}, \hat{Y})Z := \nabla_{\hat{X}}\nabla_{\hat{Y}}Z - \nabla_{\hat{Y}}\nabla_{\hat{X}}Z - \nabla_{[\hat{X},\hat{Y}]}Z,$$

where $X = \rho(\hat{X})$, $Y = \rho(\hat{Y})$ and $Z = \rho(\hat{Z})$. The tensors $\mathcal{T}_\nabla$ and Ω are called, respectively, the *torsion and curvature tensors of* ∇. They determine two torsion tensors defined by

$$\mathcal{S}(X, Y) := \mathcal{T}_\nabla(H\hat{X}, H\hat{Y}), \qquad \mathcal{T}(\dot{X}, Y) := \mathcal{T}_\nabla(V\hat{X}, H\hat{Y})$$

and three curvature tensors defined by

$$R(X, Y) := \Omega(H\hat{X}, H\hat{Y}),$$

$$P(X, \dot{Y}) := \Omega(H\hat{X}, V\hat{Y}),$$

$$Q(\dot{X}, \dot{Y}) := \Omega(V\hat{X}, V\hat{Y}),$$

where $\dot{X} = \mu(\hat{X})$ and $\dot{Y} = \mu(\hat{X})$.

3. Shen's L-process on Berwald connection. In this section, we are going to study the connection obtained by Shen's L-process on the Berwald connection. For this aim, we give a short and exact definition of the Berwald connection.

In 1926, L. Berwald introduced a connection and two curvature tensors. The Berwald connection is torsion-free, but is not necessarily metric-compatible [2]. It was Berwald who first successfully extended the notion of Riemann curvature to Finsler spaces. He also introduced a notion of non-Riemannian quantity called Berwald curvature.

The Berwald connection introduced by the following properties:

Berwald connection: Let (M, F) be an n-dimensional Finsler manifold. Then the Berwald connection $\mathfrak{D}$ is a linear connection in π^*TM, which has the following properties:

(i) $\mathfrak{D}$ is torsion-free, i.e., for all $\hat{X}$, $\hat{Y} \in C^\infty(T(TM_0))$,

$$\mathcal{T}(\hat{X}, \hat{Y}) := \mathfrak{D}_{\hat{X}}\rho(\hat{Y}) - \mathfrak{D}_{\hat{Y}}\rho(\hat{X}) - \rho\left([\hat{X}, \hat{Y}]\right) = 0. \tag{3.1}$$

(ii) $\mathfrak{D}$ is almost compatible with F in the following sence:

$$\begin{aligned}(\mathfrak{D}_{\hat{Z}}g)(X, Y) &:= \hat{Z}g(X, Y) - g(\mathfrak{D}_{\hat{Z}}X, Y) - g(X, \mathfrak{D}_{\hat{Z}}Y) = \\ &= 2F^{-1}A(\mu(\hat{Z}), X, Y) - 2\dot{A}(\rho(\hat{Z}), X, Y),\end{aligned} \tag{3.2}$$

where X, $Y \in C^\infty(\pi^*TM)$ and $\hat{Z} \in T_v(TM_0)$.

In [17], Tayebi and Najafi did not consider the Shen's C-process on Berwald connection. Here, we apply Shen's C-process on Berwald connection and find a new torsion-free connection. First, we prove the existence of this linear Finslerian connection.

Theorem 3.1. *Let (M,F) be an n-dimensional Finsler manifold. Then there is a unique linear connection D in π^*TM, which has the following properties:*
(i) *D is torsion-free in the sense of* (3.1);
(ii) *D is almost compatible with the Finsler structure in the following sense: for all $X,Y\in$ $\in C^\infty(\pi^*TM)$ and $\hat{Z}\in T_v(TM_0)$,*

$$(D_{\hat{Z}}g)(X,Y):=-2\dot{A}\left(\rho(\hat{Z}),X,Y\right)+2F^{-1}\left[A\left(\mu(\hat{Z}),X,Y\right)+\dot{A}\left(\mu(\hat{Z}),X,Y\right)\right]. \tag{3.3}$$

Proof. In a standard local coordinate system (x^i,y^i) in TM_0, we write

$$D_{\frac{\partial}{\partial x^i}}\partial_j=\Gamma^k_{ij}\partial_k,\qquad D_{\frac{\partial}{\partial y^i}}\partial_j=F^k_{ij}\partial_k.$$

Clearly, (3.1) and (3.3) are equivalent to the following:

$$\Gamma^k_{ij}=\Gamma^k_{ji}, \tag{3.4}$$

$$F^k_{ij}=0, \tag{3.5}$$

$$\frac{\partial(g_{ij})}{\partial x^k}=\Gamma^l_{ki}g_{lj}+\Gamma^l_{kj}g_{il}-2\dot{A}_{ijk}+2\Gamma^l_{km}l^m(A_{lij}+\dot{A}_{lij}), \tag{3.6}$$

$$\frac{\partial(g_{ij})}{\partial y^k}=F^s_{ik}g_{sj}+F^s_{kj}g_{is}+2F^{-1}(A_{ijk}+\dot{A}_{ijk})-2F^s_{mk}l^m\dot{A}_{ijk}. \tag{3.7}$$

Note that (3.5) and (3.7) are just the definition of A_{ijk}. We must compute Γ^k_{ij} from (3.4) and (3.6). Then making a permutation to i,j,k in (3.6), and, by using (3.4), we obtain

$$\Gamma^k_{ij}=\gamma^k_{ij}+\dot{A}^k_{ij}+g^{kl}\left\{\Gamma^m_{lb}(A_{mij}+\dot{A}_{mij})-\Gamma^m_{ib}(A_{mjl}+\dot{A}_{mlj})-\Gamma^m_{jb}(A_{mil}+\dot{A}_{mil})\right\}\ell^b, \tag{3.8}$$

where

$$\gamma^k_{ij}:=\frac{1}{2}g^{kl}\left\{\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right\} \tag{3.9}$$

and $A^k_{ij}=g^{kl}A_{ijl}$. Multiplying (3.8) by ℓ^i implies that

$$\Gamma^k_{ib}\ell^b=\gamma^k_{ib}\ell^b-(A^k_{im}+\dot{A}^k_{im})\Gamma^m_{lb}\ell^l\ell^b. \tag{3.10}$$

Contracting (3.10) with ℓ^j yields

$$\Gamma^k_{ab}\ell^a\ell^b=\gamma^k_{ab}\ell^a\ell^b. \tag{3.11}$$

By putting (3.11) in (3.10), one can obtain

$$\Gamma^k_{ib}\ell^b=\gamma^k_{ib}\ell^b-\ell^a\ell^b\gamma^m_{ab}(A^k_{mi}+\dot{A}^k_{mi}). \tag{3.12}$$

Putting (3.12) in (3.8) gives us the following:

$$\Gamma^k_{ij}=\gamma^k_{ij}+\dot{A}^k_{ij}+g^{kl}\left\{\gamma^m_{lb}(A_{mij}+\dot{A}_{mij})-\gamma^m_{ib}(A_{mlj}+\dot{A}_{mlj})-\gamma^m_{jb}(A_{mil}+\dot{A}_{mil})\right\}\ell^b+$$

$$+\gamma^s_{ab}\ell^a\ell^b\left\{(A^m_{sj}+\dot{A}^m_{sj})(A^k_{mi}+\dot{A}^k_{mi})+(A^m_{si}+\dot{A}^m_{si})(A^k_{mj}+\dot{A}^k_{mj})-\right.$$

$$\left.-(A^k_{sm}+\dot{A}^k_{sm})(A^m_{ij}+\dot{A}^m_{ij})\right\}.$$

This proves the uniqueness of D. The set $\{\Gamma^k_{ij},F^k_{ij}=0\}$, where $\{\Gamma^k_{ij}\}$ are given by (3.1), defines a linear connection D satisfying (3.1) and (3.3).

4. Curvatures of the connection D. The curvature tensor Ω of D is defined by

$$\Omega(\hat{X},\hat{Y})Z = D_{\hat{X}}D_{\hat{Y}}Z - D_{\hat{Y}}D_{\hat{X}}Z - D_{[\hat{X},\hat{Y}]}Z,$$

where $\hat{X},\hat{Y} \in C^{\infty}(T(TM_0))$ and $Z \in C^{\infty}(\pi^*TM)$. Let $\{e_i\}_{i=1}^n$ be a local orthonormal (with respect to g) frame field for the vector bundle π^*TM such that $g(e_i, e_n) = 0, i = 1,\ldots,n-1$ and

$$e_n := \frac{y}{F} = \frac{y^i}{F(x,y)}\frac{\partial}{\partial x^i} = \ell.$$

Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. These are local sections of dual bundle π^*TM. One readily finds that

$$\omega^n := \frac{\partial F}{\partial x^i} = \ell_i dx^i = \omega,$$

which is the *Hilbert form*. It is obvious that $\omega(\ell) = 0$.

Now, let us put

$$\rho = \omega^i \otimes e_i, \qquad De_i = \omega_i^{\ j} \otimes e_j, \qquad \Omega e_i = 2\Omega_i^{\ j} \otimes e_j.$$

$\{\Omega_i^{\ j}\}$ and $\{\omega_i^{\ j}\}$ are called the *curvature forms* and *connection forms* of D with respect to $\{e_i\}$.

We have $\mu := DF\ell = F\{\omega_n^{\ i} + d(\log F)\delta_n^i\} \otimes e_i$. Put $\omega^{n+i} := \omega_n^{\ i} + d(\log F)\delta_n^i$. It is easy to see that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. By definition $\rho = \omega^i \otimes e_i, \quad \mu = F\omega^{n+i} \otimes e_i$. Using the above formula for Theorem 3.1, it then re-expresses the structure equation of the new connection D as follows:

$$d\omega^i = \omega^j \wedge \omega_j^{\ i}, \tag{4.1}$$

$$dg_{ij} = g_{kj}\omega_i^{\ k} + g_{ki}\omega_j^{\ k} - 2\dot{A}_{ijk}\omega^k + 2(A_{ijk} + \dot{A}_{ijk})\omega^{n+k}. \tag{4.2}$$

Define $g_{ij.k}$ and $g_{ij|k}$ by

$$dg_{ij} - g_{kj}\omega_i^{\ k} - g_{ik}\omega_j^{\ k} = g_{ij|k}\omega^k + g_{ij.k}\omega^{n+k},$$

where $g_{ij.k}$ and $g_{ij|k}$ are, respectively, the vertical and horizontal covariant derivative of g_{ij} with respect to the connection D. This gives

$$g_{ij|k} = -2\dot{A}_{ijk},$$

$$g_{ij.k} = 2(A_{ijk} + \dot{A}_{ijk}).$$

It can be shown that $\delta^i_{j|s} = 0$ and $\delta^i_{j.s} = 0$, thus $(g^{ij}g_{jk})_{|s} = 0$ and $(g^{ij}g_{jk})_{.s} = 0$. So,

$$g^{ij}_{|s} = 2\dot{A}^{ij}_s, \qquad g^{ij}_{.s} = 2(\dot{A}^{ij}_s + A^{ij}_s).$$

Moreover, torsion freeness is equivalent to the absent of dy^k in $\{\omega_j^{\ i}\}$ namely

$$\omega_j^{\ i} = \Gamma^i_{jk}(x,y)dx^k,$$

which is equivalent to

$$d\omega_i^{\ j} - \omega_i^{\ k} \wedge \omega_k^{\ j} = \Omega_i^{\ j}. \tag{4.3}$$

Since the $\Omega_j^{\ i}$ are 2-forms on the manifold TM_0, they can be generally expanded as

$$\Omega_i^{\ j} = \frac{1}{2} R_{i\ kl}^{\ j} \omega^k \wedge \omega^l + P_{i\ kl}^{\ j} \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q_{i\ kl}^{\ j} \omega^{n+k} \wedge \omega^{n+l}. \tag{4.4}$$

The objects R, P and Q are respectively the hh-, hv- and vv-curvature tensors of the connection D. Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM$, $\dot{e}_i \in$ $\in VTM$ such that $\rho(\bar{e}_i) = e_i$, $\mu(\dot{e}_i) = Fe_i$. Let us put

$$R(\bar{e}_k, \bar{e}_l)e_i = R_{i\ kl}^{\ j} e_j, \qquad P(\bar{e}_k, \dot{e}_l)e_i = P_{i\ kl}^{\ j} e_j, \qquad Q(\dot{e}_k, \dot{e}_l)e_i = Q_{i\ kl}^{\ j} e_j.$$

The connection defined in Theorem 3.1 is torsion-free. Then we have $Q = 0$. First Bianchi identity for R is given by

$$R_{i\ kl}^{\ j} + R_{k\ li}^{\ j} + R_{l\ ik}^{\ j} = 0$$

and

$$P_{i\ kl}^{\ j} = P_{k\ il}^{\ j}. \tag{4.5}$$

Exterior differentiation of (4.3) gives the second Bianchi identity

$$d\Omega_i^{\ j} - \omega_i^{\ k} \wedge \Omega_k^{\ j} + \omega_k^{\ j} \wedge \Omega_i^{\ k} = 0. \tag{4.6}$$

We decompose the covariant derivative of the Cartan tensor on TM

$$dA_{ijk} - A_{ljk}\omega_i^{\ l} - A_{ilk}\omega_j^{\ l} - A_{ijl}\omega_k^{\ l} = A_{ijk|l}\omega^l + A_{ijk.l}\omega^{n+l}. \tag{4.7}$$

Similarly, for $\dot{A}_{ijk}$, we get

$$d\dot{A}_{ijk} - \dot{A}_{ljk}\omega_i^{\ l} - \dot{A}_{ilk}\omega_j^{\ l} - \dot{A}_{ijl}\omega_k^{\ l} = \dot{A}_{ijk|l}\omega^l + \dot{A}_{ijk.l}\omega^{n+l}. \tag{4.8}$$

It is easy to see that, $A_{ijk|l}$, $A_{ijk.l}$, $\dot{A}_{ijk|l}$ and $\dot{A}_{ijk.l}$ are symmetric with respect to indices i, j and k.

Put $\dot{A}_{ijk} = \dot{A}(e_i, e_j, e_k)$. Then

$$A_{ijk|n} = \dot{A}_{ijk}.$$

By (4.7) and (4.8), we get

$$A_{njk|l} = 0, \text{ and } \quad A_{njk.l} = -A_{jkl},$$

$$\dot{A}_{njk|l} = 0, \text{ and } \quad \dot{A}_{njk.l} = -\dot{A}_{jkl}.$$

Theorem 4.1. *Let (M, F) be a Finsler manifold. Suppose that D is the linear torsion-free connection obtained by Shen's L-process on Berwald's connection. Then the hv-curvature of D vanishes if and only if F is a Berwald metric.*

Proof. Let (M,F) be a Finsler manifold. Differentiating (4.2), and using (4.1), (4.2), (4.3), (4.7) and (4.8) leads to

$$d\omega^i = \omega^j \wedge \omega^i_j, \tag{4.9}$$

$$dg_{ij} = g_{kj}\omega^k_i + g_{ki}\omega^k_j - 2\dot{A}_{ijk}\omega^k + 2(A_{ijk} + \dot{A}_{ijk})\omega^{n+k}. \tag{4.10}$$

By differentiating of (4.10), we get

$$0 = dg_{ik}\omega^k_j + g_{ik}d\omega^k_j + dg_{jk}\omega^k_i + g_{jk}d\omega^k_i + 2(dA_{ijk} + d\dot{A}_{ijk})\omega^{n+k}+$$
$$+2(A_{ijk} + \dot{A}_{ijk})d\omega^{n+k} - 2d\dot{A}_{ijk}\omega^k - 2\dot{A}_{ijk}d\omega^k.$$

Using (4.4), (4.6) and (4.7), one can obtain

$$R_{ijkl} + R_{jikl} = -2A_{ijs}R^s_{n\ kl} - 2\dot{A}_{ijs}R^s_{n\ kl} - 4\dot{A}_{ijk|l}, \tag{4.11}$$

$$P_{ijkl} + P_{jikl} = -2\dot{A}_{ijk,l} - 2(A_{ijl|k} + \dot{A}_{ijl|k}) - 2(A_{ijs} + \dot{A}_{ijs})P_n{}^s{}_{kl}, \tag{4.12}$$

$$A_{ijk.m} + \dot{A}_{ijk.m} = 0. \tag{4.13}$$

Permuting $i,\ j,\ k$ in (4.12) yields

$$P_{jkil} + P_{kjil} = -2\dot{A}_{jki.l} - 2(A_{jkl|i} + \dot{A}_{jkl|i}) - 2(A_{jks} + \dot{A}_{jks})P_n{}^s{}_{il}, \tag{4.14}$$

$$P_{kijl} + P_{ikjl} = -2\dot{A}_{kij.l} - 2(A_{kil|j} + \dot{A}_{kil|j}) - 2(A_{kis} + \dot{A}_{kis})P_n{}^s{}_{jl}. \tag{4.15}$$

From (4.12), (4.14) and (4.15), we get

$$P_{ijkl} = -\dot{A}_{ijk,l} - \Big[(A_{ijl|k} + \dot{A}_{ijl|k}) + (A_{jkl|i} + \dot{A}_{jkl|i}) - (A_{kil|j} + \dot{A}_{kil|j})\Big] -$$
$$- \Big[(A_{ijs} + \dot{A}_{ijs})P^s_{n\ kl} + (A_{jks} + \dot{A}_{jks})P^s_{n\ il} - (A_{kis} + \dot{A}_{kis})P^s_{n\ jl}\Big]. \tag{4.16}$$

Taking a vertical derivation of $\dot{A}_{ijk}y^i = 0$ with respect to y^l implies that

$$\dot{A}_{ijk,l}y^i = -\dot{A}_{jkl}. \tag{4.17}$$

Multiplying (4.17) with y^j yields

$$\dot{A}_{ijk,l}y^iy^j = 0. \tag{4.18}$$

By contracting (4.16) with y^i and considering $A_{ijl|k}y^i = 0$, (4.18) and (4.17), we get

$$P_{njkl} = -\ddot{A}_{jkl} - (A_{jks} + \dot{A}_{jks})P^s_{n\ nl}. \tag{4.19}$$

On the other hand, multiplying (4.12) with y^iy^j implies that $P_{nnkl} = 0$. Thus, by (4.5), we have

$$P_{knnl} = 0. \tag{4.20}$$

Contracting (4.12) with y^jy^k yields

$$P_{innl} + P_{ninl} = 0. \tag{4.21}$$

By (4.20) and (4.21) it follows that

$$P_{ninl} = 0. \tag{4.22}$$

Putting (4.22) in (4.19) implies that

$$P_{njkl} = -\ddot{A}_{jkl}. \tag{4.23}$$

Let F be a Berwald metric. Thus from (4.23), we get $P_{njkl} = 0$ or, equivalently, $P^{j}_{n\,kl} = 0$. By putting it and $A_{ijk|l} = 0$ in (4.16), we get $P = 0$.

Conversely let $P = 0$. By (4.23), it follows that $\ddot{A}_{jkl} = 0$. By assumption, (4.16) reduces to following:

$$\dot{A}_{ijk,l} = -(A_{ijl|k} + \dot{A}_{ijl|k}) - (A_{jkl|i} + \dot{A}_{jkl|i}) + (A_{kil|j} + \dot{A}_{kil|j}). \tag{4.24}$$

Permuting $i,\ j,\ k$ in the above identity leads to

$$\dot{A}_{jki,l} = -(A_{jkl|i} + \dot{A}_{jkl|i}) - (A_{kil|j} + \dot{A}_{kil|j}) + (A_{ijl|k} + \dot{A}_{ijl|k}). \tag{4.25}$$

(4.24), (4.25) yields

$$A_{ijl|k} + \dot{A}_{ijl|k} = A_{kil|j} + \dot{A}_{kil|j}. \tag{4.26}$$

Contracting (4.26) with y^k implies that

$$\dot{A}_{ijl} = -\ddot{A}_{ijl}. \tag{4.27}$$

Since $\ddot{A}_{jkl} = 0$, then (4.27) reduces to $\dot{A}_{ijk} = 0$. Putting it and $P = 0$ in (4.12) imply $A_{ijk|l} = 0$. This means that F is a Berwald metric.

5. Compact Finsler manifolds. Let $\bar{\ell}$ denote the unique vector field in HTM such that $\rho(\bar{\ell}) = \ell$. We call $\bar{\ell}$ the *geodesic field* on TM_0, because it determines all geodesics and it is called a *spray*.

Let $c : [a, b] \to (M, F)$ be a unit speed C^∞ curve. The canonical lift of c to TM_0 is defined by $\hat{c} := \dfrac{dc}{dt} \in TM_0$. It is easy to see that $\rho\left(\dfrac{d\hat{c}}{dt}\right) = \ell_{\hat{c}}$. The curve c is called a *geodesic* if its canonical lift $\hat{c}$ satisfies $\dfrac{d\hat{c}}{dt} = \bar{\ell}_{\hat{c}}$, where $\bar{\ell}$ is the *geodesic field* on TM_0, i.e., $\ell \in HTM,\ \rho(\bar{\ell}) = \ell$.

Let $I_xM = \{v \in T_xM, F(v) = 1\}$ and $IM = \bigcup_{p\in M} I_xM$. The I_xM is called *indicatrix*, and it is a compact set. We can show that the projection of integral curve $\varphi(t)$ of $\bar{\ell}$ with $\varphi(0) \in IM$ is a unit speed geodesics c whose canonical lift is $\hat{c}(t) = \varphi(t)$. A Finsler manifold (M, F) is called *complete* if any unit speed geodesic $c : [a, b] \to M$ can be extended to a geodesic defined on R. This is equivalent to requiring that the geodesic field $\bar{\ell}$ restricted to IM is complete.

Let (M, F) be a Finsler manifold and c be a unit speed geodesic in M. A section $X = X(t)$ of π^*TM along $\hat{c}$ is said to be parallel if $D_{\frac{d\hat{c}}{dt}} X = 0$. For $v \in TM_0$, let us define

$$\|A\|_v = \sup A(X, Y, Z),$$

where the supremum is taken over all unit vectors of $\pi^*_v TM$. Put $\|A\|_v = \sup_{v\in IM} \|A\|_v$. Then we have the following theorem.

Theorem 5.1. *Let (M,F) be a compact Finsler manifold. Then F is a Landsberg metric if and only if*

$$\ddot{A}=0. \tag{5.1}$$

This means that, on compact manifolds $P_{njkl}=0$ if and only if F is a Landsberg metric.

Proof. Let us fix $X,\ Y,\ Z\in\pi^*TM$ at $v\in I_xM$. Suppose that $c\colon M\to R$ is the unit speed geodesic with $\dfrac{dc}{dt}(0)=v$. Let $X(t),\ Y(t)$ and $Z(t)$ denote the parallel sections along $\hat{c}$ with $X(0)=X,\ Y(0)=Y$ and $Z(0)=Z$. Put

$$\mathbf{A}(t)=A\left(X(t),Y(t),Z(t)\right),$$

$$\dot{\mathbf{A}}(t)=\dot{A}\left(X(t),Y(t),Z(t)\right),$$

$$\ddot{\mathbf{A}}(t)=\ddot{A}\left(X(t),Y(t),Z(t)\right).$$

By definition, (5.1) implies that

$$\dot{\mathbf{A}}=\frac{d\mathbf{A}}{dt} \qquad \text{and} \qquad \ddot{\mathbf{A}}=\frac{d\dot{\mathbf{A}}}{dt}. \tag{5.2}$$

Therefore, from (5.1) and (5.2), we have

$$\frac{d\dot{\mathbf{A}}}{dt}=0.$$

Then

$$\mathbf{A}(t)=t\dot{\mathbf{A}}(0)+\mathbf{A}(0).$$

Since M is compact then it is complete and $\|\dot{\mathbf{A}}\|<\infty$. Then by letting $t\to-\infty$ or $t\to\infty$, we get

$$\dot{\mathbf{A}}(0)=\dot{\mathbf{A}}(X,Y,Z)=0.$$

Thus, F is a Landsberg metric.

***Remark* 5.1.** Suppose that F satisfies (5.1). This equation is equivalent to that for any linearly parallel vector fields $u,\ v,\ w$ along a geodesic c, the following holds:

$$\frac{d}{dt}\left[\dot{\mathbf{A}}_{\dot{c}}(u,v,w)\right]=0.$$

The geometric meaning of this is that the rate of change of the Landsberg curvature is constant along any Finslerian geodesic [22].

An (α,β)-metric is a Finsler metric defined by $F:=\alpha\phi(s),\ s=\beta/\alpha$, where ϕ is a smooth function on a symmetric interval $(-b_0,b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on the base manifold (see [13, 20, 21]). There is a special class of (α,β)-metric, namely Randers metrics. A Randers metric $F=\alpha+\beta$ on a manifold M is just a Riemannian metric α perturbated by a one form β on M such that the Riemanninan length of $\beta^\sharp$ is less than 1 (see [10, 18]).

In the proof of the main theorem in [22], the authors used the condition $g^{ij}L_{ijk|s}y^s=0$ and proved that every Randers metric with closed one form β is a stretch metric if and only if it is Berwaldian. Then, we get the following corollary.

***Corollary* 5.1.** *Let (M,F) be a Finsler – Randers manifold equipped with the Finsler connection D. Then F is a Berwald metric if and only if $P_{njkl}=0$.*

The Corollary 5.1 can be considered as an extension of Theorem 5.1. We delete the condition “compact” and replace the Randers manifold instead of arbitrary manifold.

6. Shen’s C-process on Berwald connection. In this section, we are going to study the connection obtained by Shen’s C-process on the Berwald connection.

Theorem 6.1. *Let (M,F) be an n-dimensional Finsler manifold. Then there is a unique linear connection ∇ in π^*TM, which has the following properties:*

(i) ∇ *is torsion-free in the sense of* (3.1);

(ii) ∇ *is almost compatible with the Finsler structure in the following sense:*

$$(\nabla_{\hat{Z}}g)(X,Y):=2\left[A(\rho(\hat{Z}),X,Y)-\dot{A}(\rho(\hat{Z}),X,Y)\right]+2F^{-1}A(\mu(\hat{Z}),X,Y), \tag{6.1}$$

where $X,Y\in C^{\infty}(\pi^*TM)$ and $\hat{Z}\in T_v(TM_0)$.

Proof. In a standard local coordinate system (x^i,y^i) in TM_0, we write

$$\nabla_{\frac{\partial}{\partial x^i}}\partial_j=\Gamma^k_{ij}\partial_k, \qquad \nabla_{\frac{\partial}{\partial y^i}}\partial_j=F^k_{ij}\partial_k.$$

The equations (3.1) and (6.1) are equivalent to

$$\Gamma^k_{ij}=\Gamma^k_{ji}, \tag{6.2}$$

$$F^k_{ij}=0, \tag{6.3}$$

$$\frac{\partial(g_{ij})}{\partial x^k}=\Gamma^l_{ki}g_{jl}+\Gamma^l_{kj}g_{li}+2(A_{ijk}-\dot{A}_{ijk})+2\Gamma^l_{km}l^mA_{ijl}, \tag{6.4}$$

$$\frac{\partial(g_{ij})}{\partial y^k}=F^l_{ik}g_{lj}+F^l_{kj}g_{li}+2F^{-1}A_{ijk}+2F^l_{mk}l^mA_{ijl}. \tag{6.5}$$

Then making a permutation to i,j,k in (6.4), and by using (6.2), we obtain

$$\Gamma^k_{ij}=\gamma^k_{ij}-(A^k_{ij}-\dot{A}^k_{ij})+g^{kl}\left\{A_{ijm}\Gamma^m_{lb}-A_{ljm}\Gamma^m_{ib}-\Gamma^m_{jb}A_{ilm}\right\}\ell^b,$$

where γ^k_{ij} defined by (3.9). By the same argument used in Theorem 3.1, we get

$$\Gamma^k_{ij}=\gamma^k_{ij}-(A^k_{ij}-\dot{A}^k_{ij})+g^{kl}\left\{A_{ijm}\gamma^m_{lb}-A_{jlm}\gamma^m_{ib}-A_{lim}\gamma^m_{jb}\right\}\ell^b+$$
$$+\left\{A^k_{jm}A^m_{is}+A^k_{im}A^m_{js}-A^k_{sm}A^m_{ij}\right\}\gamma^s_{ab}\ell^a\ell^b. \tag{6.6}$$

This proves the uniqueness of ∇. The set $\{\Gamma^k_{ij},F^k_{ij}=0\}$, where $\{\Gamma^k_{ij}\}$ are given by (6.6), defines a linear torsion-free and almost compatible connection ∇ satisfying (3.1) and (6.1).

Here, we remark that the connection ∇ can be expressed by the following equations

$$d\omega^i=\omega^j\wedge\omega_j{}^i, \tag{6.7}$$

$$dg_{ij}=g_{ki}\omega_j{}^k+g_{jk}\omega_i{}^k+2(A_{ijk}-\dot{A}_{ijk})\omega^k+2A_{ijk}\omega^{n+k}. \tag{6.8}$$

Thus,

$$g_{ij|k}=2(A_{ijk}-\dot{A}_{ijk}), \qquad g_{ij.k}=2A_{ijk}.$$

By a simple calculation, we get the following theorem.

Theorem 6.2. *The new connection ∇ can be obtained from the Shen connection by Matsumoto's L-process.*

Thus we get the following diagram:

$$\begin{array}{ccc}
\text{Chern connection} & \xrightarrow{SC\text{-process}} & \text{Shen connection} \\
| & & | \\
ML\text{-process} & & ML\text{-process} \\
\downarrow & & \downarrow \\
\text{Berwald connection} & \xrightarrow{SC\text{-process}} & \text{the connection } \nabla.
\end{array}$$

7. Curvatures of the connection ∇.

Theorem 7.1. *Let (M, F) be an n-dimensional Finsler manifold. Suppose that ∇ is obtained from the Berwald connection by Shen's C-process. Then the hv-curvature of ∇ vanishes if and only if F is Riemannian.*

Proof. Let (M, F) be a Finsler manifold. Differentiating (6.8), and using (6.7), (6.8), (4.3), (4.7) and (4.8) leads to

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{7.1}$$

$$dg_{ij} = g_{kj}\omega_i^k + g_{ki}\omega_j^k + 2(A_{ijk} - \dot{A}_{ijk})\omega^k + 2A_{ijk}\omega^{n+k}. \tag{7.2}$$

By differentiating of (7.2), we get

$$dg_{ik}\omega_j^k + g_{ik}d\omega_j^k + dg_{jk}\omega_i^k + g_{jk}d\omega_i^k + 2(dA_{ijk} - d\dot{A}_{ijk})\omega^k +$$

$$+2(A_{ijk} - \dot{A}_{ijk})d\omega^k + 2dA_{ijk}\omega^{n+k} + 2A_{ijk}d\omega^{n+k} = 0. \tag{7.3}$$

Putting (4.4) in (7.3) implies that

$$R_{ijkl} + R_{jikl} = -2A_{ijs}R^s_{n\ kl}, \tag{7.4}$$

$$P_{ijkl} + P_{jikl} = -2\dot{A}_{ijk.l} + 2(A_{ijk.l} - A_{ijl|k}) - 2A_{ijs}P^s_{nkl}, \tag{7.5}$$

$$A_{ijk.l} = A_{ijl.k}. \tag{7.6}$$

Permuting $i,\ j,\ k$ in (7.5) yields

$$P_{jkil} + P_{kjil} = -2\dot{A}_{jki.l} + 2(A_{jki.l} - A_{jkl|i}) - 2A_{jks}P^s_{n\ il}, \tag{7.7}$$

$$P_{kijl} + P_{ikjl} = -2\dot{A}_{kij.l} + 2(A_{kij.l} - A_{kil|j}) - 2A_{kis}P_{n\ jl}^{\ s}. \tag{7.8}$$

From (7.5), (7.7) and (7.8), we have

$$P_{ijkl} = -\dot{A}_{ijk.l} + A_{ijk.l} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) -$$

$$-A_{ijs}P^s_{n\ kl} - A_{jks}P^s_{n\ il} + A_{kis}P^s_{n\ jl}. \tag{7.9}$$

Therefore,

$$P_{njkl} = -A_{jkl}. \tag{7.10}$$

By (7.9) and (7.10), it follows that the hv-curvature of F is vanishing if and only if F reduces to a Riemannian metric.

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