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INTEGRAL EQUATIONS INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION
ІНТЕГРАЛЬНІ РІВНЯННЯ, ЩО МІСТЯТЬ УЗАГАЛЬНЕНУ ФУНКЦІЮ МІТТАГ-ЛЕФФЛЕРА

The paper deals with solving the integral equation with a generalized Mittag-Leffler function $E^{\gamma,q}_{\alpha,\beta}(z)$ that defines a kernel using a fractional integral operator. The existence of the solution is justified and necessary conditions on the integral equation admiting a solution are discussed. Also, the solution of the integral equation is derived.

Розглядається розв'язність інтегрального рівняння з узагальненою функцією Міттаг-Леффлера $E^{\gamma,q}_{\alpha,\beta}(z)$, що визначає ядро з використанням дробового інтегрального оператора. Існування розв'язку обгрунтовано, обговорюються необхідні умови для інтегрального рівняння, що допускають розв'язок. Також наведено розв'язок такого інтегрального рівняння.

1. Introduction and preliminaries. The Mittag-Leffler function was defined by the Swedish mathematician, G. Mittag-Leffler [5], by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1}$$

where z is a complex variable and $\alpha \geq 0$ that occurs as the solution of fractional order differential equations. The Mittag-Leffler function is a direct generalization of exponential function, hyperbolic functions, and trigonometric functions as $E_1(z)=e^z,\ E_2(z^2)=\cosh z$ and $E_2(-z^2)=\cos z$. For $0<\alpha<1$ and $|z|<1$, it interpolates between the exponential function e^z and a geometric function $\dfrac{1}{1-z}=\sum_{k=0}^{\infty} z^k$.

Due to its vast involvement in the field of physics, engineering and applied sciences, many authors defined and studied different generalizations of Mittag-Leffler function, namely, $E_{\alpha,\beta}(z)$ introduced by Wiman [16], $E^{\gamma}_{\alpha,\beta}(z)$ suggested by Prabhakar [6], $E^{\gamma,q}_{\alpha,\beta}(z)$ defined and studied by Shukla and Prajapati [9], etc.

Shukla and Prajapati [9] defined the generalization of (1) as $E^{\gamma,q}_{\alpha,\beta}(z)$, for $\alpha,\beta,\gamma\in C$, $\mathrm{Re}\,(\alpha)$, $\mathrm{Re}\,(\beta)$, $\mathrm{Re}\,(\gamma)>0$, and $q\in(0,1)\cup N$, in the form

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{2}$$

where $(\gamma)_{qn}=\dfrac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denote the generalized Pochhammer symbol, which in particular, reduces to $q^{qn}\prod_{r=1}^{q}\left(\dfrac{\gamma+r-1}{q}\right)_n$ if $q\in N$, and, moreover, it is the generalization of the exponential function as $E^{1,1}_{1,1}(z)=\exp(z)$. Also, $E^{1,1}_{\alpha,1}(z)=E_\alpha(z)$, $E^{1,1}_{\alpha,\beta}(z)=E_{\alpha,\beta}(z)$ and $E^{\gamma,1}_{\alpha,\beta}(z)=E^{\gamma}_{\alpha,\beta}(z)$.

Further, Shukla and Prajapati [10] studied some properties of the generalized Mittag-Leffler-type function and generated integral operator

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$$E_{\alpha,\beta,w;a+}^{\gamma,q}f(x)=\int\limits_a^x (x-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,q}\{w(x-t)^{\alpha}\}f(t)dt \tag{3}$$

for $\alpha,\beta,\gamma,w\in C,\ \mathrm{Re}\,(\alpha),\ \mathrm{Re}\,(\beta),\ \mathrm{Re}\,(\gamma)>0$, and $q\in(0,1)\cup N$.

Fractional integral operators play an important role in the solution of several problems of science and engineering. Many fractional integral operators like Riemann–Liouville, Weyl, Kober, Erdelyi–Kober and Saigo operators are studied by various authors due to their application in physical, engineering and technological sciences such as reaction, diffusion, viscoelasticity, etc. A detailed account of these operators can be found in the survey paper by Srivastava and Saxena [13]. Various properties of family of Mittag-Leffler functions using fractional integral operators have been obtained by many researchers (see, for example, [1–3, 8, 10, 14, 15]).

In this paper, we apply the results obtained by Shukla and Prajapati [10] to prove the existence and uniqueness of the solution of the following integral equation, involving generalized Mittag-Leffler function (2). For $\alpha,\beta,\gamma,\lambda\in C,\ \mathrm{Re}\,(\alpha),\mathrm{Re}\,(\beta),\mathrm{Re}\,(\gamma),\mathrm{Re}\,(\lambda)>0,\ q\in(0,1)\cup N$ and for any real number $a\geq 0$,

$$\int\limits_a^x (x-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,q}\{\lambda(x-t)^{\alpha}\}f(t)dt=g(x). \tag{4}$$

In Section 2, integral on left-hand side of (4) is considered as an operator and its existence is justified. In Section 3, properties of the integral operator are derived. In Section 4, necessary condition for the solution is obtained and the integral equation (4) is solved. In Section 5, we discuss the integral equation

$$\int\limits_x^b (t-x)^{\beta-1}E_{\alpha,\beta}^{\gamma,q}\{\lambda(x-t)^{\alpha}\}f(t)dt=g(x) \tag{5}$$

for $\alpha,\beta,\gamma,\lambda\in C,\ \mathrm{Re}\,(\alpha),\mathrm{Re}\,(\beta),\mathrm{Re}\,(\gamma),\mathrm{Re}\,(\gamma)>0,\ q\in(0,1)\cup N$ and for any real number $0\leq a\leq x\leq b$.

In the sequel that follow, some additional properties of $E_{\alpha,\beta}^{\gamma,q}$ are derived, and we enumerate some terminologies and definitions, required in the investigation.

Definition 1.1. *Lebesgue measurable functions* [7]: *$L(a,b)$ denote the linear space of real (or complex) valued functions $f(x)$ on $[a,b]$ for $b>a$, i.e., f is L-integrable if*

$$L(a,b)=\left\{f:\|f\|_1\equiv\int\limits_a^b|f(t)|\,dt<\infty\right\}.$$

Definition 1.2. *Riemann–Liouville fractional integrals of order μ* [4]: *Let $f(x)\in L(a,b)$, $b>a,\ \mu\in C,\ \mathrm{Re}(\mu)>0$. Then*

$$I^{\mu}f(x)={}_aI_x^{\mu}f(x)=I_{a+}^{\mu}f(x)=\frac{1}{\Gamma(\mu)}\int\limits_a^x\frac{f(t)}{(x-t)^{1-\mu}}dt,\quad x>a, \tag{6}$$

is called fractional integral of order μ. I^{μ} is bounded. Also $I^{\mu}f=0\Rightarrow f=0$. Hence, inverse operator exists on subspace L_{μ} of L. Similarly,

$$J^{\mu} f(x) = {}_x I_b^{\mu} f(x) = I_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int\limits_x^b \frac{f(t)}{(t-x)^{1-\mu}} dt, \quad x < b.$$

We intend to maintain following properties of $E_{\alpha,\beta}^{\gamma,q}$ [10], which are employed to prove theorems in this paper.

If $\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta),\ \mathrm{Re}\,(\gamma),\ \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}(\mu) > 0$, and $q \in (0,1) \cup N$, then

$$\frac{1}{\Gamma(\mu)} \int\limits_t^x (x-s)^{\mu-1} (s-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \left\{ \lambda (s-t)^{\alpha} \right\} ds = (x-t)^{\beta-\mu-1} E_{\alpha,\beta+\mu}^{\gamma,q} \left\{ \lambda (x-t)^{\alpha} \right\}$$

and

$$\frac{1}{\Gamma(\mu)} \int\limits_t^x (s-t)^{\mu-1} (x-s)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \left\{ \lambda (x-s)^{\alpha} \right\} ds = (x-t)^{\beta-\mu-1} E_{\alpha,\beta+\mu}^{\gamma,q} \left\{ \lambda (x-t)^{\alpha} \right\}.$$

2. The integral operator. We employ the operator $E_{\alpha,\beta,\lambda;a+}^{\gamma,q} f(x)$, defined by (3) to represent left-hand side of the integral equation (4) and, hence, forth denote it as $I(\alpha,\beta,\gamma,q;\lambda)f(x)$. Therefore,

$$I(\alpha,\beta,\gamma,q;\lambda)f(x) = E_{\alpha,\beta,\lambda;a+}^{\gamma,q} f(x) = \int\limits_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \left\{ \lambda (x-t)^{\alpha} \right\} f(t) dt. \tag{7}$$

We may often use $I(\beta)$ instead of $I(\alpha,\beta,\gamma,q;\lambda)$ owing to reason that, in the operator $I(\alpha,\beta,\gamma,q;\lambda)$, all other parameters remain unaltered except β.

In the later part of the paper we discuss the integral equation (5) for which we define the operator as follows:

$$I^{*}(\alpha,\beta,\gamma,q;\lambda)f(x) = \int\limits_x^b (t-x)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \big\{ \lambda (x-t)^{\alpha} \big\} f(t) dt. \tag{8}$$

To justify the existence of the integral defined by the left-hand side of the equation (4), we employ the following theorem from [10].

Theorem 2.1 (existence of the operator)**.** *Let* $f \in L(a,b),\ b > a;\ \alpha,\beta,\gamma,\lambda \in C,\ \mathrm{Re}\,(\alpha),\ \mathrm{Re}\,(\beta),\ \mathrm{Re}\,(\gamma),\ \mathrm{Re}\,(\lambda) > 0$, *and* $q \in (0,1) \cup N$. *Then*

$$\int\limits_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \left\{ \lambda (x-t)^{\alpha} \right\} f(t) dt$$

defines a function in L. *Moreover, it is bounded on* L *and*

$$\big\| I(\alpha,\beta,\gamma,q;\lambda) f \big\|_1 \le B \|f\|_1,$$

where

$$B = (b-a)^{\mathrm{Re}\,(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_{qn}|}{|\Gamma(\alpha n + \beta)|\, [\mathrm{Re}\,(\alpha) n + \mathrm{Re}\,(\beta)]} \frac{\left| \lambda (b-a)^{\mathrm{Re}(\alpha)} \right|^n}{n!}.$$

3. Properties of the operator.

Theorem 3.1 (composition property). *If* $\alpha, \beta, \gamma, \beta', \lambda \in C,\ q \in (0,1) \cup N,\ \operatorname{Re}(\alpha),\ \operatorname{Re}(\beta),$ $\operatorname{Re}(\gamma),\ \operatorname{Re}(\lambda),\ \operatorname{Re}(\beta') > 0,$ *then the relation*

$$I(\alpha,\beta,\gamma,q;\lambda) I\left(\alpha,\beta',\gamma',q;\lambda\right) = I\left(\alpha,\beta+\beta',\gamma+\gamma',q;\lambda\right) \tag{9}$$

is valid for any summable function $f \in L(a,b)$. *In particular,*

$$I(\alpha,\beta,\gamma,q;\lambda) I^{-1}\left(\alpha,\beta',\gamma',q;\lambda\right) f(x) = I^{\beta+\beta'} f(x). \tag{10}$$

Proof. Let $f \in L$ and $x \in (a,b)$. Then

$$I(\alpha,\beta,\gamma,q;\lambda) I\left(\alpha,\beta',\gamma',q;\lambda\right) f(x) =$$
$$= \int\limits_a^x (x-u)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \lambda(x-u)^{\alpha} du \int\limits_a^u (u-t)^{\beta'-1} E_{\alpha,\beta'}^{\gamma',q} \lambda(u-t)^{\alpha} f(t) dt.$$

Changing the order of integration, which is justified by Fubini's theorem, we get

$$I(\alpha,\beta,\gamma,q;\lambda) I\left(\alpha,\beta',\gamma',q;\lambda\right) f(x) =$$
$$= \int\limits_a^x f(t) dt \int\limits_a^u (u-t)^{\beta'-1} (x-u)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \left\{\lambda(x-u)^{\alpha}\right\} E_{\alpha,\beta'}^{\gamma',q} \left\{\lambda(u-t)^{\alpha}\right\} du.$$

Further simplification yields

$$= (x-t)^{\beta+\beta'-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\gamma)_{qm} (\gamma')_{qn}}{\Gamma(\alpha m+\beta)\Gamma(\alpha n+\beta')} \frac{\lambda^{m+n} (x-t)^{\alpha m+\alpha n} \Gamma(\alpha m+\beta)\Gamma(\alpha n+\beta')}{m! n! \Gamma(\alpha(m+n)+\beta+\beta')} =$$
$$= (x-t)^{\beta+\beta'-1} E_{\alpha,\beta+\beta'}^{\gamma+\gamma',q} \left\{\lambda(x-t)^{\alpha}\right\},$$

which leads to (9). On substituting $\gamma' = -\gamma$ in (9), this reduces to (10), in accordance with the following remark.

Remark **3.1.** It should be noted that for $\gamma = 0$, the operator defined by (7), coincides with the Riemann – Liouville fractional integral of order β,

$$I(\alpha,\beta,0,q;\lambda) f(x) = I^{\beta} f(x). \tag{11}$$

Similarly, under same techniques, used in Theorem 3.1, we prove commutative property.

Theorem 3.2 (commutative property). *If* $\alpha, \beta, \gamma, \mu, \lambda \in C,\ q \in (0,1) \cup N$ *and* $\operatorname{Re}(\alpha),\ \operatorname{Re}(\beta),$ $\operatorname{Re}(\gamma),\ \operatorname{Re}(\lambda) > 0,$ *then, for almost all* x *belongs to* (a,b) *and* $f \in L,$

$$I^{\mu} I(\beta) f(x) = I(\beta) I^{\mu} f(x), \tag{12}$$

that justifies that the operator $I(\beta)$ *commutes with* I^{μ}.

One can easily prove this theorem by using the results (6) and (7). Hence, details are avoided. Further, we will need the following property given by Shukla and Prajapati [10] to obtain the solution of the integral equation (4).

Theorem 3.3 (shifting property)**.** *Let* $\alpha, \beta, \gamma, \mu, \lambda \in C,\ q \in (0,1) \cup N,\ \mathrm{Re}\,(\mu) > -\mathrm{Re}\,(\beta)$ *and* $\mathrm{Re}\,(\alpha) > 0,\ \mathrm{Re}\,(\beta) > 0,\ \mathrm{Re}\,(\gamma) > 0,\ \mathrm{Re}\,(\lambda) > 0,\ \mathrm{Re}\,(\mu) > 0.$ *Then*

$$I^{\mu} I(\alpha, \beta, \gamma, q; \lambda) = I(\alpha, \beta + \mu, \gamma, q; \lambda). \tag{13}$$

It can be observed that the composition of fraction integral operator I^{μ} and the operator $I(\beta)$ result in to shifting of the second parameter β of (7) by the order of fraction integral operator μ, where as all other parameters remain unaltered. Thus, we call this property as a shifting property.

4. Solution of the integral equation. To obtain the solution of integral equation (4), we need to prove the following lemma.

Lemma 4.1. *For* $\alpha, \beta, \gamma, \lambda \in C,\ \mathrm{Re}\,(\alpha), \mathrm{Re}\,(\beta), \mathrm{Re}\,(\gamma), \mathrm{Re}\,(\lambda) > 0,$ *and* $f \in L,$

$$I^{-\beta} I(\alpha, \beta, \gamma, q; \lambda) f(x) = f(x) + \alpha\lambda(\gamma)_q \int\limits_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha+1}^{\gamma+q,q} \left\{\lambda(x-t)^{\alpha}\right\} f(t)dt.$$

Proof. By shifting property (13), we get

$$I^{1-\beta} I(\alpha, \beta, \gamma, q; \lambda) f(x) = I(\alpha, \beta + 1 - \beta, \gamma, q; \lambda) f(x) = \int\limits_a^x E_{\alpha,1}^{\gamma,q} \left\{\lambda(x-t)^{\alpha}\right\} f(t)dt.$$

Therefore,

$$I(\alpha, \beta, \gamma, q; \lambda) f(x) = I^{\beta-1} \int\limits_a^x E_{\alpha,1}^{\gamma,q} \left\{\lambda(x-t)^{\alpha}\right\} f(t)dt =$$

$$= I^{\beta} \frac{d}{dx} \int\limits_a^x E_{\alpha,1}^{\gamma,q} \left\{\lambda(x-t)^{\alpha}\right\} f(t)dt =$$

$$= I^{\beta} \int\limits_a^x (\gamma)_q E_{\alpha,1+\alpha}^{\gamma+q,q} \left\{\lambda(x-t)^{\alpha}\right\} \lambda\alpha(x-t)^{\alpha-1} f(t)dt =$$

$$= I^{\beta} \left[f(x) + \alpha\lambda(\gamma)_q \int\limits_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha+1}^{\gamma+q,q} \left\{\lambda(x-t)^{\alpha}\right\} f(t)dt \right]. \tag{14}$$

Lemma 4.1 is proved.

Theorem 4.1 (the necessary condition for existence of solution)**.** *The existence of* $I^{-\beta} g$ *in* L *is a necessary condition for the integral*

$$\int\limits_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,q} \left\{\lambda(x-t)^{\alpha}\right\} f(t)dt = g(t) \tag{15}$$

to admit a solution f *in* L *.*

Proof. Suppose the integral equation (15) has a solution $f \in L$. Then the equation (14) can be written as

$$I(\alpha,\beta,\gamma,q;\lambda)f(x) = I^{\beta}\left[f(x) + \alpha\lambda(\gamma)_q \int\limits_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha+1}^{\gamma+q,q}\{\lambda(x-t)^{\alpha}\} f(t)dt\right] = g(x). \tag{16}$$

For $f \in L$ and by Theorem 2.1, it can be observed that the integral in (16) exists in L. Consequently, $I^{-\beta}g$ exists in L. This completes the justification.

Once the necessary conditions are justified, inwhat follow is the solution of the integral equation (4).

Theorem 4.2 (the solution)**.** *If* $\mathrm{Re}\,(\mu) > \mathrm{Re}\,(\beta) > 0$ *and* $I^{-\mu}g$ *exists in* L, *then the integral equation*

$$\int\limits_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,q}\{\lambda(x-t)^{\alpha}\} f(t)dt = g(t), \tag{17}$$

for $a < x < b$, *possesses a solution or a class of equivalent solutions* $f \in L$ *given by*

$$f(x) = \int\limits_a^x (x-t)^{\mu-\beta-1} E_{\alpha,\mu-\beta}^{-\gamma,q}\{\lambda(x-t)^{\alpha}\} I^{-\mu}g(t)dt. \tag{18}$$

Proof. The equation (17) can be written as

$$I(\alpha,\beta,\gamma,q;\lambda)f(x) = g(x). \tag{19}$$

On writing equation (18) as

$$I(\alpha,\mu-\beta,-\gamma,q;\lambda)I^{-\mu}g(x) = f(x). \tag{20}$$

Substituting for $f(x)$ from result (20) to the left-hand side of result (19), we get

$$I(\alpha,\beta,\gamma,q;\lambda)f(x) = I(\alpha,\beta,\gamma,q;\lambda)I(\alpha,\mu-\beta,-\gamma,q;\lambda)I^{-\mu}g(x). \tag{21}$$

By composition property (9), we write

$$I(\alpha,\beta,\gamma,q;\lambda)f(x) = I(\alpha,\mu,0,q;\lambda)I^{-\mu}g(x).$$

By the Remark 3.1 and invoking (11), we arrive at

$$I(\alpha,\beta,\gamma,q;\lambda)f(x) = I^{\mu}I^{-\mu}g(x) = g(x).$$

Thus, $f(x)$ as defined in (18) is proved to be a solution of the integral equation (4).

Corollary **4.1.** *Under the conditions of the above theorem,* (17) *and* (18) *imply each other.*

It can be easily verified by Theorem 3.2 that the operator $I\,(\beta)$ commutes with I^{μ}. Now, to prove (17) imply (18), we substitute the value of $g(x)$ from (19) to the left-hand side of (20), we have

$$I(\alpha, \mu-\beta, -\gamma, q; \lambda) I^{-\mu} I(\alpha, \beta, \gamma, q; \lambda) f(x).$$

By commutative property (12), above result becomes

$$I(\alpha, \mu-\beta, -\gamma, q; \lambda) I(\alpha, \beta, \gamma, q; \lambda) I^{-\mu} f(x).$$

On applying result (21), we obtain $f(x)$.

The converse can also be proved in a similar manner.

***Remark* 4.1.** If $q = 1$, Theorem 4.2 reduces to the solution of integral equation due to Prabhakar [6].

***Remark* 4.2.** For $\alpha = 1$, $a = 0$ and γ is a positive integer, Theorem 4.2 yield the transform pair due to Wimp [17].

5. The integral equation (5). The integral operator defined in (8), assists in discussing (5). Actually, it can be verified that all results analogous to Theorems 2.1, 3.1 and 3.2 hold for $I^*(\alpha, \beta, \gamma, q; \lambda)$, because J^{μ} plays the same role as I^{μ} does for $I(\alpha, \beta, \gamma, q; \lambda)$.

Theorem 5.1. *The existence of* $J^{-\beta}g$ *in* L *is a necessary condition for the integral*

$$I^*(\alpha, \beta, \gamma, q; \lambda) = \int\limits_x^b (t-x)^{\beta-1} E_{\alpha,\beta}^{\gamma,q}\{\lambda(x-t)^{\alpha}\} f(t)dt = g(x)$$

to admit a solution f *in* L. *Whereas, the existence of* $J^{-r}g$ *for* $\operatorname{Re}(r) > \operatorname{Re}(\beta) > 0$ *is a sufficient condition for equation* (5) *to admit a unique solution.*

Theorem 5.2. *If* $\operatorname{Re}(r) > \operatorname{Re}(\beta) > 0$ *and* $J^{-r}g$ *exists in* L, *then the integral equations*

$$\int\limits_x^b (t-x)^{\beta-1} E_{\alpha,\beta}^{\gamma,q}\{\lambda(x-t)^{\alpha}\} f(t)dt = g(x) \qquad (22)$$

and

$$f(x) = \int\limits_x^b (t-x)^{r-\beta-1} E_{\alpha,r-\beta}^{-\gamma,q}\left\{\lambda(t-x)^{\alpha}\right\} I^{-r} g(t)dt$$

imply each other.

Owing to the similarity of the proof to Theorem 4.1 and Corollary 4.1, details are avoided.

***Remark* 5.1.** For $q = 1$, result of integral equations becomes an apparent special case due to Prabhakar [6].

***Remark* 5.2.** When $\alpha = 1$, and parameters are specialized, (22) reduces to the integral equations due to Srivastava [11, 12].

6. Some additional result for $E^{\gamma,q}_{\alpha,\beta}$.

Theorem 6.1. *If* $\alpha, \beta, \gamma \in C,\ \mathrm{Re}\,(\alpha), \mathrm{Re}\,(\beta), \mathrm{Re}\,(\gamma) > 0$, *and* $q \in N$, *then*

$$\left(qz\frac{d}{dz} + \gamma\right)E^{\gamma,q}_{\alpha,\beta}(z) = \gamma E^{\gamma+1,q}_{\alpha,\beta}(z). \tag{23}$$

Proof. We have

$$\left(qz\frac{d}{dz} + \gamma\right)E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}(nq+\gamma) = \sum_{n=0}^{\infty}\frac{(\gamma)(\gamma+1)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} = \gamma E^{\gamma+1,q}_{\alpha,\beta}(z).$$

***Remark* 6.1.** On setting $q = 1$ in (23), this gives result due to Prabhakar [6] and for $q = 1$, $\gamma = 1$ yields relations proved by Wiman [16].

7. Conclusion. Since long the integral equations with special functions are not studied, in this paper, we obtained some properties of integral equation through generalized Mittag-Leffler function. This work may be useful in the study on integral equation and special functions.

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References

1. R. Desai, I. A. Salehbhai, A. K. Shukla, *Note on generalized Mittag-Leffler function,* SpringerPlus, **5** (2016).
2. A. A. Kilbas, M. Saigo, *On Mittag-Leffler type function, fractional calculas operators and solutions of integral equations,* Integral Transforms and Spec. Functions, **4**, № 4, 355 – 370 (1996).
3. A. A. Kilbas, M. Saigo, R. K. Saxena, *Generalized Mittag-Leffler function and generalized fractional calculus operators,* Integral Transforms and Spec. Functions, **15**, № 1, 31 – 49 (2004).
4. K. S. Miller, B. Ross, *An introduction to fractional calculus and fractional differential equations*, Wiley, New York (1993).
5. G. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, C. R. Acad. Sci. Paris, **137**, 554 – 558 (1903).
6. T. R. Prabhakar, *A singular equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J., **19**, 7 – 15 (1971).
7. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives — theory and applications,* Gordon and Breach, Switzerland (1993).
8. R. K. Saxena, J. P. Chauhan, R. K. Jana, A. K. Shukla, *Further results on the generalized Mittag-Leffler function operator,* J. Inequal. and Appl., **2015**, № 75 (2015).
9. A. K. Shukla, J. C. Prajapati, *On a generalization of Mittag-Leffler function and its properties*, J. Math. Anal. and Appl., **336**, № 2, 797 – 811 (2007).
10. A. K. Shukla, J. C. Prajapati, *On a generalization of Mittag-Leffler function and generalized integral operator,* Math. Sci. Rec. J., **12**, № 12, 283 – 290 (2008).
11. K. N. Srivastava, *A class of integral equations involving Laguree polynomials as kernel*, Proc. Edinburgh Math. Soc., **15**, 33 – 36 (1966).
12. K. N. Srivastava, *On integral equations involving Whittaker's function*, Proc. Glasgow Math. Assoc., **7**, 125 – 127 (1966).
13. H. M. Srivastava, R. K. Saxena, *Operators of fractional integration and their applications,* Appl. Math. and Comput., **118**, 1 – 52 (2001).
14. H. M. Srivastava, Z. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. and Comput., **211**, № 1, 198 – 210 (2009).
15. Z. Tomovski, R. Hilfer, H. M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions,* Integral Transforms and Spec. Functions, **21**, № 11, 797 – 814 (2010).
16. A. Wiman, *Über den fundamentalsatz in der theorie der funktionen* $E(x)$, Acta Math., **29**, 191 – 201 (1905).
17. J. Wimp, *Two integral transform pairs involving hypergeometric functions,* Proc. Glasgow Math. Assoc., **7**, 42 – 44 (1965).

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