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CHENEY – SHARMA TYPE OPERATORS ON A TRIANGLE WITH TWO AND THREE CURVED EDGES

ОПЕРАТОРИ ТИПУ ЧЕЙНІ – ШАРМИ НА ТРИКУТНИКУ З ДВОМА ТА ТРЬОМА ВИГНУТИМИ СТОРОНАМИ

We construct some Cheney – Sharma type operators defined on a triangle with two and three curved edges, their product and Boolean sum. We study their interpolation properties and the degree of exactness.

Побудовано деякі оператори типу Чейні – Шарми, визначені на трикутнику з двома та трьома вигнутими сторонами, визначено їхній добуток і булеву суму. Також вивчено їхні інтерполяційні властивості та ступінь точності.

1. Introduction. There have been constructed interpolation operators of Lagrange, Hermite and Birkhoff type on a triangle with all straight sides, starting with the paper [5] of R. E. Barnhil, G. Birkhoff and W. J. Gordon, and in many others papers (see, e.g., [4, 6, 9, 14]). Then, were considered interpolation operators on triangles with curved sides (one, two or all curved sides), many of them in connection with their applications in computer aided geometric design and in finite element analysis (see, e.g, [1 – 3, 7, 8, 15, 20]).

Also the Bernstein-type operators were used as interpolation operators both on triangles with straight sides (see, e.g., [10, 13, 17 – 19]) and with curved sides (see, e.g., [11, 12]).

The aim of this paper is to construct some Cheney – Sharma type operators that have some interpolatory properties on a triangle with two and three curved edges. They are extension of the Cheney – Sharma type operators of second type, given by E. W. Cheney and A. Sharma in [16], to the case of a curved side. There will be studied the interpolation properties and degree of exactness.

Let $m \in \mathbb{N}$ and β a nonnegative parameter. The Cheney – Sharma operators of second kind $Q_m : C([0,1]) \to C([0,1])$, introduced in [16], are given by

$$(Q_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f\left(\frac{i}{m}\right), \tag{1.1}$$

$$q_{m,i}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}. \tag{1.2}$$

2. Triangle with all curved sides. ***2.1. Univariate operators.*** In [12], we have the triangle $\tilde{T}_h$ with all curved sides, which has the vertices $V_1 = (0,h)$, $V_2 = (h,0)$ and $V_3 = (0,0)$, and the three curved sides γ_1, γ_2 (along the coordinate axis), and γ_3 (opposite to the vertex V_3). γ_1 is defined by $(x, f_1(x))$, with $f_1(0) = f_1(h) = 0$, $f_1(x) \le 0$, for $x \in [0,h]$; γ_2 is defined by $(g_2(y), y)$ with $g_2(0) = g_2(h) = 0$, $g_2(y) \le 0$, for $y \in [0,h]$ and γ_3 is defined by the one-to-one functions f_3 and g_3, where g_3 is the inverse of the function f_3, i.e., $y = f_3(x)$ and $x = g_3(y)$ with $x, y \in [0,h]$ and $f_3(0) = g_3(0) = h$ (see Fig. 1).

Let F be a real-valued function defined on $\tilde{T}_h$ and $(x, f_1(x))$, $(x, f_3(x))$, respectively, $(g_2(y), y)$, $(g_3(y), y)$ the points in which the parallel lines to the coordinate axes, passing through the point

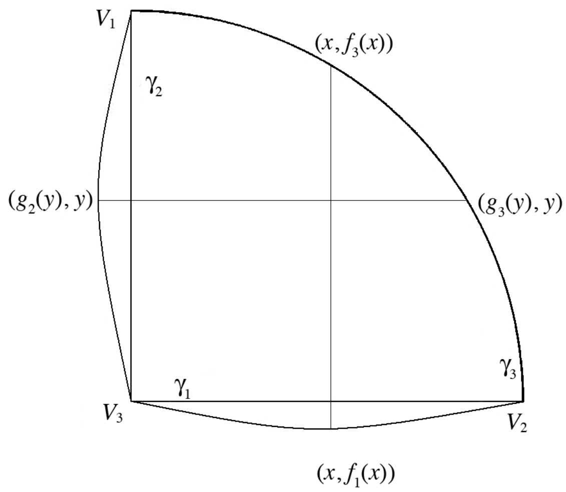

Fig. 1. Triangle $\tilde{T}_h$.

$(x, y) \in \tilde{T}_h$, intersect the sides γ_1, γ_2 and γ_3. We consider the uniform partitions of the intervals $[g_2(y), g_3(y)]$ and $[f_1(x), f_3(x)]$, $x, y \in [0, h]$:

$$\Delta_m^x = \left\{ g_2(y) + i\frac{g_3(y) - g_2(y)}{m} \;\middle|\; i = \overline{0, m} \right\},$$

respectively,

$$\Delta_n^y = \left\{ f_1(x) + j\frac{f_3(x) - f_1(x)}{n} \;\middle|\; j = \overline{0, n} \right\}.$$

For $m, n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}_+$, we consider the following extensions of the Cheney – Sharma operator given in (1.1):

$$\big(Q_m^x F\big)(x, y) = \sum_{i=0}^{m} q_{m,i}(x, y) F\left(g_2(y) + i\frac{g_3(y) - g_2(y)}{m}, y \right) \tag{2.1}$$

with

$$q_{m,i}(x, y) = \binom{m}{i} \frac{\dfrac{x - g_2(y)}{g_3(y) - g_2(y)} \left(\dfrac{x - g_2(y)}{g_3(y) - g_2(y)} + i\beta \right)^{i-1}}{(1 + m\beta)^{m-1}} \times$$

$$\times \left[1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)} \right] \left[1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)} + (m - i)\beta \right]^{m-i-1},$$

respectively,

$$\big(Q_n^y F\big)(x, y) = \sum_{j=0}^{n} q_{n,j}(x, y) F\left(x, f_1(x) + j\frac{f_3(x) - f_1(x)}{n} \right) \tag{2.2}$$

with

$$q_{n,j}(x,y) = \binom{n}{j} \frac{\dfrac{y-f_1(x)}{f_3(x)-f_1(x)} \left(\dfrac{y-f_1(x)}{f_3(x)-f_1(x)} + j\alpha \right)^{j-1}}{(1+n\alpha)^{n-1}} \times$$

$$\times \left[1 - \frac{y-f_1(x)}{f_3(x)-f_1(x)}\right] \left[1 - \frac{y-f_1(x)}{f_3(x)-f_1(x)} + (n-j)\alpha\right]^{n-j-1}.$$

Theorem 2.1. *If F is a real-valued function defined on $\tilde{T}_h$, then*:
1) $Q_m^x F = F$ *on* $\gamma_2 \cup \gamma_3$,
2) $Q_n^y F = F$ *on* $\gamma_1 \cup \gamma_3$,
3) $\big(Q_m^x e_{ij}\big)(x,y) = x^i y^j,\ i = 0,1,\ j \in \mathbb{N}$,
4) $\big(Q_n^y e_{ij}\big)(x,y) = x^i y^j,\ i \in \mathbb{N}, j = 0,1$.

Proof. 1. We write

$$\big(Q_m^x F\big)(x,y) = \frac{1}{(1+m\beta)^{m-1}} \Bigg\{ \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)}\right] \times$$

$$\times \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)} + m\beta\right]^{m-1} F(g_2(y),y) +$$

$$+ \frac{x-g_2(y)}{g_3(y)-g_2(y)} \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)}\right] \times$$

$$\times \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x-g_2(y)}{g_3(y)-g_2(y)} + i\beta \right)^{i-1} \times$$

$$\times \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)} + (m-i)\beta\right]^{m-i-1} \times$$

$$\times F\left(g_2(y) + i\frac{g_3(y)-g_2(y)}{m}, y\right) +$$

$$+ \frac{x-g_2(y)}{g_3(y)-g_2(y)} \left[\frac{x-g_2(y)}{g_3(y)-g_2(y)} + m\beta\right]^{m-1} F(g_3(y),y) \Bigg\}.$$

So,

$$\big(Q_m^x F\big)(g_2(y),y) = F(g_2(y),y),$$

$$\big(Q_m^x F\big)(g_3(y),y) = F(g_3(y),y).$$

2. We have

$$\big(Q_n^y F\big)(x,y) = \frac{1}{(1+n\alpha)^{n-1}} \Bigg\{ \left[1 - \frac{y-f_1(x)}{f_3(x)-f_1(x)}\right] \times$$

$$\times \left[1 - \frac{y-f_1(x)}{f_3(x)-f_1(x)} + n\alpha\right]^{n-1} F(x, f_1(x)) +$$

$$+\frac{y-f_1(x)}{f_3(x)-f_1(x)}\left[1-\frac{y-f_1(x)}{f_3(x)-f_1(x)}\right]\times$$

$$\times\sum_{j=0}^{n}\binom{n}{j}\left(\frac{y-f_1(x)}{f_3(x)-f_1(x)}+j\alpha\right)^{j-1}\times$$

$$\times\left[1-\frac{y-f_1(x)}{f_3(x)-f_1(x)}+(n-j)\alpha\right]^{n-j-1}\times$$

$$\times F\left(x, f_1(x)+j\frac{f_3(x)-f_1(x)}{n}\right)+\frac{y-f_1(x)}{f_3(x)-f_1(x)}\times$$

$$\times\left[\frac{y-f_1(x)}{f_3(x)-f_1(x)}+n\alpha\right]^{n-1}F(x,f_3(x))\Bigg\}.$$

So,

$$\left(Q_n^y F\right)(x,f_1(x))=F(x,f_1(x)),$$

$$\left(Q_n^y F\right)(x,f_3(x))=F(x,f_3(x)).$$

The proof for conditions 3 and 4 follows by the property $\operatorname{dex}(Q_m)=1$ (proved in [16]).
Theorem 2.1 is proved.

2.2. Product operators. Let $P_{mn}^1=Q_m^x Q_n^y$, respectively, $P_{nm}^2=Q_n^y Q_m^x$ be the product of the operators Q_m^x and Q_n^y.

We have

$$\left(P_{mn}^1 F\right)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}q_{m,i}(x,y)q_{n,j}(x_i,y)F\left(x_i, f_1(x_i)+j\frac{f_3(x_i)-f_1(x_i)}{n}\right),$$

$$x_i=g_2(y)+i\frac{g_3(y)-g_2(y)}{m},$$

and

$$\left(P_{nm}^2 F\right)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}q_{m,i}(x,y_j)q_{n,j}(x,y)F\left(g_2(y_j)+i\frac{g_3(y_j)-g_2(y_j)}{m}, y_j\right),$$

$$y_j=f_1(x)+j\frac{f_3(x)-f_1(x)}{n}.$$

Theorem 2.2. *If F is a real-valued function defined on $\tilde{T}_h$, then*:
1) $\left(P_{mn}^1 F\right)(V_3)=F(V_3)$, $\left(P_{mn}^1 F\right)=F$ *on* Γ_3,
2) $\left(P_{nm}^2 F\right)(V_3)=F(V_3)$, $\left(P_{nm}^2 F\right)=F$ *on* Γ_3.

Proof. The proof follows from the properties

$$\left(P_{mn}^1 F\right)(x,0)=\left(Q_m^x F\right)(x,0),$$

$$\left(P_{mn}^1 F\right)(0,y)=\left(Q_n^y F\right)(0,y),$$

$$\big(P^1_{mn}F\big)(x, f_3(x)) = F(x, f_3(x)), \quad x, y \in [0, h],$$

and

$$\big(P^2_{nm}F\big)(x, 0) = \big(Q^x_m F\big)(x, 0),$$

$$\big(P^2_{nm}F\big)(0, y) = \big(Q^y_n F\big)(0, y),$$

$$\big(P^2_{nm}F\big)(g_3(y), y) = F(g_3(y), y), \quad x, y \in [0, h],$$

which can be verified by a straightforward computation.

Theorem 2.2 is proved.

2.3. Boolean sum operators. We consider the Boolean sums of the operators Q^x_m and Q^y_n, i.e.,

$$S^1_{mn} = Q^x_m \oplus Q^y_n = Q^x_m + Q^y_n - Q^x_m Q^y_n,$$

respectively,

$$S^2_{nm} = Q^y_n \oplus Q^x_m = Q^y_n + Q^x_m - Q^y_n Q^x_m.$$

Theorem 2.3. *If F is a real-valued function defined on $\tilde{T}_h$, then*

$$S^1_{mn}\big|_{\partial\tilde{T}} = F\big|_{\partial\tilde{T}}$$

and

$$S^2_{nm}\big|_{\partial\tilde{T}} = F\big|_{\partial\tilde{T}}.$$

Proof. As

$$\big(S^1_{mn}F\big)(x, f_1(x)) = \big(Q^x_m F\big)(x, f_1(x)),$$

$$\big(S^1_{mn}F\big)(g_2(y), y) = \big(Q^y_n F\big)(g_2(y), y),$$

$$\big(S^1_{mn}F\big)(x, f_3(x)) = F(x, f_3(x)),$$

the proof follows.

3. Triangle with two curved sides. ***3.1.*** For $f_1(x) = 0,\ x \in [0, h]$, the triangle $\tilde{T}_h$ becomes a triangle with two curved sides (see Fig. 2).

We suppose that F is a real-valued function defined on $\tilde{T}_h$ and $(g_2(y), y),\ (g_3(y), y)$, respectively, $(x, 0),\ (x, f_3(x))$ the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in \tilde{T}_h$, intersect the sides $\gamma_1,\ \gamma_2$, and γ_3.

We consider the uniform partitions of the intervals $[g_2(y),\ g_3(y)]$ and $[0, f_3(x)],\ x, y \in [0, h]$:

$$\Delta^x_m = \left\{ g_2(y) + i\frac{g_3(y) - g_2(y)}{m} \;\middle|\; i = \overline{0, m} \right\},$$

respectively,

$$\Delta^y_n = \left\{ \frac{j}{n} f_3(x) \;\middle|\; j = \overline{0, n} \right\}.$$

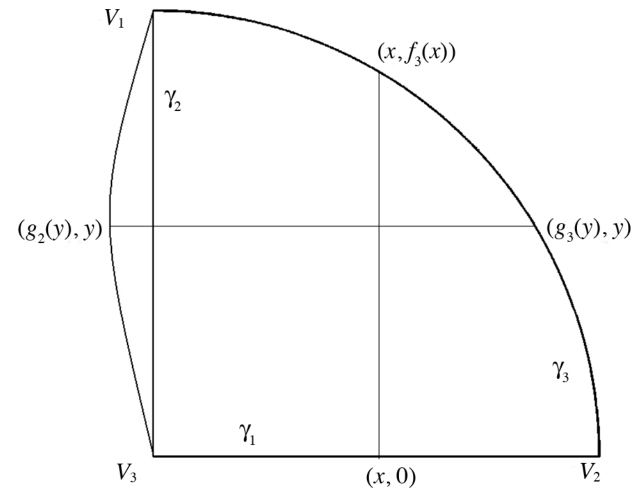

Fig. 2. Triangle $\tilde{T}_h$ with two curved sides.

For $m, n \in \mathbb{N},\ \alpha, \beta \in \mathbb{R}_+$, we have the Cheney – Sharma operator Q_m^x defined in (2.1) and, respectively,

$$\big(Q_n^y F\big)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, \frac{j}{n} f_3(x)\right)$$

with

$$q_{n,j}(x,y) = \binom{n}{j} \frac{\dfrac{y}{f_3(x)}\left(\dfrac{y}{f_3(x)} + j\alpha\right)^{j-1}\left(1 - \dfrac{y}{f_3(x)}\right)\left[1 - \dfrac{y}{f_3(x)} + (n-j)\alpha\right]^{n-j-1}}{(1+n\alpha)^{n-1}}.$$

Theorem 3.1. *If F is a real-valued function defined on $\tilde{T}_h$, then*:
1) $Q_m^x F = F$ *on* $\gamma_2 \cup \gamma_3$,
2) $Q_n^y F = F$ *on* $\gamma_1 \cup \gamma_3$,
3) $\big(Q_m^x e_{ij}\big)(x,y) = x^i y^j,\ i = 0, 1,\ j \in \mathbb{N}$,
4) $\big(Q_n^y e_{ij}\big)(x,y) = x^i y^j,\ i \in \mathbb{N},\ j = 0, 1$.

Proof. The proof for condition 1 is made in previous section.

2. We have

$$\big(Q_n^y F\big)(x,y) = \frac{1}{(1+n\alpha)^{n-1}} \Bigg\{ \left(1 - \frac{y}{f_3(x)}\right)\left[1 - \frac{y}{f_3(x)} + n\alpha\right]^{n-1} F(x,0) +$$

$$+ \frac{y}{f_3(x)}\left(1 - \frac{y}{f_3(x)}\right) \sum_{j=0}^{n} \binom{n}{j} \left(\frac{y}{f_3(x)} + j\alpha\right)^{j-1} \times$$

$$\times \left[1 - \frac{y}{f_3(x)} + (n-j)\alpha\right]^{n-j-1} F\left(x, \frac{j}{n} f_3(x)\right) +$$

$$+ \frac{y}{f_3(x)}\left(\frac{y}{f_3(x)} + n\alpha\right)^{n-1} F(x, f_3(x)) \Bigg\}.$$

So,

$$\big(Q_n^y F\big)(x,0) = F(x,0)),$$

$$\big(Q_n^y F\big)(x,f_3(x)) = F(x,f_3(x)).$$

Theorem 3.1 is proved.

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the product of the operators Q_m^x and Q_n^y. We have

$$\big(P_{mn}^1 F\big)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y) q_{n,j}(x_i,y) F\left(x_i, \frac{j}{n} f_3(x_i)\right),$$

$$x_i = g_2(y) + i\frac{g_3(y) - g_2(y)}{m},$$

and

$$\big(P_{nm}^2 F\big)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}\left(x, \frac{j}{n} f_3(x)\right) q_{n,j}(x,y)\times$$

$$\times F\left(g_2\left(\frac{j}{n} f_3(x)\right) + i\frac{g_3\left(\frac{j}{n} f_3(x)\right) - g_2\left(\frac{j}{n} f_3(x)\right)}{m}, \frac{j}{n} f_3(x)\right).$$

Theorem 3.2. *If F is a real-valued function defined on $\tilde{T}_h$, then*:
1) $\big(P_{mn}^1 F\big)(V_3) = F(V_3)$, $\big(P_{mn}^1 F\big) = F$ *on* Γ_3,
2) $\big(P_{nm}^1 F\big)(V_3) = F(V_3)$, $\big(P_{nm}^2 F\big) = F$ *on* Γ_3.

Proof. The proof follows from the properties

$$\big(P_{mn}^1 F\big)(x,0) = \big(Q_m^x F\big)(x,0),$$

$$\big(P_{mn}^1 F\big)(g_2(y),y) = \big(Q_n^y F\big)(g_2(y),y),$$

$$\big(P_{mn}^1 F\big)(x,f_3(x)) = F(x,f_3(x)), \quad x,y \in [0,h],$$

and

$$\big(P_{nm}^2 F\big)(x,0) = \big(Q_m^x F\big)(x,0),$$

$$\big(P_{nm}^2 F\big)(g_2(y),y) = \big(Q_n^y F\big)(g_2(y),y),$$

$$\big(P_{nm}^2 F\big)(g_3(y),y) = F(g_3(y),y), \quad x,y \in [0,h],$$

which can be verified by a straightforward computation.

Theorem 3.2 is proved.

We consider the Boolean sums of the operators Q_m^x and Q_n^y, i.e.,

$$S_{mn}^1 = Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y,$$

respectively,

$$S_{nm}^2 = Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x.$$

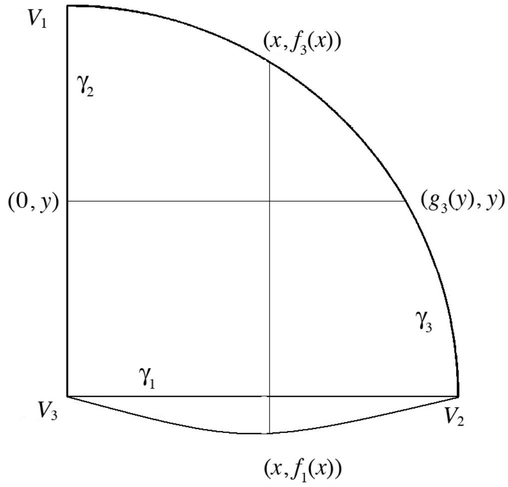

Fig. 3. Triangle $\tilde{T}_h$ with two curved sides.

Theorem 3.3. *If F is a real-valued function defined on $\tilde{T}_h$, then*

$$S^1_{mn}\big|_{\partial\tilde{T}} = F|_{\partial\tilde{T}}$$

and

$$S^2_{nm}\big|_{\partial\tilde{T}} = F|_{\partial\tilde{T}}.$$

Proof. As

$$\big(S^1_{mn}F\big)(x,0) = \big(Q^x_m F\big)(x,0),$$

$$\big(S^1_{mn}F\big)(g_2(y),y) = \big(Q^y_n F\big)(g_2(y),y),$$

$$\big(S^1_{mn}F\big)(x,f_3(x)) = F(x,f_3(x)),$$

the proof follows.

3.2. For $g_2(y) = 0,\ y \in [0,h]$, the triangle $\tilde{T}_h$ also becomes a triangle with two curved sides (see Fig. 3).

Also, we suppose that F is a real-valued function defined on $\tilde{T}_h$ and $(0,y)$, $(g_3(y),y)$, respectively, $(x,f_1(x))$, $(x,f_3(x))$ are the points in which the parallel lines to the coordinate axes, passing through the point $(x,y) \in \tilde{T}_h$, intersect the sides γ_1, γ_2, and γ_3.

We consider the uniform partitions of the intervals $[0, g_3(y)]$ and $[f_1(x), f_3(x)]$, $x, y \in [0,h]$:

$$\Delta^x_m = \left\{ \frac{i}{m} g_3(y) \;\middle|\; i = \overline{0,m} \right\},$$

respectively,

$$\Delta^y_n = \left\{ f_1(x) + j\frac{f_3(x) - f_1(x)}{n} \;\middle|\; j = \overline{0,n} \right\}.$$

For $m, n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}_+$, we have

$$\big(Q_m^x F\big)(x,y) = \sum_{i=0}^{m} q_{m,i}(x,y) F\left(\frac{i}{m} g_3(y), y\right)$$

with

$$q_{m,i}(x,y) = \binom{m}{i} \frac{\dfrac{x}{g_3(y)}\left(\dfrac{x}{g_3(y)} + i\beta\right)^{i-1}\left(1 - \dfrac{x}{g_3(y)}\right)\left[1 - \dfrac{x}{g_3(y)} + (m-i)\beta\right]^{m-i-1}}{(1+m\beta)^{m-1}},$$

respectively, the operator Q_n^y from (2.2).

Theorem 3.4. *If F is a real-valued function defined on $\tilde{T}_h$, then*:

1) $Q_m^x F = F$ *on* $\gamma_2 \cup \gamma_3$,

2) $Q_n^y F = F$ *on* $\gamma_1 \cup \gamma_3$,

3) $\big(Q_m^x e_{ij}\big)(x,y) = x^i y^j,\ i = 0,1,\ j \in \mathbb{N}$,

4) $\big(Q_n^y e_{ij}\big)(x,y) = x^i y^j,\ i \in \mathbb{N},\ j = 0,1$.

Proof. 1. We have

$$\big(Q_m^x F\big)(x,y) = \frac{1}{(1+m\beta)^{m-1}}\left\{\left(1 - \frac{x}{g_3(y)}\right)\left[1 - \frac{x}{g_3(y)} + m\beta\right]^{m-1} F(0,y) + \right.$$

$$+ \frac{x}{g_3(y)}\left(1 - \frac{x}{g_3(y)}\right)\sum_{i=0}^{m}\binom{m}{i}\left(\frac{x}{g_3(y)} + i\beta\right)^{i-1} \times$$

$$\times\left[1 - \frac{x}{g_3(y)} + (m-1)\beta\right]^{m-i-1} F\left(\frac{i}{m} g_3(y), y\right) +$$

$$\left. + \frac{x}{g_3(y)}\left(\frac{x}{g_3(y)} + m\beta\right)^{m-1} F(g_3(y), y)\right\}.$$

So,

$$\big(Q_m^x F\big)(0,y) = F(0,y)),$$

$$\big(Q_m^x F\big)(g_3(y),y) = F(g_3(y),y).$$

The proof for condition 2 is made in previous section.

The product operators will be

$$\big(P_{mn}^1 F\big)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y) q_{n,j}\left(\frac{i}{m} g_3(y), y\right) \times$$

$$\times F\left(\frac{i}{m} g_3(y), f_1\left(\frac{i}{m} g_3(y)\right) + j\frac{f_3\left(\frac{i}{m} g_3(y)\right) - f_1\left(\frac{i}{m} g_y(x)\right)}{n}\right)$$

and

$$\big(P^2_{nm}F\big)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}q_{m,i}(x,y_j)q_{n,j}(x,y)F\left(\frac{i}{m}g_3(y),y_j\right),$$

$$y_j=f_1(x)+j\,\frac{f_3(x)-f_1(x)}{n}.$$

Theorem 3.5. *If F is a real-valued function defined on* $\tilde{T}_h$, *then*:
1) $\big(P^1_{mn}F\big)(V_3)=F(V_3),\ \big(P^1_{mn}F\big)=F$ *on* Γ_3,
2) $\big(P^1_{nm}F\big)(V_3)=F(V_3),\ \big(P^2_{nm}F\big)=F$ *on* Γ_3.

Proof. The proof follows from the properties

$$\big(P^1_{mn}F\big)(x,f_1(x))=\big(Q^x_mF\big)(x,f_1(x)),$$
$$\big(P^1_{mn}F\big)(0,y)=\big(Q^y_nF\big)(0,y),$$
$$\big(P^1_{mn}F\big)(x,f_3(x))=F(x,f_3(x)),\quad x,y\in[0,h],$$

and

$$\big(P^2_{nm}F\big)(x,f_1(x))=\big(Q^x_mF\big)(x,f_1(x)),$$
$$\big(P^2_{nm}F\big)(0,y)=\big(Q^y_nF\big)(0,y),$$
$$\big(P^2_{nm}F\big)(g_3(y),y)=F(g_3(y),y),\quad x,y\in[0,h],$$

which can be verified by a straightforward computation.

For the Boolean sums we have the following theorem.

Theorem 3.6. *If* F *is a real-valued function defined on* $\tilde{T}_h$, *then*

$$S^1_{mn}\big|_{\partial\tilde{T}}=F|_{\partial\tilde{T}}$$

and

$$S^2_{nm}\big|_{\partial\tilde{T}}=F|_{\partial\tilde{T}}.$$

Proof. As

$$\big(S^1_{mn}F\big)(x,f_1(x))=\big(Q^x_mF\big)(x,f_1(x)),$$
$$\big(S^1_{mn}F\big)(0,y)=\big(Q^y_nF\big)(0,y),$$
$$\big(S^1_{mn}F\big)(x,f_3(x))=F(x,f_3(x)),$$

the proof follows.

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