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I. Ali (School Basic Sci., Indian Inst. Technology Indore, Simrol, India)

BOUNDS FOR THE RIGHT SPECTRAL RADIUS OF QUATERNIONIC MATRICES *

ГРАНИЧНІ ОЦІНКИ ДЛЯ ПРАВОГО СПЕКТРАЛЬНОГО РАДІУСА МАТРИЦЬ КВАТЕРНІОНІВ

In this paper, we present bounds for the sum of the moduli of right eigenvalues of a quaternionic matrix. As a consequence, we obtain bounds for the right spectral radius of a quaternionic matrix. We also present a minimal ball in 4D spaces which contains all the Geršgorin balls of a quaternionic matrix. As an application, we introduce the estimation for the right eigenvalues of quaternionic matrices in the minimal ball. Finally, we suggest some numerical examples to illustrate of our results.

Знайдено граничні оцінки для сум модулів правих власних значень кватерніонних матриць. Як наслідок отримано оцінки для правого спектрального радіуса таких матриць. У чотиривимірних просторах знайдено мінімальний шар, який містить всі шари Гершгоріна матриці кватерніонів. Як застосування запропоновано оцінку для правих власних значень матриць кватерніонів. Також наведено приклади для ілюстрації цих результатів.

1. Introduction. The problems over a quaternion division algebra have received much attention in the literature due to their applications in pure and applied sciences, such as the quantum physics, control theory, altitude control, computer graphics and signal processing (see, for example, [1, 2, 4 – 6, 12, 14, 20 – 22] and the references therein). There are many research paper published on the location and estimation of the left and right eigenvalues of a quaternionic matrix [8, 16, 20, 22, 23]. The stability of linear difference/differential equations with quaternionic matrix coefficients is based on the location of right eigenvalues of their corresponding quaternionic block matrices [10, 11, 15]. The upper bound for the left and right spectral radius of a quaternionic matrix has proposed by F. Zhang [22] in terms of the operator norm of a quaternionic matrix. Bounds for the sum of the left eigenvalues norms are derived with the help of localization theorems for left eigenvalues of a quaternionic matrix [8]. The first attempts to locate the zeros of quaternionic polynomials were given by G. Opfer [9] by direct calculation.

In the first part of the paper, we present bounds for the sum of the absolute values of right eigenvalues of a quaternionic matrix. We further discuss bounds for the right spectral radius of a quaternionic matrix by applying the above theory. In the second part of the paper, we provide a minimal ball which contains all the Geršgorin balls of a quaternionic matrix. Then we give localization theorems for right eigenvalues of a quaternionic matrix with the help of the above minimal ball.

The paper is organized as follows. Section 2 reviews some existing results. Section 3 discusses upper bounds for the sum of the right eigenvalues norms and the right spectral radius of a quaternionic

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matrix. Finally, Section 4 presents a minimal ball and location for right eigenvalues of a quaternionic matrix.

2. Preliminaries. Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The set of real quaternions is defined by

$$\mathbb{H} := \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \colon q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. This relation implies that $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The conjugate of $q \in \mathbb{H}$ is $\overline{q} := q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ and the modulus of q is $|q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. $\Im(a)$ denotes the imaginary part of $a \in \mathbb{C}$. The real part of a quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as $\Re(q) = q_0$. Let $p, q \in \mathbb{H}$. Then (a) $|q| = |\overline{q}|$ and $\overline{pq} = \overline{q}\,\overline{p}$; (b) $|pq| = |qp| = |p||q|$; (c) $\mathbf{j}c = \overline{c}\mathbf{j}$ or $\mathbf{j}c\overline{\mathbf{j}} = \overline{c}$ for every $c \in \mathbb{C}$; (d) $p^{-1} = \dfrac{\overline{p}}{|p|^2}$ if $p \neq 0$, and $|\rho^{-1}p\rho| = |p|$ for all $\rho \in \mathbb{H} \setminus \{0\}$.

The collection of all n-column vectors with elements in $\mathbb{H}$ is denoted by $\mathbb{H}^n$. For $x \in \mathcal{K}^n$, where $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the transpose of x is x^T. If $x = [x_1, \ldots, x_n]^T$, then the conjugate of x is defined as $\overline{x} = [\overline{x_1}, \ldots, \overline{x_n}]^T$ and the conjugate transpose of x is defined as $x^H = [\overline{x_1}, \ldots, \overline{x_n}]$. For $x, y \in \mathbb{H}^n$, the inner product is defined as $\langle x, y\rangle := y^H x$ and the norm of x is defined as $\|x\| := \sqrt{\langle x, x\rangle}$. The sets of $m \times n$ real, complex, and quaternionic matrices are denoted by $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$, and $M_{m\times n}(\mathbb{H})$, respectively. When $m = n$, these sets are denoted by $M_n(\mathcal{K})$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

For $A \in M_{m\times n}(\mathcal{K})$, the conjugate, transpose, and conjugate transpose of A are defined as $\overline{A} = (\overline{a_{ij}}) \in M_{m\times n}(\mathcal{K})$, $A^T = (a_{ji}) \in M_{n\times m}(\mathcal{K})$, and $A^H = (\overline{A})^T \in M_{n\times m}(\mathcal{K})$, respectively. A square matrix $A \in M_n(\mathbb{H})$ is said to be Hermitian if $A^H = A$. We define the Frobenius norm on $A \in M_n(\mathbb{H})$ by

$$\|A\|_F := \left(\operatorname{trace} A^H A\right)^{1/2}.$$

Let $p, q \in \mathbb{H}$. Then p and q are said to be similar, denoted by $p \sim q$, if

$$p \sim q \Leftrightarrow \exists\, 0 \neq r \in \mathbb{H} \quad \text{such that} \quad p = r^{-1}qr. \tag{1}$$

The set

$$[p] := \{u \in \mathbb{H} \colon u = \rho^{-1}p\rho \ \text{ for all } \ 0 \neq \rho \in \mathbb{H}\} \tag{2}$$

is called an equivalence class of $p \in \mathbb{H}$.

For any quaternionic matrix $A = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k} \in M_n(\mathbb{H}), B_k \in M_n(\mathbb{R}),\ k = 1, 2, 3, 4$, A can be uniquely expressed as $A = (B_1 + B_2\mathbf{i}) + (B_3 + B_4\mathbf{i})\mathbf{j} = A_1 + A_2\mathbf{j}, A_1, A_2 \in M_n(\mathbb{C})$. Define a function $\Psi \colon M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ by

$$\Psi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}.$$

The matrix Ψ_A is called the complex adjoint matrix of A. Unlike the complex matrix, there are two types of eigenvalues of a quaternionic matrix, namely left and right.

Definition 2.1. *Let $A \in M_n(\mathbb{H})$. Then the left, right, and the standard right eigenvalues, respectively, are given by*

$$\Lambda_l(A) := \{\lambda \in \mathbb{H} \colon Ax = \lambda x \ \textit{for some nonzero} \ x \in \mathbb{H}^n\},$$

$$\Lambda_r(A) := \{\lambda \in \mathbb{H} \colon Ax = x\lambda \ \textit{for some nonzero} \ x \in \mathbb{H}^n\}$$

and

$$\Lambda_s(A) := \{\lambda \in \mathbb{C} \colon Ax = x\lambda \ \textit{for some nonzero} \ x \in \mathbb{H}^n, \ \Im(\lambda) \geq 0\}.$$

Definition 2.2. *Let $A \in M_n(\mathbb{H})$. Then the right spectral radius of A is defined as*

$$\rho_r(A) := \max\{|\lambda| \colon \lambda \in \Lambda_r(A)\}.$$

Definition 2.3. *Let $A \in M_n(\mathbb{H})$. Then A is said to be η-Hermitian if $A = (A^\eta)^H$, where $A^\eta = \eta^H A \eta$ and $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.*

Definition 2.4. *A matrix $A \in M_n(\mathbb{H})$ is said to be invertible if there exists $B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.*

Definition 2.5. *Let $A \in M_n(\mathbb{H})$. Then A is said to be a central closed matrix if there exists an invertible matrix T such that $T^{-1}AT = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, where $\mu_i \in \mathbb{R}$, $1 \leq i \leq n$.*

We recall the following results for the development of our theory.

Lemma 2.1 [19]**.** *Let $A \in M_n(\mathbb{H})$ be a central closed matrix and suppose that the standard right eigenvalues of A are $\mu_1, \mu_2, \ldots, \mu_n$. Then* $\mathrm{trace}(A) = \sum_{i=1}^n \mu_i$.

Theorem 2.1 ([13], Theorem 3.1)**.** *Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a central closed matrix. Then all the standard right eigenvalues of A are located in the following ball*:

$$G(A) = \left\{ z \in \mathbb{H} \colon \left| z - \frac{\mathrm{trace}(A)}{n} \right| \leq \xi(A) \right\},$$

where

$$\xi(A) = \sqrt{\frac{n-1}{2n-1}} \sqrt{\frac{n-1}{n}\eta + \sqrt{\eta^2 - \frac{2n-1}{n^2} \, F(A)}},$$

$$\eta = \left(\|A\|_F^2 - \left| \frac{\mathrm{trace}(A)}{n} \right|^2 \right), \quad F(A) = \|AA^H\|_F^2 - \|A^2\|_F^2.$$

Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then define the deleted absolute row sums of A as

$$r_i(A) := \sum_{j=1, \, j \neq i}^{n} |a_{ij}|, \quad 1 \leq i \leq n.$$

We also define the n Geršgorin balls as follows:

$$G_i(A) := \{z \in \mathbb{H} \colon |z - a_{ii}| \leq r_i(A)\}, \quad 1 \leq i \leq n.$$

3. Bounds for the sum of the right eigenvalue norms for a quaternionic matrix. In view of Definition 2.2, one can compute the right spectral radius of a quaternionic matrix by means of the following simple procedure.

Let $A \in M_n(\mathbb{H})$.

Factorize $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in M_n(\mathbb{C})$.

Write the complex adjoint matrix $\Psi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$.

Find $\Lambda(\Psi_A) = \{\mu_1, \ldots, \mu_n, \overline{\mu_1}, \ldots, \overline{\mu_n}\}$.

Divide $\Lambda(\Psi_A)$ into two sets Λ_1, Λ_2 such that $\Lambda_1 \cup \Lambda_2 = \Lambda(\Psi_A)$, the eigenvalues with positive imaginary part belong to Λ_1 and the elements of Λ_2 are the conjugates of the ones of Λ_1. Consequently, from Definition 2.1, Λ_1 is the set of the standard right eigenvalues of A.

Then the right spectral radius of A is given as

$$\rho_r(A) := \max_{1 \leq i \leq n} \{|\mu_i| : \mu_i \in \Lambda_s(A) := \Lambda_1\}. \tag{3}$$

One can also compute all the right eigenvalues of a quaternionic matrix with the help of standard right eigenvalues of that matrix. In view of the above points, we can obtain the right spectrum of A as follows:

$$\Lambda_r(A) = \bigcup_{i=1}^{n} [\mu_i], \quad \mu_i \in \Lambda_s(A) := \Lambda_1.$$

The Geršgorin theorem for right eigenvalues of a quaternionic matrix has proved by F. Zhang [22] which is as follows.

Lemma 3.1 ([22], Theorem 7). *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. For every right eigenvalue μ of A there exists a nonzero quaternion β such that $\beta^{-1}\mu\beta$ (which is also a right eigenvalue) is contained in the union of n Geršgorin balls $G_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)\}$, $1 \leq i \leq n$, that is,*

$$\left\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\right\} \cap \left(\bigcup_{i=1}^{n} G_i(A)\right) \neq \varnothing.$$

In particular, when μ is real, it is contained in a Geršgorin ball.

First, in this section, we derive bounds for the sum of the absolute values of right eigenvalue of a quaternionic matrix with the help of Theorem 3.1 which are as follows.

Theorem 3.1. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. If λ_i $(1 \leq i \leq n)$ are right eigenvalues of A such that they lie within n distinct Geršgorin balls $G_i(A)$, respectively, then we have the following inequalities*:

$$\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|, \tag{4}$$

$$\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij}| + \sum_{i=1}^{n} \left(\left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)|. \tag{5}$$

Moreover, if μ_i, $1 \leq i \leq n$, are standard right eigenvalues of A, then we have the following inequalities:

$$\sum_{i=1}^{n} |\mu_i| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|, \tag{6}$$

$$\sum_{i=1}^{n} |\mu_i| \leq \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij}| + \sum_{i=1}^{n} \left(\left| a_{ii} - \frac{\mathrm{trace}(A)}{n} \right| \right) + |\mathrm{trace}(A)| \,. \tag{7}$$

Proof. *Inequality* (4): Since λ_i, $1 \leq i \leq n$, are n right eigenvalues of A such that they lie within n distinct Geršgorin balls $G_i(A) = \{z \in \mathbb{H} \colon |z - a_{ii}| \leq r_i(A)\}$, respectively. Now, without loss of generality, we can consider as follows:

$$\lambda_i \in G_i(A) \quad \text{and} \quad G_i(A) \neq G_j(A), \quad 1 \leq i, j \leq n, \quad i \neq j.$$

By applying Theorem 3.1, we have

$$|\lambda_i - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad 1 \leq i \leq n.$$

This implies that

$$|\lambda_i| \leq |a_{ii}| + \sum_{j=1, j\neq i}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{ij}|.$$

Therefore,

$$\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

Inequality (5): The Geršgorin balls $G_i(A) = \{z \in \mathbb{H} \colon |z - a_{ii}| \leq r_i(A)\}$, $1 \leq i \leq n$, have the centres a_{ii}, respectively. Based on the particle and centre gravity theorem, each $G_i(A)$, $1 \leq i \leq n$, can be considered as a particle or a rigid body. Then the centre of all particle or rigid body is

$$\frac{1}{n} \sum_{i=1}^{n} a_{ii} = \frac{\mathrm{trace}(A)}{n}.$$

Now, we get

$$\left| \lambda_i - \frac{\mathrm{trace}(A)}{n} \right| = \left| \lambda_i - a_{ii} + a_{ii} - \frac{\mathrm{trace}(A)}{n} \right| \leq |\lambda_i - a_{ii}| + \left| a_{ii} - \frac{\mathrm{trace}(A)}{n} \right|,$$

$$\left| \lambda_i - \frac{\mathrm{trace}(A)}{n} \right| \leq r_i(A) + \left| a_{ii} - \frac{\mathrm{trace}(A)}{n} \right|.$$

This implies that

$$|\lambda_i| \leq r_i(A) + \left| a_{ii} - \frac{\mathrm{trace}(A)}{n} \right| + \left| \frac{\mathrm{trace}(A)}{n} \right|.$$

Therefore, we obtain

$$\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} r_i(A) + \sum_{i=1}^{n} \left(\left| a_{ii} - \frac{\mathrm{trace}(A)}{n} \right| \right) + \sum_{i=1}^{n} \left| \frac{\mathrm{trace}(A)}{n} \right|.$$

Thus, we have the following desired result:

$$\sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij}| + \sum_{i=1}^{n} \left(\left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)| \,.$$

Inequality (6): Since $\lambda_i,\ 1 \leq i \leq n$, are right eigenvalues of A, so from Lemma 3 of [3] there exist $\rho_i \in \mathbb{H} \setminus \{0\},\ 1 \leq i \leq n$, such that $\rho_i^{-1}\lambda_i\rho_i = \mu_i$, where $\mu_,\ 1 \leq i \leq n$, are the standard right eigenvalues of A. This implies that $\lambda_i = \rho_i\mu_i\rho_i^{-1}$. From the inequality (4), we get

$$\sum_{i=1}^{n} |\rho_i\mu_i\rho_i^{-1}| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|.$$

Since $|\rho_i\mu_i\rho_i^{-1}| = |\mu_i|,\ 1 \leq i \leq n$. Therefore, $\sum_{i=1}^{n} |\mu_i| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|$.

Similarly, from the inequality (5), we have the desired inequality (7), that is,

$$\sum_{i=1}^{n} |\mu_i| \leq \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij}| + \sum_{i=1}^{n} \left(\left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)| \,.$$

Theorem 3.1 is proved.

Now, from Theorem 3.1, we make the following observations for upper bounds of the right spectral radius:

From (3), we can easily see that $\rho_r(A) \leq \sum_{i=1}^{n} |\mu_i|,\ \mu_i \in \Lambda_s(A)$. By applying inequality (6), we have upper bound of the right spectral radius of A which is as follows:

$$\rho_r(A) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|.$$

By applying inequality (7), we obtain upper bound of the right spectral radius of A in terms of the trace of A which is as follows:

$$\rho_r(A) \leq \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ij}| + \sum_{i=1}^{n} \left(\left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right) + |\text{trace}(A)| \,.$$

The upper bound of the left and right spectral radius of a quaternionic matrix has derived in [22] in terms of the spectral norm of the quaternionic matrix. However, the spectral norm of a quaternionic matrix is expensive to compute. Here, our bounds are in terms of moduli of entries and trace of a quaternionic matrix which are much easier to compute than the spectral norm.

W. Junliang and Z. Yan [8] have been given Schur's inequality for right eigenvalues of a quaternionic matrix which is as follows.

Lemma 3.2 ([8], Corollary 2.1). *Let* $A := (a_{ij}) \in M_n(\mathbb{H})$ *and* $\mu_1, \mu_2, \ldots, \mu_n$ *be standard right eigenvalues of* A. *Then we have the following inequality:*

$$\sum_{i=1}^{n} |\mu_i|^2 \leq \|A\|_F^2 := \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2. \tag{8}$$

From Lemma 3.2, it is clear that the inequalities (6) and (7) are different from the inequality (8). As applications of Theorem 3.1 and Lemma 3.2, we derive sharper estimations of the right spectral radius in terms of the trace and the Frobenius norm.

Let μ_m be a standard right eigenvalue of maximum modulus. Then, from inequality (6), we can be written as

$$|\mu_m| \le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| - \sum_{i\neq m}^{n}|\mu_i|.$$

By applying the arithmetic-geometric mean inequality, we obtain

$$|\mu_m| \le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| - (n-1)\prod_{i\neq m}^{n}|\mu_i|^{\frac{1}{n-1}} \le$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| - (n-1)\frac{\prod_{i=1}^{n}|\mu_i|^{\frac{1}{n-1}}}{|\mu_m|^{\frac{1}{n-1}}}.$$

From [21] (Theorem 8.1(4)), we have

$$|\mu_m| \le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| - (n-1)\frac{(|A|_q)^{\frac{1}{2n-2}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|\right)^{\frac{1}{n-1}}},$$

where $|A|_q$ is the q-determinant of the quaternionic matrix A. The definition of the right spectral radius gives

$$\rho_r(A) \le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| - (n-1)\frac{(|A|_q)^{\frac{1}{2n-2}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|\right)^{\frac{1}{n-1}}}. \tag{9}$$

Similarly, inequality (7) yields

$$\rho_r(A) \le \xi_1 - (n-1)\frac{(|A|_q)^{\frac{1}{2n-2}}}{(\xi_1)^{\frac{1}{n-1}}}, \tag{10}$$

where $\xi_1 = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}|a_{ij}| + \sum_{i=1}^{n}\left(\left|a_{ii} - \dfrac{\operatorname{trace}(A)}{n}\right|\right) + |\operatorname{trace}(A)|$.

From the above procedure, Lemma 3.2 yields

$$\rho_r(A) \le \left[\|A\|_F^2 - (n-1)\left(\frac{|A|_q}{\|A\|_F^2}\right)^{\frac{1}{n-1}}\right]^{1/2}. \tag{11}$$

Let $A \in M_n(\mathbb{H})$. Then the discrete-time quaternionic system $w(t+1) = Aw(t)$ is said to be asymptotically stable if and only if $\Lambda_r(A) \subset S_{\mathbb{H}} = \{q \in \mathbb{H}\colon |q| < 1\}$. We present application of bounds of the right spectral radius for the stability of a discrete-time quaternionic system. From the above definition, if $\rho_r(A) < 1$, then the system $w(t+1) = Aw(t)$ is asymptotically stable.

We now give a numerical example to verify our theoretical results.

***Example* 3.1.** Let us consider a quaternionic matrix $A = \begin{bmatrix} 2+\mathbf{i} & \mathbf{k} & \mathbf{j} \\ 0 & -\mathbf{i} & \mathbf{j} \\ 0 & 0 & 1+\mathbf{j} \end{bmatrix}$. It is clear that $2+\mathbf{i}$, $-\mathbf{i}$ and $1+\mathbf{j}$ are three right eigenvalues of A. Thus the standard right eigenvalues of A are $2+\mathbf{i}$, $\mathbf{i}$ and $1+\mathbf{i}$. Here, we obtain

$$\sum_{i=1}^{3} |\lambda_i| = 1 + \sqrt{2} + \sqrt{5} \leq \sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}| = 4 + \sqrt{2} + \sqrt{5}.$$

Furthermore, we also get

$$\sum_{i=1}^{3} |\lambda_i| = 1 + \sqrt{2} + \sqrt{5} \leq 3 + \sqrt{10} + \frac{2\sqrt{19} + \sqrt{2}}{3} = \delta,$$

where $\delta = \sum_{i=1}^{3}\sum_{j=1, j\neq i}^{3} |a_{ij}| + \sum_{i=1}^{3}\left(\left|a_{ii} - \dfrac{\mathrm{trace}(A)}{3}\right|\right) + |\mathrm{trace}(A)|$. Thus, Theorem 3.1 is verified.

We next verify the results of the right spectral radius. The right spectral radius and the q-determinant of A are

$$\rho_r(A) = 2.2361 \quad \text{and} \quad |A|_q = 10,$$

respectively. From (9), we have

$$\sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}| - (2)\frac{(|A|_q)^{\frac{1}{4}}}{\left(\sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|\right)^{\frac{1}{2}}} = 6.3644.$$

Thus, the inequality (9) is verified. From (10), we obtain

$$\xi_1 - (2)\frac{(|A|_q)^{\frac{1}{4}}}{(\xi_1)^{\frac{1}{2}}} = 8.3881,$$

where $\xi_1 = \sum_{i=1}^{3}\sum_{j=1, j\neq i}^{3} |a_{ij}| + \sum_{i=1}^{3}\left(\left|a_{ii} - \dfrac{\mathrm{trace}(A)}{3}\right|\right) + |\mathrm{trace}(A)|$. Hence, the inequality (10) is verified. Since $\|A\|_F^2 = 11$, so, from (11), we get

$$\left[\|A\|_F^2 - (2)\left(\frac{|A|_q}{\|A\|_F^2}\right)^{\frac{1}{2}}\right]^{1/2} = 3.0155.$$

Finally, the inequality (11) is verified.

4. Location of right eigenvalues of a quaternionic matrix. Firstly, in this section, we find a minimal ball in 4D spaces which containing all the Geršgorin balls of a quaternionic matrix.

Theorem 4.1. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then there must be a minimal ball in 4D spaces containing all the Geršgorin balls of A:*

$$\eta(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{\mathrm{trace}(A)}{n} \right| \leq \max_{1\leq i\leq n}\left[r_i(A) + \left| a_{ii} - \frac{\mathrm{trace}(A)}{n} \right| \right] \right\}.$$

Proof. From Theorem 3.1, we have the following n Geršgorin balls for A:

$$G_i(A) = \{q \in \mathbb{H} \colon |q - a_{ii}| \le r_i(A)\}, \quad 1 \le i \le n.$$

Based on the particle and centre gravity theorem, each $G_i(A)$, $1 \le i \le n$, can be treated as a particle or a rigid body. Then the centre of all particles or rigid bodies is $\dfrac{1}{n}\sum_{i=1}^{n} a_{ii} = \dfrac{\text{trace}(A)}{n}$. To find the smallest ball, we set the following model:

$$\min \left| q - \frac{\text{trace}(A)}{n} \right| \quad \text{such that} \quad |q_i - a_{ii}| \le r_i(A), \quad 1 \le i \le n.$$

Since

$$\left| q_i - \frac{\text{trace}(A)}{n} \right| = \left| q_i - a_{ii} + a_{ii} - \frac{\text{trace}(A)}{n} \right| \le |q_i - a_{ii}| + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right|,$$

$$\left| q_i - \frac{\text{trace}(A)}{n} \right| \le r_i(A) + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right|,$$

the solution to the above model is that

$$\left| q - \frac{\text{trace}(A)}{n} \right| = \max_{1 \le i \le n} \left| q_i - \frac{\text{trace}(A)}{n} \right|,$$

$$\left| q - \frac{\text{trace}(A)}{n} \right| \le \max_{1 \le i \le n} \left[r_i(A) + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right] =: R.$$

Thus, all the Geršgorin balls of A must belong to the smallest ball with radius R and center at $\dfrac{\text{trace}(A)}{n}$. If we denote smallest ball by $\eta(A)$, then

$$\eta(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{\text{trace}(A)}{n} \right| \le \max_{1 \le i \le n} \left[r_i(A) + \left| a_{ii} - \frac{\text{trace}(A)}{n} \right| \right] \right\}.$$

Theorem 4.1 is proved.

Now, we turn to locate the right eigenvalues in the minimal ball $\eta(A)$. In fact, a right eigenvalue is not necessarily contained in a minimal ball $\eta(A)$. For example, consider a quaternionic matrix $A = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix}$. Here, $-\mathbf{i}$ is a right eigenvalue of A but it is not contained in minimal ball $\eta(A)$, that is,

$$-\mathbf{i} \notin \eta(A) := \left\{ q \in \mathbb{H} \colon \left| q - \frac{\mathbf{i} + \mathbf{j}}{2} \right| \le \frac{1}{2}\sqrt{2} \right\}.$$

Fortunately, we have the following theorem for right eigenvalues of a quaternionic matrix.

Theorem 4.2. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. For every right eigenvalue λ of A there exists a nonzero quaternion α such that $\alpha^{-1}\lambda\alpha$ (which is also a right eigenvalue) is contained in the minimal ball $\eta(A)$, that is,*

$$\{\alpha^{-1}\lambda\alpha \colon 0 \ne \alpha \in \mathbb{H}\} \cap \eta(A) \ne \varnothing.$$

Proof. The proof follows from Theorems 3.1 and 4.1.

By definitions, Hermitian and η-Hermitian matrices have all the real diagonal entries. η-Hermitian matrices arise widely in applications [7, 17, 18]. Thus, we present the following result.

Theorem 4.3. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and $a_{ii} \in \mathbb{R}$ for all i. Then all the right eigenvalues of A are contained in $\eta(A)$.*

Proof. Since all the right eigenvalues of a quaternionic matrix with all real diagonal entries are contained in the union of n Geršgorin balls $G_i(A),\ 1 \leq i \leq n$. Therefore, the proof follows from Theorem 4.1.

We now provide a numerical example to show the effectiveness of our result.

***Example* 4.1.** Let us consider a quaternionic matrix $A = \begin{bmatrix} 1 & \mathbf{j} & \mathbf{k} \\ 1+\mathbf{i} & 2 & \mathbf{i}+\mathbf{j} \\ 1-\mathbf{j} & \mathbf{j}+\mathbf{k} & 4 \end{bmatrix}$. Then the complex adjoint matrix of A is given as

$$\Psi_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & \mathbf{i} \\ 1+\mathbf{i} & 2 & \mathbf{i} & 0 & 0 & 1 \\ 1 & 0 & 4 & -1 & 1+\mathbf{i} & 0 \\ 0 & -1 & \mathbf{i} & 1 & 0 & 0 \\ 0 & 0 & -1 & 1-\mathbf{i} & 2 & -\mathbf{i} \\ 1 & -1+\mathbf{i} & 0 & 1 & 0 & 4 \end{bmatrix}.$$

The spectrum of Ψ_A is $\Lambda(\Psi_A) = \{3.7627 + 1.0148\mathbf{i},\ 2.6089 + 0.9307\mathbf{i},\ 0.6283 + 0.6807\mathbf{i},\ 3.7627 - 1.0148\mathbf{i},\ 2.6089 - 0.9307\mathbf{i},\ 0.6283 - 0.6807\mathbf{i}\}$. Therefore, the right spectrum of A is $\Lambda_r(A) = [3.7627 + 1.0148\mathbf{i}] \cup [2.6089 + 0.9307\mathbf{i}] \cup [0.6283 + 0.6807\mathbf{i}]$. From Theorem 4.1, we obtain

$$\eta(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{\operatorname{trace}(A)}{n} \right| \leq \max_{1 \leq i \leq n} \left[r_i(A) + \left| a_{ii} - \frac{\operatorname{trace}(A)}{n} \right| \right] \right\},$$

$$\eta(A) = \left\{ q \in \mathbb{H} : \left| q - \frac{7}{3} \right| \leq \max_{1 \leq i \leq n} \left[\frac{10}{3}, \frac{6\sqrt{2}+1}{3}, \frac{6\sqrt{2}+5}{3} \right] \right\},$$

$$\eta(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{7}{3} \right| \leq \frac{6\sqrt{2}+5}{3} \right\}.$$

By Theorem 4.3, we know that all the right eigenvalues of A should be contained in $\eta(A)$. It is clear that all the standard right eigenvalues $\mu_1 = 3.7627 + 1.0148\mathbf{i}$, $\mu_2 = 2.6089 + 0.9307\mathbf{i}$ and $\mu_3 = 0.6283 + 0.6807\mathbf{i}$ are contained in $\eta(A)$. Here, we can also easily see that

$$\left| \rho^{-1}\mu_1\rho - \frac{7}{3} \right| = \left| \mu_1 - \frac{7}{2} \right|, \quad \left| \alpha^{-1}\mu_2\alpha - \frac{7}{3} \right| = \left| \mu_2 - \frac{7}{2} \right|,$$

$$\left| \beta^{-1}\mu_3\beta - \frac{7}{3} \right| = \left| \mu_3 - \frac{7}{2} \right| \quad \forall \rho, \alpha, \beta \in \mathbb{H} \setminus \{0\}.$$

Hence all the right eigenvalues of A are contained in $\eta(A)$. Thus, Theorem 4.3 is verified.

We are now ready to establish some results on a quaternionic matrix. In general, similar quaternionic matrices may have different traces follows from the following example.

***Example* 4.2.** Let $A = \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ 0 & -\mathbf{i} \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{i} - \frac{1}{2}\mathbf{k} & \frac{1}{2}\mathbf{i} \\ -\frac{1}{2}\mathbf{i} & -\mathbf{i} - \frac{1}{2}\mathbf{k} \end{bmatrix}$. Then $B = U^H A U$, where $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathbf{i} \\ -\mathbf{j} & 1 \end{bmatrix}$ is an unitary matrix but $\operatorname{trace}(A) = 0$ while $\operatorname{trace}(B) = -\mathbf{k}$.

However, the following result is true.

Theorem 4.4. *Let* $A := (a_{ij}) \in M_n(\mathbb{H})$ *be a central closed matrix. If* A *is similar to quaternionic matrix* B. *Then* B *is also a central closed matrix. Moreover,* $\operatorname{trace}(A) = \operatorname{trace}(B)$.

Proof. Since A is similar to B, then there exists a nonsingular quaternion matrix P such that $A = PBP^{-1}$. Also A is central closed matrix, then there exists a nonsingular quaternionic matrix Q such that $A = QDQ^{-1}$, where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with the real standard right eigenvalues λ_i. From the above, we have

$$QDQ^{-1} = PBP^{-1} \Rightarrow B = P^{-1}DQ^{-1}P.$$

Setting $P^{-1}Q = T$, then we obtain $B = TDT^{-1}$. Hence, B is also a central closed matrix.

For second part: From Lemma 2.1, we have $\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i = \operatorname{trace}(D)$. Moreover, $\operatorname{trace}(B) = \sum_{i=1}^{n} \lambda_i = \operatorname{trace}(D)$. It follows from the above that $\operatorname{trace}(A) = \operatorname{trace}(B)$.

Theorem 4.5. *Let* $A := (a_{ij}) \in M_n(\mathbb{H})$ *be central closed matrix. If* $B_1, B_2, \ldots, B_s$ *are similar to* A, *then we can derive a minimal ball in* $4D$ *spaces which contain all Geršgorin balls of at least one matrix among* $B_1, B_2, \ldots, B_s$ *and* A. *That is,*

$$G_{\min}(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{\operatorname{trace}(A)}{n} \right| \le \min_{1\le k\le s} \left\{ \max_{1\le i\le n} \left[r_i(A) + \left| a_{ii} - \frac{\operatorname{trace}(A)}{n} \right| \right], \right.\right.$$
$$\left.\left. \max_{1\le i\le n} \left[r_i(B_k) + \left| b_{ii} - \frac{\operatorname{trace}(A)}{n} \right| \right] \right\} \right\}.$$

Proof. Since $B_1, B_2, \ldots, B_s$ are similar to A, then from Theorem 4.4, we have $\operatorname{trace}(B_k) = \operatorname{trace}(A)$, $1 \le k \le s$. It reveals that the balls $\eta(B_1), \eta(B_2), \ldots, \eta(B_s)$ and $\eta(A)$ are concentric balls whose centers at $\dfrac{\operatorname{trace}(A)}{n}$. Therefore, it only needs us to find the ball with the smallest radius from $s+1$ concentric balls.

If we denote $G_{\min}(A)$ a ball with the smallest radius, then we have

$$G_{\min}(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{\operatorname{trace}(A)}{n} \right| \le \min_{1\le k\le s} \left\{ \max_{1\le i\le n} \left[r_i(A) + \left| a_{ii} - \frac{\operatorname{trace}(A)}{n} \right| \right], \right.\right.$$
$$\left.\left. \max_{1\le i\le n} \left[r_i(B_k) + \left| b_{ii} - \frac{\operatorname{trace}(A)}{n} \right| \right] \right\} \right\}.$$

From the above it is clear that the radius and center of the smallest ball can be determined by entries of $B_1, B_2, \ldots, B_s$ and A.

Theorem 4.6. *Let* $A := (a_{ij}) \in M_n(\mathbb{H})$ *be central closed matrix and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be* n *right eigenvalues of* A. *If* $B_1, B_2, \ldots, B_s$ *are similar to* A, *then* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are contained in* $G_{\min}(A)$.

Proof. Since central closed quaternionic matrices have all real right eigenvalues, then, from Theorem 3.1, all the right eigenvalues of A are contained in $\bigcup_{i=1}^{n} G_i(A)$. Therefore, from Theorem 4.5, we have the required result.

Example **4.3.** Consider a central closed quaternionic matrix $A = \begin{bmatrix} 1 & -\mathbf{i} & -\mathbf{j} & \mathbf{k} \\ \mathbf{i} & 1 & -2\mathbf{k} & \mathbf{j} \\ \mathbf{j} & 2\mathbf{k} & 7 & -\mathbf{i} \\ -\mathbf{k} & -\mathbf{j} & \mathbf{i} & 1 \end{bmatrix}$.

Then the complex adjoint matrix of A is given as

$$\Psi_A = \begin{bmatrix} 1 & -\mathbf{i} & 0 & 0 & 0 & 0 & -1 & \mathbf{i} \\ \mathbf{i} & 1 & 0 & 0 & 0 & 1 & -2\mathbf{i} & 1 \\ 0 & 0 & 7 & -\mathbf{i} & 1 & 2\mathbf{i} & 0 & 0 \\ 0 & 0 & \mathbf{i} & 1 & -\mathbf{i} & -1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{i} & 1 & \mathbf{i} & 0 & 0 \\ 0 & -1 & -2\mathbf{i} & -1 & -\mathbf{i} & 1 & 0 & 0 \\ -1 & 2\mathbf{i} & 0 & 0 & 0 & 0 & 7 & \mathbf{i} \\ -\mathbf{i} & 1 & 0 & 0 & 0 & 0 & -\mathbf{i} & 1 \end{bmatrix}.$$

The spectrum of Ψ_A is $\Lambda(\Psi_A) = \{-1, 1, 2, 8\}$. Consequently, the right spectrum of A is $\Lambda_r(A) = \{-1, 1, 2, 8\}$. Since A is a central closed matrix, so A is similar to the diagonal matrix $D = \operatorname{diag}(-1, 1, 2, 8)$. From Theorem 4.6, we get

$$G_{\min}(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{5}{2} \right| \le \min \left[\max_{1 \le i \le n} \left\{ \frac{9}{2}, \frac{11}{2}, 9 \right\}, \max_{1 \le i \le n} \left\{ \frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{11}{2} \right\} \right] \right\},$$

$$G_{\min}(A) = \left\{ q \in \mathbb{H} \colon \left| q - \frac{5}{2} \right| \le \frac{11}{2} \right\}.$$

Here, all the right eigenvalues of A are contained in $G_{\min}(A)$. Hence, Theorem 4.6 is verified.

Finally, we present a numerical example which shows that our inclusion region $G_{\min}(A)$ (defined in Theorem 2.1) is potentially sharper than the inclusion region $G(A)$ (defined in Theorem 4.5) for some quaternionic matrices.

***Example* 4.4.** Let us consider a central closed quaternionic matrix

$$A = \begin{bmatrix} 1 & 3 + 9\mathbf{i} - 12\mathbf{j} + 10\mathbf{k} & 13\mathbf{i} - 10\mathbf{j} - 7\mathbf{k} \\ 3 - 9\mathbf{i} + 12\mathbf{j} - 10\mathbf{k} & 3 & 5\mathbf{i} - 7\mathbf{j} + 6\mathbf{k} \\ -13\mathbf{i} + 10\mathbf{j} + 7\mathbf{k} & -5\mathbf{i} + 7\mathbf{j} - 6\mathbf{k} & 2 \end{bmatrix}.$$

Then, the complex adjoint matrix of A is given as

$$\Psi_A = \begin{bmatrix} 1 & 3 + 9\mathbf{i} & 13\mathbf{i} & 0 & -12 + 10\mathbf{i} & -10 - 7\mathbf{i} \\ 3 - 9\mathbf{i} & 3 & 5\mathbf{i} & 12 - 10\mathbf{i} & 0 & -7 + 6\mathbf{i} \\ -13\mathbf{i} & -5 & 2 & 10 + 7\mathbf{i} & 7 - 6 & 0 \\ 0 & 12 + 10\mathbf{i} & 10 - 7\mathbf{i} & 1 & 3 - 9\mathbf{i} & -13\mathbf{i} \\ -12 - 10\mathbf{i} & 0 & 7 + 6\mathbf{i} & 3 + 9\mathbf{i} & 3 & -5\mathbf{i} \\ -10 + 7\mathbf{i} & -7 - 6\mathbf{i} & 0 & 13\mathbf{i} & 5\mathbf{i} & 2 \end{bmatrix}.$$

The right spectrum of A is $\Lambda_r(A) = \{-25.3430, 1.4493, 29.8937\}$. We can easily see that A is similar to $D := \operatorname{diag}(-25.3430, 1.4493, 29.8937)$. From Theorems 2.1 and 4.6, we have the following balls:

$$G(A) = \{z \in \mathbb{H} \colon |z - 2| \le 31.8957\} \text{ and } G_{\min}(A) = \{z \in \mathbb{H} \colon |z - 2| \le 27.8937\}.$$

From the above balls, it is clear that $G_{\min}(A) \subset G(A)$. Thus our estimation is could be sharp for some quaternionic matrices.

References

1. S. L. Adler, *Quaternionic quantum mechanics and quantum fields*, Oxford Univ. Press, New York (1995).
2. S. S. Ahmad, I. Ali *Bounds for eigenvalues of matrix polynomials over quaternion division algebra*, Adv. Appl. Clifford Algebras, **26**, № 4, 1095 – 1125 (2016).
3. J. L. Brenner, *Matrices of quaternions*, Pacif. J. Math., **1**, 329 – 335 (1951).
4. A. Bunse-Gerstner, R. Byers, V. Mehrmann, *A quaternion QR algorithm*, Numer. Math., **55**, 83 – 95 (1989).
5. J. H. Conway, D. A. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, A. K. Peters Natick (2002).
6. T. L. Hankins, *Sir William Rowan Hamilton*, The Johns Hopkins Univ. Press, Baltimore (1980).
7. R. A. Horn, F. Zhang, *A generalization of the complex Autonne – Takagi factorization to quaternion matrices*, Linear and Multilinear Algebra, **60**, 1239 – 1244 (2012).
8. W. Junliang, Z. Yan, *Estimate for the lower bound of rank and the upper bound of eigenvalues norms sum of given quaternion matrix*, Comput. Math. and Appl., **59**, 3160 – 3166 (2010).
9. G. Opfer, *Polynomials and Vandermonde matrices over the field the quaternions*, Electron. Trans. Numer. Anal., **36**, 9 – 16 (2009).
10. R. Pereira, *Quaternionic polynomials and behavioral systems*, Ph. D. Thesis, Univ. Aveira (2006).
11. R. Pereira, P. Rocha, *On the determinant of quaternionic polynomial matrices and its application to system stability*, Math. Methods Appl. Sci., **31**, 99 – 122 (2008).
12. R. Pereira, P. Rocha, P. Vettori, *Algebraic tools for the study of quaternionic behavioral systems*, Linear Algebra and Appl., **400**, 121 – 140 (2005).
13. S. Quaisar, L. Zou, *Distribution for the standard eigenvalues of quaternion matrices*, Int. Math. Forum, **7**, 831 – 838 (2012).
14. L. Rodman, *Stability of invariant subspaces of quaternion matrices*, Complex Anal. Oper. Theory, **6**, 1069 – 1119 (2012).
15. L. Rodman, *Topics in quaternion linear algebra*, Princeton Univ. Press, Princeton, NJ (2014).
16. W. So, *Quaternionic left eigenvalue problem*, Southeast Asian Bull. Math., **29**, 555 – 565 (2005).
17. C. C. Took, D. P. Mandic, F. Zhang, *On the unitary diagonalisation of a special class of quaternion matrices*, Appl. Math. Lett., **24**, 1806 – 1809 (2011).
18. C. C. Took, D.P. Mandic, *Augmented second-order statistics of quaternion random signals*, Signal Processing, **91**, 214 – 224 (2011).
19. B. Tu, *The generalization of schur theorem over quaternion division ring*, Chinese Ann. Math., **2**, 130 – 138 (1988).
20. J. Wu, *Distribution and estimation for eigenvalues of real quaternion matrices*, Comput. Math. and Appl., **55**, 1998 – 2004 (2008).
21. F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra and Appl., **251**, 21 – 57 (1997).
22. F. Zhang, *Geršgorin type theorems for quaternionic matrices*, Linear Algebra and Appl., **424**, 139 – 155 (2007).
23. L. Zou, Y. Jiang, J. Wu, *Location for the right eigenvalues of quaternion matrices*, J. Appl. and Math. Comput., **38**, 71 – 83 (2012).

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