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ON $p(x)$-KIRCHHOFF-TYPE EQUATION INVOLVING $p(x)$-BIHARMONIC OPERATOR VIA GENUS THEORY

ПРО $p(x)$-РІВНЯННЯ ТИПУ КІРХГОФА ІЗ $p(x)$-БІГАРМОНІЧНИМ ОПЕРАТОРОМ З ТОЧКИ ЗОРУ ТЕОРІЇ РОДУ

The paper deals with the existence and multiplicity of nontrivial weak solutions for the $p(x)$-Kirchhoff-type problem

$$-M\left(\int\limits_{\Omega}\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta^2_{p(x)}u=f(x,u)\quad\text{in}\quad\Omega,$$

$$u=\Delta u=0\quad\text{on}\quad\partial\Omega.$$

By using variational approach and Krasnoselskii's genus theory, we prove the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type equation.

Розглядаються проблеми існування та множинності нетривіальних слабких розв'язків $p(x)$-задачі типу Кірхгофа

$$-M\left(\int\limits_{\Omega}\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta^2_{p(x)}u=f(x,u)\quad\text{в}\quad\Omega,$$

$$u=\Delta u=0\quad\text{на}\quad\partial\Omega.$$

Використовуючи варіаційний підхід та теорію роду Красносельського, ми доводимо існування та множинність розв'язків для $p(x)$-рівняння типу Кірхгофа.

1. Introduction. In this paper, we are interested in the following problem:

$$-M\left(\int\limits_{\Omega}\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta^2_{p(x)}u=f(x,u)\quad\text{in}\quad\Omega,$$
$$u=\Delta u=0\quad\text{on}\quad\partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 2$, with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)}u = \Delta\big(|\Delta u|^{p(x)-2}\Delta u\big)$ is the $p(x)$-biharmonic operator, p is a continuous function on $\overline{\Omega}$ with $1 < p(x) < N$.

We assume that $M(t)$ and $f(x,t)$ satisfy the following assumptions:

(M$_1$) $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function and satisfies the (polynomial growth) condition

$$m_1t^{\beta-1}\leq M(t)\leq m_2t^{\alpha-1}$$

for all $t>0$ and $m_1,\ m_2$ real numbers such that $0<m_1\leq m_2$ and $\alpha\geq\beta>1$;

(f_1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$d_1|t|^{s(x)-1} \leq f(x,t) \leq d_2|t|^{q(x)-1}$$

for all $t \geq 0$ and for all $x \in \overline{\Omega}$, where $d_1,\ d_2$ are positive constants and $s, q \in C(\overline{\Omega})$ such that $1 < s(x) < q(x) < p^*(x) < \dfrac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$;

(f_2) f is an odd function according to t, that is,

$$f(x,t) = -f(x,-t)$$

for all $t \in \mathbb{R}$ and for all $x \in \overline{\Omega}$.

The problem (1.1) is related to the stationary problem of a model presented by Kirchhoff [16]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of equation (1.2) is that the equation contains a nonlocal coefficient $\dfrac{\rho_0}{h} + \dfrac{E}{2L}\displaystyle\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\dfrac{1}{2L}\displaystyle\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise identity. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension.

The operator $\Delta^2_{p(x)} u = \Delta\big(|\Delta u|^{p(x)-2}\Delta u\big)$ is said to be the $p(x)$-biharmonic, and becomes p-biharmonic when $p(x) = p$ (a constant). The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [19], electrorheological fluids [20] or image restoration [1].

In recent years, elliptic problems involving p-Kirchhoff-type operators have been studied in many papers, we refer to [2, 4], in which the authors have used different methods to get the existence of solutions for (1.1) in the case when $p(x) = p$ is a constant.

The study of the Kirchhoff-type equations has already been extended to the case involving the p-Laplacian operator given by the formula $\Delta_p u = \operatorname{div}\big(|\Delta u|^{p-2}\Delta u\big)$ [8]

$$-M\left(\int_\Omega |\nabla u|^p\, dx\right)^{p-1} \Delta_p u = f(x,u) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

we point out that establishing conditions on M and f for which Kirchhoff-type equations possess solutions is the key argument.

In the case $p(x)$-Laplacian operator, in [3], the authors studied the Kirchhoff-type equation

$$-M\left(\int\limits_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\Delta_{p(x)}u=f(x,u)\quad\text{in}\quad\Omega,$$
$$u=0\quad\text{on}\quad\partial\Omega,\tag{1.3}$$

by using the Krasnoselskii's genus theory, they showed the existence and multiplicity of the solutions of the the problem (1.3).

Motivated by the above papers and the results in [17, 18], we consider (1.1) to study the existence and multiplicity of the solutions.

This paper is organized as follows. In Section 2, we present some necessary preliminary results on variable exponent Sobolev spaces. Next, we give the main results and proofs about the existence and multiplicity of the solutions.

2. Preliminaries. In order to deal with $p(x)$-biharmonic operator problems, we need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ and some properties of $p(x)$-biharmonic operator, which we will use later (for details see [21, 22]).

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega)=\left\{u\colon\ \Omega\longrightarrow\mathbb{R},\ \text{measurable and}\ \int\limits_{\Omega}|u(x)|^{p(x)}\,dx<\infty\right\},$$

where $p\in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega})=\left\{h\in C(\overline{\Omega})\colon\ h(x)>1\ \ \forall x\in\overline{\Omega}\right\}.$$

Denote

$$p^+=\max_{x\in\overline{\Omega}}p(x),\qquad p^-=\min_{x\in\overline{\Omega}}p(x),$$

and, for all $x\in\overline{\Omega}$ and $k\geq 1$,

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-p(x)}, & \text{if}\quad p(x)<N,\\ +\infty, & \text{if}\quad p(x)\geq N,\end{cases}$$

and

$$p_k^*(x)=\begin{cases}\dfrac{Np(x)}{N-kp(x)}, & \text{if}\quad kp(x)<N,\\ +\infty, & \text{if}\quad kp(x)\geq N.\end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm:

$$|u|_{p(x)}=\inf\left\{\mu>0;\ \int\limits_{\Omega}\left|\frac{u(x)}{\mu}\right|^{p(x)}dx\leq 1\right\},$$

and the space $\left(L^{p(x)}(\Omega),|.|_{p(x)}\right)$ is a Banach.

Proposition 2.1 [9, 21]. *The space $\left(L^{p(x)}(\Omega), |.|_{p(x)}\right)$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$, i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Hölder's type inequality

$$\left| \int\limits_{\Omega} uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}$$

holds true.

Furthermore, if we define the mapping $\rho\colon L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int\limits_{\Omega} |u|^{p(x)}\, dx,$$

then the following relations hold.

Proposition 2.2 [21, 22].

(i) $|u|_{p(x)} < 1\ (= 1,\ > 1) \Leftrightarrow \rho(u) < 1\ (= 1,\ > 1)$,

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$,

(iii) $|u_n - u|_{p(x)} \longrightarrow 0 \Leftrightarrow \rho(u_n - u) \longrightarrow 0$.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \colon D^\alpha u \in L^{p(x)}(\Omega),\ \ |\alpha| \leq k \right\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multiindex and $|\alpha| = \sum_{i=1}^{N} \alpha_i$.

The space $W^{k,p(x)}(\Omega)$ is equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer to [9, 10, 13, 15].

Proposition 2.3 [9]. *Let $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$. Then there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Consider the function space X defined by

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega).$$

Then X is a separable and reflexive Banach space equipped with the norm

$$\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}.$$

***Remark* 2.1.** According to [7], the norm $\|u\|_{2,p(x)}$ is equivalent to the norm $|\Delta u|_{p(x)}$ in the space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Consequently, the norms $\|.\|_{2,p(x)}, \|.\|$ and $|\Delta .|_{p(x)}$ are equivalent.

Proposition 2.4 [14]. *If we put*

$$J(u) = \int\limits_\Omega |\Delta u|^{p(x)} dx,$$

then, for all $u, u_n \in X$, *the following relations hold true*:

(i) $\|u\| < 1 \ (= 1; \ > 1) \Longleftrightarrow J(u) < 1 \ (= 1; \ > 1)$,

(ii) $\|u\| > 1 \Longrightarrow \|u\|^{p^-} \leq J(u) \leq \|u\|^{p^+}$,

for all $u_n \in X$, *we have*

(iii) $\|u_n\| \longrightarrow 0 \Longleftrightarrow J(u_n) \longrightarrow 0$,

(iv) $\|u_n\| \longrightarrow \infty \Longleftrightarrow J(u_n) \longrightarrow \infty$.

Proposition 2.5 [12]. *Let* X *be a Banach space and*

$$\Lambda(u) = \int\limits_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx.$$

The functional $\Lambda \colon X \longrightarrow \mathbb{R}$ *is convex. The mapping* $\Lambda' \colon X \longrightarrow X'$ (Λ' *is the Fréchet derivative of* Λ) *is a strictly monotone, bounded homeomorphism and of* (S_+), *namely*,

$$u_n \rightharpoonup u \quad (\textit{weakly}) \qquad \textit{and} \qquad \overline{\lim_{n\to\infty}} \langle \Lambda'(u_n), u_n - u\rangle \leq 0 \quad \textit{implies} \quad u_n \longrightarrow u \quad (\textit{strongly}),$$

where $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

Definition 2.1. *We say that* $u \in X$ *is a weak solution of* (1.1) *if*

$$M\left(\int\limits_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx\right) \int\limits_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi\, dx = \int\limits_\Omega f(x,u)\varphi\, dx$$

for all $\varphi \in X$.

We associate to the problem (1.1) the energy functional, defined as $I \colon X \longrightarrow \mathbb{R}$,

$$I(u) = \widehat{M}\left(\int\limits_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx\right) - \int\limits_\Omega F(x,u)\, dx,$$

where $\widehat{M}(t) = \displaystyle\int_0^t M(s)\, ds$ and $F(x,u) = \displaystyle\int_0^u f(x,t)\, dt$.

Standard arguments show that $I \in C^1(X,\mathbb{R})$ and

$$\langle I'(u), v\rangle = \lim_{h\to 0}\frac{I(u+hv)-I(u)}{h} =$$

$$= M\left(\int\limits_{\Omega}\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\int\limits_{\Omega}|\Delta u|^{p(x)-2}\Delta u\Delta v\,dx - \int\limits_{\Omega} f(x,u)v\,dx$$

for any $u, v \in X$.

Hence, we can notice that critical points of functional I are the weak solutions for problem (1.1).

For simplicity, we use d_i, to denote the general nonnegative or positive constant (the exact value may change from line to line).

3. Main results and proofs. We present some basic notions on the Krasnoselskii's genus (see [5, 6]) that we will use in the proof of our main results.

Let Y be a real Banach space. Set

$$\Re = \big\{E \subset Y\backslash\{0\} : E \text{ is compact and } E = -E\big\}.$$

Definition 3.1 [6, 23]**.** *Let* $E \in \Re$ *and* $Y = \mathbb{R}^k$. *The genus* $\gamma(E)$ *of* E *is defined by*

$$\gamma(E) = \min\big\{k \geq 1;\ \textit{there exists an odd continuous mapping}\ \phi\colon E \to \mathbb{R}^k\backslash\{0\}\big\}.$$

If such a mapping does not exist for any $k > 0$, we set $\gamma(E) = \infty$. Note also that if E is a subset, which consists of finitely many pairs of points, then $\gamma(E) = 1$. Moreover, from definition, $\gamma(\varnothing) = 0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$-dimensional sphere via an odd map.

Now, we will give some results of Krasnoselskii's genus, which are necessary throughout the present paper.

Theorem 3.1 [6, 23]**.** *Let* $Y = \mathbb{R}^N$ *and* $\partial\Omega$ *be the boundary of an open, symmetric, and bounded subset* $\Omega \subset \mathbb{R}^N$ *with* $0 \in \Omega$. *Then* $\gamma(\partial\Omega) = N$.

***Corollary* 3.1** [6, 23]**.** $\gamma(S^{N-1}) = N$ (*recall the notation* S^{N-1} *which stands for the unit sphere in* $\mathbb{R}^N$).

***Remark* 3.1** [6, 23]**.** If Y is of infinite dimension and separable and S is the unit sphere in Y, then $\gamma(S) = \infty$.

Definition 3.2 [6, 23]**.** *We say that the functional satisfies the Palais–Smale condition (PS) if every sequence* $(u_n) \subset Y$ *such that*

$$|I(u_n)| \leq C \qquad \textit{and} \qquad I'(u_n) \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty$$

contains a convergent subsequence in the norm of Y.

The first result of the present paper is the following theorem.

Theorem 3.2. *Suppose* (M_1), (f_1), *and* (f_2) *hold. If* $p(x) < q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and* $q^+ < \beta p^-$, *then the problem* (1.1) *has infinitely many solutions.*

The following result obtained by Clark in [11] is the main idea, which we use in the proof of Theorem 3.2.

Theorem 3.3. *Let* $J \in C^1(X,\mathbb{R})$ *be a functional satisfying the (PS) condition. Furthermore, let us suppose that*:

(i) *J is bounded from below and even,*

(ii) *there is a compact set $K \in \Re$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points, and their corresponding critical values are less than $J(0)$.

Lemma 3.1. *Suppose* (M_1), (f_1), *and $q^+ < \beta p^-$ hold. Then I is bounded from below.*

Proof. From (M_1) and (f_1), we have

$$I(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right) - \int_\Omega F(x,u)\,dx \geq$$

$$\geq m_1 \int_0^{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx} \rho^{\beta-1}d\rho - \frac{d_2}{q^-}\int_\Omega |u|^{q(x)}\,dx =$$

$$= \frac{m_1}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)^\beta - \frac{d_2}{q^-}\int_\Omega |u|^{q(x)}\,dx$$

and by Propositions 2.2, 2.3, and 2.4, for all $u \in X$, we get

$$I(u) \geq \frac{m_1}{\beta(p^+)^\beta}\big(\alpha(\|u\|)\big)^\beta - \frac{d_2 C^{q^+}}{q^-}\|u\|^{q^+}, \tag{3.1}$$

where $\alpha\colon [0,+\infty[\longrightarrow \mathbb{R}$ is defined by

$$\alpha(t) = \begin{cases} t^{p^+}, & \text{if} \quad t \leq 1, \\ t^{p^-}, & \text{if} \quad t > 1. \end{cases}$$

As $\beta p^+ \geq \beta p^- > q^+$, I is bounded from below.

Lemma 3.2. *Suppose* (M_1), (f_1), *and $q^+ < \beta p^-$ hold. Then I satisfies the (PS) condition.*

Proof. Let (u_n) in X be a sequence such that

$$I(u_n) \longrightarrow c \qquad \text{and} \qquad I'(u_n) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

From (3.2), we have $|I(u_n)| \leq d_3$. This fact, combined with (3.1), implies that

$$d_3 \geq I(u_n) \geq \frac{m_1}{\alpha(p^+)^\alpha}\|u_n\|^{\beta p^-} - \frac{d_4}{q^-}\|u_n\|^{q^+},$$

where $\|u_n\| > 1$. Because $q^+ < \beta p^-$, I is coercive, we deduce that (u_n) is bounded in X. Hence, there exists a subsequence, still denoted by $(u_n) \subset X$ and $u \in X$ such that

$$u_n \rightharpoonup u \qquad \text{as} \quad n \to \infty \quad \text{in} \quad X.$$

From Proposition 2.3, we obtain

$$u_n \longrightarrow u \quad \text{in} \quad L^{q(x)}(\Omega),$$

$$u_n \longrightarrow u \quad \text{a.e.} \quad \Omega.$$

Then by (3.2), we have $\langle I'(u_n), u_n - u\rangle \longrightarrow 0$. Thus,

$$\left\langle I'(u_n), u_n - u\right\rangle = M\left(\int\limits_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int\limits_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u)\,dx-$$

$$-\int\limits_\Omega f(x,u_n)(u_n-u)\,dx \longrightarrow 0.$$

By (f_1) and Proposition 2.1, it follows that

$$\left|\int\limits_\Omega f(x,u_n)(u_n-u)\,dx\right| \le d_2\int\limits_\Omega |u_n|^{q(x)-1}|u_n-u|dx \le$$

$$\le d_6\left||u_n|^{q(x)-1}\right|_{q'(x)}|u_n-u|_{q(x)}.$$

Since (u_n) converges strongly to u in $L^{q(x)}(\Omega)$, that is, $|u_n-u|_{q(x)} \to 0$ as $n\to\infty$, we get

$$\int\limits_\Omega f(x,u_n)(u_n-u)\,dx \to 0.$$

Hence,

$$M\left(\int\limits_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)\to 0.$$

From (M_1), it follows

$$\int\limits_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)\to 0.$$

By Proposition 2.5, we get that $u_n\to u$ in X.

***Proof of Theorem* 3.2.** We consider (see [5])

$$\Re_k = \{E\subset\Re\colon \gamma(E)\ge k\},$$

$$c_k = \inf_{E\in\Re_k}\sup_{u\in E} I(u), \quad k=1,2,\ldots,$$

then we have

$$-\infty < c_1 \le c_2 \le \ldots \le c_k \le c_{k+1} \le \ldots.$$

Now, we will prove that $c_k<0$ for every $k\in\mathbb{N}$. Since X is a separable Banach space, for any $k\in\mathbb{N}$, we can choose a k-dimensional linear subspace X_k of X such that $X_k\subset C_0^\infty(\Omega)$. As the norms on X_k are equivalent, there exists $r_k \in (0,1)$ such that $u \in X_k$ with $\|u\| \le r_k$ implies $|u|_{L^\infty} \le \delta$.

Set $S^k_{r_k} = \{u \in X_k : \|u\| = r_k\}$. By the compactness of $S^k_{r_k}$ and condition (f$_1$), there exists a constant $\eta_k > 0$ such that

$$\int_\Omega F(x,u)\,dx \ge \frac{d_1}{s^+}\int_\Omega |u|^{s(x)}\,dx \ge \eta_k \quad \forall u \in S^k_{r_k}. \tag{3.3}$$

From (M$_1$) and (f$_1$), for $u \in S^k_{r_k}$ and $t \in (0,1)$, we have

$$I(tu) = \widehat{M}\left(\int_\Omega \frac{|\Delta tu|^{p(x)}}{p(x)}\,dx\right) - \int_\Omega F(x,tu)\,dx \le$$

$$\le m_2\left(\int_\Omega \frac{|\Delta tu|^{p(x)}}{p(x)}\right) - \frac{d_1}{s^+}\int_\Omega |tu|^{s(x)}\,dx \le$$

$$\le \frac{m_2}{\alpha(p^-)^\alpha} t^{\alpha p^-} r_k^{\alpha p^-} - t^{s^+}\eta_k. \tag{3.4}$$

Since $s^+ < q^- \le q^+ < \beta p^- \le \alpha p^-$, we can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$I(t_k u) \le -\varepsilon_k < 0 \quad \forall u \in S^k_{r_k},$$

that is,

$$I(u) \le -\varepsilon_k < 0 \quad \forall u \in S^k_{t_k r_k}.$$

It is clear that $\gamma(S^k_{t_k r_k}) = k$, so $c_k \le -\varepsilon_k < 0$. Therefore, by Lemma 3.1, Lemma 3.2 and above results, we can apply Theorem 3.3 to obtain that the functional I admits at least k pairs of distinct critical points, and since k is arbitrary, we obtain infinitely many critical points of I.

Theorem 3.4. *Suppose* (M$_1$), (f$_1$), *and* (f$_2$) *hold. If* $q(x) < p(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *then the problem* (1.1) *has a sequence of solution* $\{\pm u_k : k = 1,2,\ldots\}$ *such that* $I(\pm u_k) < 0$.

Proof. We follow the same steps applied in the proof of the Lemma 3.1, and the fact $q^+ < p^-$, we prove that I is coercive. Because I is weak lower semicontinuous, I attaints its minimum on X, that is, (1.1) has a solution. By the coercivity of I, we know that I satisfies (PS) condition on X. And from condition (f$_2$), I is even.

In the rest of the proof, since we use the same arguments which we used in the proof of the Theorem 3.2, we omit the discussions here.

Hence, if we follow the similar processes as we did in (3.3) and (3.4), and the fact $s^+ < q^- \le$ $\le q^+ < p^- < \alpha p^-$, we can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$I(u) \le -\varepsilon_k < 0 \quad \forall u \in S^k_{t_k r_k}.$$

Clearly, $\gamma(S^k_{t_k r_k}) = k$, so $c_k \le -\varepsilon_k < 0$. By Krasnoselskii's genus, each c_k is a critical value of I, then there is a sequence of solutions $\{\pm u_k : k = 1,2,\ldots\}$ such that $I(\pm u_k) < 0$.

References

1. Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math., **66**, № 4, 1383 – 1406 (2006).
2. B. Cheng, X. Wu, J. Liu, *Multiplicity of nontrivial solutions for Kirchhoff type problems*, Boundary Value Problems, **2010**, Article ID 268946 (2010), 13 p.
3. M. Avci, B. Cekic, R. A. Mashiyev, *Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory*, Math. Meth. Appl. Sci., **34**, 1751 – 1759 (2011).
4. J. J. Sun, C. L. Tang, *Existence and multiplicity of solutions for Kirchhoff type equations*, Nonlinear Anal., **74**, 1212 – 1222 (2011).
5. K. C. Chang, *Critical point theory and applications*, Shanghai Sci. and Technol. Press, Shanghai (1986).
6. M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, MacMillan, New York (1964).
7. A. Zang, Y. Fu, *Interpolation inequalities for derivatives in variable exponent Lebesgue – Sobolev spaces*, Nonlinear Anal., **69**, 3629 – 3636 (2008).
8. F. J. S. A. Corrêa, G. M. Figueiredo, *On a elliptic equation of p-Kirchhoff type via variational methods*, Bull. Austr. Math. Soc., **74**, 263 – 277 (2006).
9. X. L. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. and Appl., **263**, 424 – 446 (2001).
10. M. Milhailescu, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplacian operator*, Nonlinear Anal., **67**, 1419 – 1425 (2007).
11. D. C. Clark, D. Gilbarg, *A variant of the Ljusternik – Schnirelman theory*, Indiana Univ. Math. J., **22**, № 1, 65 – 74 (1972).
12. A. R. El Amrouss, F. Moradi, M. Moussaoui, *Existence of solutions for fourth-order PDEs with variable exponents*, Electron. J. Different. Equat., **2009**, № 153, 1 – 13 (2009).
13. X. L. Fan, X. Fan, *A Knobloch-type result for $p(x)$-Laplacian systems*, J. Math. and Appl., **282**, 453 – 464 (2003).
14. A. R. El Amrouss, A. Ourraoui, *Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator*, Bol. Soc. Parana. Mat., **(3)31**, № 1, 179 – 192 (2013).
15. J. H. Yao, *Solution for Neumann boundary problems involving the $p(x)$-Laplacian operators*, Nonlinear Anal., **68**, 1271 – 1283 (2008).
16. G. Kirchhoff, *Mechanik*, Teubner, Leipzig (1883).
17. N. T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, Electron. J. Qual. Theory Different. Equat., **42**, 1 – 13 (2012).
18. G. A. Afrouzi, M. Mirzapour, *Eigenvalue problems for $p(x)$-Kirchhoff type equations*, Electron. J. Different. Equat., **2013**, № 253 (2013).
19. V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR, Ser. Mat., **50**, № 4, 675 – 710 (1986).
20. E. Acerbi, G. Mingione, *Gradient estimate for the $p(x)$-Laplacian system*, J. reine und angew. Math., **584**, 117 – 148 (2005).
21. O. Kovãčik, J. Rãkosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41(116)**, 592 – 618 (1991).
22. X. L. Fan, J. S. Shen, D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}$*, J. Math. Anal. and Appl., **262**, 749 – 760 (2001).
23. A. Ambrosetti, A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge Stud. Adv. Math., **14** (2007).

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