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ON THE THEORY OF INTEGRAL MANIFOLDS FOR SOME DELAYED PARTIAL DIFFERENTIAL EQUATIONS WITH NONDENSE DOMAIN

ДО ТЕОРІЇ ІНТЕГРАЛЬНИХ МНОГОВИДІВ ДЛЯ ДЕЯКИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЗАПІЗНЕННЯМ У НЕЩІЛЬНІЙ ОБЛАСТІ

Integral manifolds are very useful in studying dynamics of nonlinear evolution equations. In this paper, we consider the nondensely-defined partial differential equation

$$\frac{du}{dt} = (A + B(t))u(t) + f(t, u_t), \quad t \in \mathbb{R}, \tag{1}$$

where $(A, D(A))$ satisfies the Hille – Yosida condition, $(B(t))_{t\in\mathbb{R}}$ is a family of operators in $\mathcal{L}(\overline{D(A)}, X)$ satisfying some measurability and boundedness conditions, and the nonlinear forcing term f satisfies $\|f(t,\phi) - f(t,\psi)\| \leq \varphi(t)\|\phi - \psi\|_{\mathcal{C}}$; here, φ belongs to some admissible spaces and $\phi,\ \psi \in \mathcal{C} := C([-r, 0], X)$. We first present an exponential convergence result between the stable manifold and every mild solution of (1). Then we prove the existence of center-unstable manifolds for such solutions.

Our main methods are invoked by the extrapolation theory and the Lyapunov – Perron method based on the admissible functions properties.

Інтегральні многовиди мають велике значення при вивченні динаміки нелінійних еволюційних рівнянь. Ми розглядаємо нещільно визначене диференціальне рівняння з частинними похідними

$$\frac{du}{dt} = (A + B(t))u(t) + f(t, u_t), \quad t \in \mathbb{R}, \tag{1}$$

де $(A, D(A))$ задовольняє умову Хілла – Йосіди, $(B(t))_{t\in\mathbb{R}}$ є сім'єю операторів у $\mathcal{L}(\overline{D(A)}, X)$, яка задовольняє деякі умови вимірюваності та обмеженості, а нелінійний доданок f задовольняє умову $\|f(t,\phi) - f(t,\psi)\| \leq \varphi(t)\|\phi - \psi\|_{\mathcal{C}}$, де φ належить до деяких допустимих просторів і $\phi,\ \psi \in \mathcal{C} := C([-r, 0], X)$. Ми насамперед пропонуємо деякий результат, що стосується експоненціальної збіжності між стійким многовидом та будь-яким слабким розв'язком рівняння (1). Далі ми доводимо існування центральних нестійких многовидів для таких розв'язків.

Наші методи доведення посилаються в основному на теорію екстраполяції та метод Ляпунова – Перрона, що базується на властивостях допустимих функцій.

1. Introduction. In this paper, we study some integral manifolds properties of the abstract delayed Cauchy problem

$$\frac{du}{dt} = (A + B(t))u(t) + f(t, u_t), \quad t \geq s, \tag{1.1}$$

$$u_s = \Phi \in \mathcal{C},$$

where $\big(A, D(A)\big)$ is a nondensely defined linear operator on a Banach space X, $B(t)$, $t \in \mathbb{R}$ is a family of linear operators in $\mathcal{L}(\overline{D(A)}, X)$, $f : \mathbb{R}\times\mathcal{C} \to X$ is a nonlinear operator, $\mathcal{C} := C([-r, 0], X)$ and the history function u_t is defined for $\theta \in [-r, 0]$ by $u_t(\theta) = u(t + \theta)$. Throughout all this work, we suppose that A is a Hille – Yosida operator, that is

(H_1) There exists $w \in \mathbb{R}$ and $M \geq 1$ such that $(w, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > w, \tag{1.2}$$

where $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda > w$. Without loss of generality, one assumes that $M = 1$. Otherwise, we can renorm the space X with an equivalent norm for which we obtain the estimation (1.2) with $M = 1$.

Integral manifolds theory plays an important role in the understanding of evolution equations dynamics. Many works on various types of equations were done in the literature (see, for example, [1, 4, 10]). Regarding the case of partial differential equations without delay, we refer the reader, for instance, to [3], where authors investigate invariant manifolds for flows in Banach spaces, by virtue of the Lyapunov – Perron method. This subject was also of great interest in the case of delayed partial differential equations. We quote, for example, [2], where the authors investigate inertial manifolds for retarded semilinear parabolic equations by the Lyapunov – Perron method.

Nevertheless, it is sometimes more convenient in applications, in many contexts, to consider equations with nondense domain such as in diffusion phenomena and population dynamics. For instance, we refer the reader to [5, 16, 22, 23]. Concerning the nonautonomous case, several results about the existence and behaviour of solutions have been studied (see [9, 17, 21] and references therein). Particularly, many results on the existence of integral manifolds were developed in the context of the following differential equation:

$$\frac{du}{dt} = A(t)u(t) + f(t, u_t), \quad t \in [s, +\infty),$$

$$u_s = \Phi,$$

where $A(t)$, $t \in \mathbb{R}$, is a family of possibly unbounded linear operators on a Banach space X and $f : \mathbb{R} \times \mathcal{C} \to X$ is a continuous function. The fixed point theory based on the uniform Lipschitzness of the nonlinear term f was the most powerful tool to investigate such problems. Unfortunately, in real situations such as some complicated reaction-diffusion phenomena, the function f which can represent the population size or the source of a material is frequently depending on time (see, for instance, [19, 20]).

In recent years, authors have established interesting results in the case of densely defined differential equations without delays (see [6, 11, 12]), by investigating the existence of integral manifolds in view of the Lyapunov – Perron method and the contribution of admissible spaces, without needing the uniform Lipschitzness of f. More recently, the existence of integral manifolds for densely defined and delayed differential equations were studied by [7, 8]. Note that the investigation of integral manifolds for delayed differential equations with nondense domain and where the nonlinear operator f is not uniformly Lipshitzian was not studied until the author, in [14], investigates the existence of unstable manifolds for (1.1) and states an attraction result for such unstable manifolds. Then, he investigates in [15] the existence of stable and center-stable manifolds for (1.1) on the positive half line.

Motivated by all these works, we aim to prove an attractiveness result between the mild solution and the stable manifold of (1.1) on the whole line $\mathbb{R}$. Furthermore, we prove the existence of a center-unstable manifold for (1.1).

2. Admissible spaces, mild solutions and integral manifolds. We first recall the following notions and properties of admissible spaces.

Definition 2.1 [8, 13]**.** *Let $\mathcal{B}$ denote the Borel algebra and λ the Lebesgue measure on $\mathbb{R}$. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ-nullfunctions) is called a Banach function space (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if*

1) *E is Banach lattice with respect to a norm $\|.\|_E$, i.e., $(E, \|.\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(.)| \leq |\varphi(.)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$,*

2) *the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure and $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$, and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$,*

3) *$E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R})$, i.e., for each seminorm p_n of $L_{1,\mathrm{loc}}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n}\|f\|_E$ for all $f \in E$.*

Definition 2.2 [8, 13]**.** *The Banach function space E is called admissible if*

(i) *there is a constant $M \geq 1$ such that for every compact interval $[a, b] \in \mathbb{R}$ we have*

$$\int_a^b |\varphi(t)|dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E,$$

(ii) *for $\varphi \in E$, the function $\Theta_1\varphi$ defined by $\Theta_1\varphi(t) = \int_t^{t+1} \varphi(\tau)d\tau$ belongs to E,*

(iii) *E is T^+_τ- and T^-_τ-invariant, where T^+_τ and T^-_τ are defined for $\tau \in \mathbb{R}$ by*

$$T^+_\tau\varphi(t) = \varphi(t-\tau) \quad \text{for} \quad t \in \mathbb{R},$$

$$T^-_\tau\varphi(t) = \varphi(t+\tau) \quad \text{for} \quad t \in \mathbb{R}.$$

Moreover, there are constants Q, R such that $\|T^+_\tau\| \leq Q$, $\|T^-_\tau\| \leq R$ for all $\tau \in \mathbb{R}$.

***Remark* 2.1.** If $\mathbf{S}(\mathbb{R}) := \left\{\xi \in L_{1,\mathrm{loc}}(\mathbb{R}) \colon \sup_{t\in\mathbb{R}} \int_t^{t+1} |\xi(\tau)|d\tau < \infty\right\}$ endowed with the norm $\|\xi\|_{\mathbf{S}} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |\xi(\tau)|d\tau$ and E is an admissible Banach function space, it is easy to show that $E \hookrightarrow \mathbf{S}(\mathbb{R})$.

Proposition 2.1 [8, 13]**.** *Let E be an admissible Banach function space. Then the following assertions hold:*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R})$ such that $\varphi \geq 0$ and $\Theta_1\varphi \in E$, where Θ_1 is defined as in Definition* 2.2 (ii). *For $\tau > 0$ we define $\Theta'_\tau\varphi$ and $\Theta''_\tau\varphi$ by*

$$\Theta'_\tau\varphi(t) = \int_{-\infty}^t e^{-\tau(t-s)}\varphi(s)ds,$$

$$\Theta''_\tau\varphi(t) = \int_t^{+\infty} e^{-\tau(s-t)}\varphi(s)ds.$$

Then $\Theta'_\tau\varphi$ *and* $\Theta''_\tau\varphi$ *belong to* E. *Particularly, if* $\sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\sigma)|d\sigma<\infty$ (*this will be satisfied if* $\varphi\in E$ (*see Remark* 2.1)), *then* $\Theta'_\tau\varphi$ *and* $\Theta''_\tau\varphi$ *are bounded. Moreover, we have*

$$\|\Theta'_\tau\varphi\|_\infty\leq\frac{Q}{1-e^{-\tau}}\|\Theta_1\varphi\|_\infty \quad \text{and} \quad \|\Theta''_\tau\varphi\|_\infty\leq\frac{R}{1-e^{-\tau}}\|\Theta_1\varphi\|_\infty.$$

(b) *E contains exponentially decaying functions* $\psi(t)=e^{-\alpha|t|}$ *for* $t\in\mathbb{R}$ *and any fixed constant* $\alpha>0$.

(c) *E does not contain exponentially growing functions* $f(t)=e^{b|t|}$ *for* $t\in\mathbb{R}$ *and any constant* $b>0$.

Definition 2.3 [8, 13]. *Let E be an admissible Banach function space and φ be a positive function belonging to E. A function $f:\mathbb{R}\times\mathcal{C}\to X$ is said to be φ-Lipschitz if f satisfies*

(i) $\|f(t,0)\|\leq\varphi(t)$ *for all* $t\in\mathbb{R}$,

(ii) $\|f(t,\phi_1)-f(t,\phi_2)\|\leq\varphi(t)\|\phi_1-\phi_2\|_{\mathcal{C}}$ *for all* $t\in\mathbb{R}$ *and all* $\phi_1,\phi_2\in\mathcal{C}$.

***Remark* 2.2.** One can remark that if $f(t,\phi)$ is φ-Lipschitz then $\|f(t,\phi)\|\leq\varphi(t)(1+\|\phi\|_{\mathcal{C}})$ for all $\phi\in\mathcal{C}$ and $t\in\mathbb{R}$.

In the following, we will assume that

(H_2) $f:\mathbb{R}\times\mathcal{C}\to X$ is φ-Lipschitz, where φ is a positive function belonging to an admissible space E.

We now introduce the following concept.

Definition 2.4. *A family of bounded linear operators* $\{U(t,s)\}_{t\geq s}$ *on a Banach space X is a strongly continuous, exponential bounded evolution family if*

(i) $U(t,t)=Id$ *and* $U(t,r)U(r,s)=U(t,s)$ *for all* $t\geq r\geq s$,

(ii) *the map* $(t,s)\mapsto U(t,s)x$ *is continuous for every* $x\in X$,

(iii) *there are constants* K, $c\geq 0$ *such that* $\|U(t,s)x\|\leq Ke^{c(t-s)}\|x\|$ *for all* $t\geq s$ *and* $x\in X$.

Let

(H_3) $t\mapsto B(t)x$ is strongly measurable for every $x\in X_0:=\overline{D(A)}$ and there exists a function $l\in L^1_{\rm loc}(\mathbb{R})$ such that $\|B(.)\|\leq l(.)$.

By [9], if we consider the homogeneous equation

$$\frac{d}{dt}u(t)=(A+B(t))u(t), \qquad t\geq s, \quad u(s)=x\in X_0, \tag{2.1}$$

then $t\mapsto U_B(t,s)x$ is the unique mild solution on $[s,+\infty)$ of the initial value problem (2.1). Note that $\{U_B(t,s)\}_{t\geq s}$ is an evolution family on X_0. Now, let

$$\mathcal{C}_A:=\{\Phi\in\mathcal{C}:\Phi(0)\in\overline{D(A)}\}.$$

The following result gives a representation of mild solutions of (1.1) in terms of the evolution family $\{U_B(t,s)\}_{t\geq s}$.

Theorem 2.1 [14]. *Assume that* $(H_1)-(H_3)$ *hold. Let* $s\in\mathbb{R}$ *and* $\Phi\in\mathcal{C}_A$. *Then equation* (1.1) *has a unique mild solution* $u\in C([s,+\infty[,X_0)$, *given by*

$$u(t)=U_B(t,s)\Phi(0)+\lim_{\lambda\to\infty}\int_s^t U_B(t,\tau)\lambda R(\lambda,A)f(\tau,u_\tau)d\tau \quad \text{for} \quad t\geq s, \tag{2.2}$$

$$u_s=\Phi.$$

Furthermore, for every $t \ge s$, $\lim_{\lambda\to\infty} \int_s^t U_B(t,\tau)\lambda R(\lambda, A) f(\tau, u_\tau) d\tau \in X_0$ *exists uniformly on compact sets in* $\mathbb{R}$.

Note that we have the following notion of exponential trichotomy for evolution families.

Definition 2.5 [18]. *An evolution family* $\{U(t,s)\}_{t\ge s}$ *on the Banach space* X *is said to have an exponential trichotomy on* $\mathbb{R}$ *if there are three families of projections* $\{P_j(t)\}$, $t \in \mathbb{R}$, $j = 1,2,3$, *and positive constants* L, γ, ζ *with* $\gamma < \zeta$ *satisfying the following conditions*:

(i) $K := \sup_{t\in\mathbb{R}} \|P_j(t)\| < \infty$, $j = 1,2,3$;

(ii) $P_1(t) + P_2(t) + P_3(t) = Id$ *for* $t \in \mathbb{R}$ *and* $P_j(t)P_i(t) = 0$ *for all* $j \ne i$;

(iii) $P_j(t)U(t,s) = U(t,s)P_j(s)$ *for* $t \ge s$ *and* $j = 1,2,3$;

(iv) $U_|(t,s)$ *are homeomorphisms from* $\operatorname{Im} P_j(s)$ *onto* $\operatorname{Im} P_j(t)$, *for all* $t \ge s$ *and* $j = 2,3$, *respectively*;

(v) *for all* $t, s \in \mathbb{R}$ *and* $x \in X$,

$$\|U(t,s)P_3(s)x\| \le Le^{\gamma|t-s|}\|P_3(s)x\|,$$

and, if $t \ge s$ *and* $x \in X$, *we have*

$$\|U(t,s)P_1(s)x\| \le Le^{-\zeta(t-s)}\|P_1(s)x\|,$$

$$\|[U_|(t,s)]^{-1}P_2(t)x\| \le Le^{-\zeta(t-s)}\|P_2(t)x\|.$$

The projections $\{P_j(t)\}$, $t \in \mathbb{R}$, $j = 1,2,3$, are called the trichotomy projections, and the constants L, γ, ζ are the trichotomy constants.

Note that the evolution family $\{U(t,s)\}_{t\ge s}$ is said to have an exponential dichotomy if the family of projections $P_3(t)$ is trivial. That is, $P_3(t) = 0$ for every $t \in \mathbb{R}$.

We now give the following concept of integral manifolds for (1.1) on the whole line $\mathbb{R}$.

Definition 2.6 [14, 15]. *A set* $\mathcal{M} \subset \mathbb{R} \times \mathcal{C}_A$ *is said to be an unstable* (*respectively*, *a stable*) *manifold for solutions of* (1.1) *if*, *for every* $t \in \mathbb{R}$, *the phase space* $\mathcal{C}_A$ *is decomposed into a direct sum* $\mathcal{C}_A = \operatorname{Im}\mathcal{P}(t) \oplus \operatorname{Ker}\mathcal{P}(t)$ *such that*

$$\sup_{t\in\mathbb{R}} \|\mathcal{P}(t)\| < \infty,$$

and there exists a family of Lipschitz continuous mappings

$$\mathcal{G}_t\colon \operatorname{Im}\mathcal{P}(t) \to \operatorname{Ker}\mathcal{P}(t), \quad t \in \mathbb{R},$$

with Lipschitz constants $\operatorname{Lip}(\mathcal{G}_t)$ *independent of* t *such that*

a) $\mathcal{M} = \{(t, \xi + \mathcal{G}_t(\xi)) \in \mathbb{R} \times (\operatorname{Im}\mathcal{P}(t) \oplus \operatorname{Ker}\mathcal{P}(t)) \mid t \in \mathbb{R},\ \xi \in \operatorname{Im}\mathcal{P}(t)\}$, *and we denote*

$$\mathcal{M}_t := \{\xi + \mathcal{G}_t(\xi)\colon (t, \xi + \mathcal{G}_t(\xi)) \in \mathcal{M}\},$$

b) $\mathcal{M}_t$ *is homeomorphic to* $\operatorname{Im}\mathcal{P}(t)$ *for all* $t \in \mathbb{R}$,

c) *for each* $s \in \mathbb{R}$ *and* $\xi \in \mathcal{M}_s$, *there corresponds one and only one solution* $u(t)$ *of* (1.1) *on* $(-\infty, s]$(*respectively*, *on* $[s-r, +\infty)$) *satisfying the conditions that* $u_s = \xi$ *and* $\sup_{t\le s}\|u_t\|_{\mathcal{C}} < \infty$ (*respectively*, $\sup_{t\ge s}\|u_t\|_{\mathcal{C}} < \infty$); *furthermore*, *any two solutions* $u_1(t)$ *and* $u_2(t)$ *of* (1.1) *corresponding to different initial functions* ξ_1, $\xi_2 \in \mathcal{M}_s$ *attract each other exponentially in the sense that*, *there exist positive constants* ν *and* C_ν *independent of* s *such that for* $t \le s$ (*respectively*, $t \ge s$)

$$\left\|u_{1t} - u_{2t}\right\|_{\mathcal{C}} \le C_\nu e^{-\nu|s-t|}\left\|(\mathcal{P}(s)\xi_1)(0) - (\mathcal{P}(s)\xi_2)(0)\right\|,$$

d) $\mathcal{M}$ *is invariant under equation* (1.1); *that is, if* $u(t)$, $t \in \mathbb{R}$, *is a solution of* (1.1) *satisfying the conditions that* $u_s \in \mathcal{M}_s$ *and* $\sup_{t\leq s}\|u_t\|_{\mathcal{C}} < \infty$ (*respectively,* $\sup_{t\geq s}\|u_t\|_{\mathcal{C}} < \infty$), *then* $u_t \in \mathcal{M}_t$ *for all* $t \in \mathbb{R}$.

Now, suppose that

(H_4) The evolution family $\{U_B(t,s)\}_{t\geq s}$ has an exponential dichotomy with projections $P_B(t)$, $t \in \mathbb{R}$ and constants L $\mu > 0$.

In that case, the corresponding Green function $\Gamma_B(t,s)$, $(t,s) \in \mathbb{R}^2$, given by

$$\Gamma_B(t,s) = \begin{cases} P_B(t)U_B(t,s), & t > s, \\ -\big[U_{|B}(s,t)\big]^{-1}(Id - P_B(s)), & t < s, \end{cases}$$

satisfies

$$\big\|\Gamma_B(t,s)\big\| \leq L(1+K)e^{-\mu|t-s|} \quad \text{for all} \quad t \neq s.$$

In order to construct unstable and stable manifolds for (1.1), we have considered in [14, 15] the families of projections $\big(\mathbb{P}_B(t)\big)_{t\in\mathbb{R}}$ and $\big(\tilde{P}_B(t)\big)_{t\in\mathbb{R}_+}$ defined on $\mathcal{C}_A$, respectively, by

$$(\mathbb{P}_B(t)\xi)(\theta) = \big[U_{B|}(t,t+\theta)\big]^{-1}(I - P_B(t))\xi(0) \quad \text{for all} \quad \theta \in [-r,0]$$

and

$$(\tilde{P}_B(t)\xi)(\theta) = U_B(t-\theta,t)P_B(t)\xi(0) \quad \text{for all} \quad \theta \in [-r,0].$$

Let us collect some results about the existence and uniqueness of solutions for (1.1) related to the choice of the family of projections on $\mathcal{C}_A$.

Theorem 2.2 [14]. *Assume that* (H_1)–(H_4) *hold. Set*

$$H := \frac{L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-\mu}}. \tag{2.3}$$

Then, *if* $H < 1$, *there exists one and only one solution of equation* (1.1) *on* $(-\infty, s]$ *given by*

$$u(t) = [U_{B|}(s,t)]^{-1}\mu_0 + \lim_{\lambda\to\infty}\int_{-\infty}^{s}\Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma$$

for some $\mu_0 \in \operatorname{Ker} P_B(s)$ *such that* $\sup_{t\leq s}\|u_t\|_{\mathcal{C}} < \infty$. *Besides*, *for any two solutions* $u(t)$, $v(t)$ *corresponding to different initial functions* ξ_1 $\xi_2 \in \operatorname{Im}\mathbb{P}_B(s)$, *we have the following estimate*:

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\nu e^{-\nu(s-t)}\|\xi_1(0) - \xi_2(0)\| \quad \textit{for all} \quad t \leq s,$$

where ν *is a positive constant satisfying*

$$0 < \nu < \mu + \ln(1 - L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty),$$

and

$$C_\nu := \frac{Le^{\mu r}}{1 - \dfrac{L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-(\mu-\nu)}}}.$$

***Remark* 2.3.** We can prove in a similar way as in Theorem 3.5 of [15] that if we consider $P_B(t)$ defined on the whole line $\mathbb{R}$ then we have under the same assumptions of Theorem 2.2, more precisely, under conditions $(H_1)-(H_4)$ and if $H < 1$, defined by (2.3), the existence and uniqueness of the solution for equation (1.1) on $[s, +\infty)$ given by

$$u(t) = U_B(t,s)\mu_0 + \lim_{\lambda\to\infty} \int\limits_s^{+\infty} \Gamma_B(t,\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma) d\sigma$$

for some $\mu_0 \in \operatorname{Im} P_B(s)$ such that $\sup_{t\geq s} \|u_t\|_{\mathcal{C}} < \infty$, and we have the following estimate for any two solutions $u(t)$, $v(t)$ corresponding to different initial functions ξ_1 $\xi_2 \in \operatorname{Im} \tilde{P}_B(s)$:

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\nu e^{-\nu(t-s)} \|\xi_1(0) - \xi_2(0)\| \quad \text{for all} \quad t \geq s,$$

where ν and C_ν are defined as above.

To get the existence of stable manifolds for (1.1) on the whole line $\mathbb{R}$, we prove the following result, which shows that property (d) of Definition 2.6 holds.

Proposition 2.2. *Let $(\mathcal{P}(t))_{t\in\mathbb{R}}$ be a family of projections on the phase space $\mathcal{C}_A$. Let $\mathcal{G}_t$, $t \in \mathbb{R}$, be defined by*

$$\mathcal{G}_t(\xi)(\theta) = \lim_{\lambda\to\infty} \int\limits_t^{+\infty} \Gamma(t-\theta,\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma) d\sigma \quad \textit{for all} \quad \theta \in [-r, 0],$$

where $\mathcal{P}(t)u_t = \xi$, and let $E = \big\{\xi + \mathcal{G}_t(\xi) \in (\operatorname{Im}\mathcal{P}(t) \oplus \operatorname{Ker}\mathcal{P}(t)),\ t \in \mathbb{R}\big\}$. Then E is invariant under (1.1)*. That is, if $u(t)$, $t \in \mathbb{R}$, is a solution of* (1.1) *satisfying $u_s \in E$ and $\sup_{t\geq s} \|u_t\|_{\mathcal{C}} < \infty$, then $u_t \in E$ for all $t \in \mathbb{R}$.*

Proof. First, let $t \geq s$. Then the result follows analogically as in Theorem 3.7 of [15], by taking $s \in \mathbb{R}$. Now, let $t \leq s$, then, for $t - r \leq \tau \leq t \leq s$, we have according to Remark 2.3

$$u_\tau(-\theta) = U_B(\tau-\theta,\tau)\mu_0 + \lim_{\lambda\to\infty} \int\limits_\tau^{+\infty} \Gamma_B(\tau-\theta,\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma) d\sigma \quad \text{for} \quad \mu_0 \in \operatorname{Im} P_B(\tau).$$

Furthermore, it follows from Theorem 2.1 that

$$u(t) = U_B(t,\tau)u(\tau) + \lim_{\lambda\to\infty} \int\limits_\tau^{t} U_B(t,\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma) d\sigma =$$

$$= U_B(t,\tau)\left(\mu_0 + \lim_{\lambda\to\infty} \int\limits_\tau^{+\infty} \Gamma_B(\tau,\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma) d\sigma\right) +$$

$$+ \lim_{\lambda\to\infty} \int\limits_\tau^{t} U_B(t,\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma) d\sigma$$

$$\text{for} \quad \mu_0 \in \operatorname{Im} P_B(\tau) = U_B(t,\tau)\mu_0 -$$

$$-U_B(t,\tau)\lim_{\lambda\to\infty}\int\limits_{\tau}^{+\infty}[U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma+$$

$$+\lim_{\lambda\to\infty}\int\limits_{\tau}^{t}U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma=$$

$$=U_B(t,\tau)\mu_0-\lim_{\lambda\to\infty}\int\limits_{\tau}^{t}U_B(t,\sigma)(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma-$$

$$-\lim_{\lambda\to\infty}\int\limits_{\tau}^{+\infty}[U_{B|}(\sigma,t)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma+$$

$$+\lim_{\lambda\to\infty}\int\limits_{\tau}^{t}U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma=$$

$$=U_B(t,\tau)\mu_0+\lim_{\lambda\to\infty}\int\limits_{\tau}^{+\infty}\Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)d\sigma.$$

Proposition 2.2 is proved.

Consequently, by virtue of Proposition 2.2 and [14, 15], the following result yields.

Theorem 2.3. *Assume that* (H_1)–(H_4) *hold. Let*

$$H<\min\left\{1,\frac{e^{\mu r}}{1+L}\right\},\tag{2.4}$$

where H *is defined as in* (2.3). *Then*, *there exist a stable and an unstable manifold for solutions of equation* (1.1) *on the whole line* $\mathbb{R}$.

3. On the exponential attractiveness of stable manifolds. With the established theory of stable manifolds for the differential equation (1.1) (see Theorem 2.3), we aim to prove that the stable manifold $\mathcal{S}=\{\mathcal{S}_t\}_{t\in\mathbb{R}}$ exponentially attracts all mild solutions of (1.1).

Theorem 3.1. *Assume that* (H_1) – (H_4) *and* (2.4) *hold. Let*

$$\tilde{a}:=Le^{\mu r}\big(\operatorname{Lip}(\mathcal{G}_s)(1+LK)+a(1+K)\big)<1,$$

where a *is a constant taking according to properties of admissible spaces. Then the stable manifold* $\mathcal{S}=\{\mathcal{S}_t\}_{t\in\mathbb{R}}$ *exponentially attracts all mild solutions of* (1.1). *In the sense that for every mild solution* $u(.)$ *of* (1.1) *with initial condition* u_s, *there exists a solution* $\bar{u}(.)$ *in* $\mathcal{S}$ (*that is,* $\bar{u}_t\in\mathcal{S}_t$, *for all* $t\geq s$) *and constants* α $\beta>0$, *such that*

$$\|u_t-\bar{u}_t\|_{\mathcal{C}}\leq\alpha e^{-\beta(t-s)}\|u_s-\bar{u}_s\|_{\mathcal{C}}\quad\textit{for all}\quad t\geq s.$$

Proof. For $s\in\mathbb{R}$, we define the following space:

$$\mathcal{C}_{s,\mu}:=\big\{x\in C\big([s-r,\infty),X\big)\text{ such that }\sup_{t\geq s-r}e^{\mu(t-s)}\|x(t)\|<\infty\big\},$$

equipped with the norm $|x|_\mu := \sup_{t\geq s-r} e^{\mu(t-s)}\|x(t)\|$. Clearly that $\mathcal{C}_{s,\mu}$ is a Banach space. To find our desired result, we will prove that there exists $\bar{u}(.) = u(.) + w(.)$ such that $w \in \mathcal{C}_{s,\mu}$.

One can see that $\bar{u}(.)$ is a solution of (1.1) if and only if $w(.)$ is a solution of the equation

$$w(t) = U_B(t,s)w(s) + \lim_{\lambda\to\infty}\int_s^t U_B(t,\tau)\lambda R(\lambda,A)(f(\tau,u_\tau+w_\tau) - f(\tau,u_\tau))d\tau. \tag{3.1}$$

Putting $\bar{f}(t,w_t) = f(t,u_t+w_t) - f(t,u_t)$, we can show that $\bar{f}\colon \mathbb{R}\times\mathcal{C} \to X$ is φ-Lipschitz. Moreover, we have $\bar{f}(t,0) = 0$. Hence, we rewrite equation (3.1) as

$$w(t) = U_B(t,s)w(s) + \lim_{\lambda\to\infty}\int_s^t U_B(t,\tau)\lambda R(\lambda,A)\bar{f}(\tau,w_\tau)d\tau. \tag{3.2}$$

It follows from Remark 2.3 that solutions of (3.2) are bounded on $[s-r,+\infty)$ if and only if

$$w(t) = U_B(t,s)\mu_0 + \lim_{\lambda\to\infty}\int_s^{+\infty} \Gamma_B(t,\tau)\lambda R(\lambda,A)\bar{f}(\tau,w_\tau)d\tau \tag{3.3}$$

for some $\mu_0 \in \operatorname{Im} P_B(s)$ and $t \geq s$.

Now, let us find $\mu_0 \in \operatorname{Im} P_B(s)$, such that $\bar{u}_s = u_s + w_s \in \mathcal{S}_s$. This means

$$\left(\tilde{P}_B(s)(u_s+w_s)\right)(\theta) = \mathcal{G}_s\big((I-\tilde{P}_B(s))(u_s+w_s)\big)(\theta), \tag{3.4}$$

which gives

$$\mu_0 = (\tilde{P}_B(s)w_s)(0)\mathcal{G}_s\big((I-\tilde{P}_B(s))(u_s+w_s)\big)(0) - P_B(s)u(s). \tag{3.5}$$

In order to look for $\bar{u}(.)$ satisfying (1.1) and $\bar{u}_s \in \mathcal{S}_s$, we will find solutions for the following system in the Banach space $\mathcal{C}_{s,\mu}$:

$$(Cw)(t) = \begin{cases} U_B(t,s)\left(\mathcal{G}_s\big((I-\tilde{P}_B(s))(u_s+w_s)\big)(0) - P_B(s)u(s)\right) + \\ + \lim\limits_{\lambda\to\infty}\displaystyle\int_s^{+\infty} \Gamma_B(t,\tau)\lambda R(\lambda,A)\bar{f}(\tau,w_\tau)d\tau \quad \text{for} \quad t\geq s, \\ U_B(2s-t,s)\left(\mathcal{G}_s\big((I-\tilde{P}_B(s))(u_s+w_s)\big)(0) - P_B(s)u(s)\right) + \\ + \lim\limits_{\lambda\to\infty}\displaystyle\int_s^{+\infty} \Gamma_B(2s-t,\tau)\lambda R(\lambda,A)\bar{f}(\tau,w_\tau)d\tau \quad \text{for} \quad s-r\leq t\leq s. \end{cases}$$

Taking $s-r \leq t \leq s$, we have

$$e^{\mu(t-s)}\|(Cw)(t)\| \leq e^{\mu(t-s)}\|U_B(2s-t,s)\mu_0\| +$$

$$+e^{\mu(t-s)}\lim_{\lambda\to\infty}\int_s^{+\infty} \|\Gamma_B(2s-t,\tau)\lambda R(\lambda,A)\bar{f}(\tau,w_\tau)\|d\tau \leq$$

$$\leq Le^{2\mu(t-s)}\|\mu_0\|+$$

$$+L(1+K)e^{\mu r}|w|_\mu\left(\int\limits_s^{2s-t} e^{-2\mu(s-t)}\varphi(\tau)d\tau+\int\limits_{2s-t}^{\infty} e^{-2\mu(\tau-s)}\varphi(\tau)d\tau\right).$$

Using the fact that φ belongs to some admissible spaces, it yields that there exists $a>0$ such that, for all $s-r\leq t\leq s$,

$$e^{\mu(t-s)}\|(Cw)(t)\|\leq L\|\mu_0\|+aL(1+K)e^{\mu r}|w|_\mu.$$

Concerning $t\geq s$, we get

$$e^{\mu(t-s)}\|(Cw)(t)\|\leq$$

$$\leq e^{\mu(t-s)}\|U_B(t,s)\mu_0\|+e^{\mu(t-s)}\lim_{\lambda\to\infty}\int\limits_s^{+\infty}\|\Gamma_B(t,\tau)\lambda R(\lambda,A)\bar{f}(\tau,w_\tau)\|d\tau\leq$$

$$\leq L\|\mu_0\|+L(1+K)e^{\mu r}|w|_\mu\left(\int\limits_s^t\varphi(\tau)d\tau+\int\limits_t^{+\infty}e^{-2\mu(\tau-t)}\varphi(\tau)d\tau\right)\leq$$

$$\leq L\|\mu_0\|+aL(1+K)e^{\mu r}|w|_\mu.$$

Consequently, we obtain

$$|Cw|_\mu\leq L\|\mu_0\|+aL(1+K)e^{\mu r}|w|_\mu.$$

Furthermore, by virtue of the Lipschitz condition on $\mathcal{G}_s$, we have in view of (3.5)

$$\|\mu_0\|\leq\|\mathcal{G}_s((I-\tilde{P}_B(s))u_s)(0)-P_B(s)u(s)\|+$$

$$+\|\mathcal{G}_s((I-\tilde{P}_B(s))(u_s+w_s))(0)-\mathcal{G}_s((I-\tilde{P}_B(s))u_s)(0)\|\leq$$

$$\leq\|\mathcal{G}_s((I-\tilde{P}_B(s))u_s)-\tilde{P}_B(s)u_s\|_{\mathcal{C}}+\mathrm{Lip}(\mathcal{G}_s)(1+LK)e^{\mu r}|w|_\mu.$$

Hence,

$$|Cw|_\mu\leq L\|\mathcal{G}_s\big((I-\tilde{P}_B(s))u_s\big)-\tilde{P}_B(s)u_s\|_{\mathcal{C}}+L\,\mathrm{Lip}(\mathcal{G}_s)(1+LK)e^{\mu r}|w|_\mu+$$

$$+aL(1+K)e^{\mu r}|w|_\mu\leq$$

$$\leq L\|\mathcal{G}_s((I-\tilde{P}_B(s))u_s)-\tilde{P}_B(s)u_s\|_{\mathcal{C}}+\tilde{a}|w|_\mu. \tag{3.6}$$

This means that Cw belongs to $\mathcal{C}_{s,\mu}$.

Now, we propose to prove that C is a contraction. Let $v,\ w\in\mathcal{C}_{s,\mu}$ and $t\in[s-r,s]$, then

$$e^{\mu(t-s)}\|(Cv)(t)-(Cw)(t)\|\leq Le^{\mu(t-s)}e^{-\mu(s-t)}\|\epsilon_0-\zeta_0\|+$$

$$+L(1+K)e^{\mu(t-s)}\int\limits_s^{+\infty}e^{-\mu|2s-t-\tau|}\varphi(\tau)\|v_\tau-w_\tau\|_{\mathcal{C}}d\tau\leq$$

$$\leq L\|\epsilon_0-\zeta_0\|+aL(1+K)e^{\mu r}|v-w|_\mu.$$

For $t \geq s$,

$$e^{\mu(t-s)}\|(Cv)(t)-(Cw)(t)\| \leq L\|\epsilon_0-\zeta_0\|+$$

$$+L(1+K)e^{\mu(t-s)}\int\limits_{s}^{+\infty} e^{-\mu|t-\tau|}\varphi(\tau)\|v_\tau-w_\tau\|_{\mathcal{C}}d\tau \leq$$

$$\leq L\|\epsilon_0-\zeta_0\|+aL(1+K)e^{\mu r}|v-w|_\mu.$$

Consequently,

$$|C_v-C_w|_\mu \leq L\|\epsilon_0-\zeta_0\|+aL(1+K)e^{\mu r}|v-w|_\mu.$$

Furthermore,

$$\|\epsilon_0-\zeta_0\| = \|\mathcal{G}_s\left(\big(I-\tilde{P}_B(s)\big)(u_s+v_s)\right)(0)-\mathcal{G}_s((I-\tilde{P}_B(s))(u_s+w_s))(0)\| \leq$$

$$\leq \operatorname{Lip}(\mathcal{G}_s)\|(I-\tilde{P}_B(s))(v_s-w_s)\|_{\mathcal{C}} \leq \operatorname{Lip}(\mathcal{G}_s)(1+LK)e^{\mu r}|v-w|_\mu.$$

Hence, we get

$$|Cv-Cw|_\mu \leq L\operatorname{Lip}(\mathcal{G}_s)(1+LK)e^{\mu r}|v-w|_\mu+aL(1+K)e^{\mu r}|v-w|_\mu \leq$$

$$\leq Le^{\mu r}(\operatorname{Lip}(\mathcal{G}_s)(1+LK)+a(1+K))|v-w|_\mu.$$

Since $\tilde{a}<1$, C is a contraction on $\mathcal{C}_{s,\mu}$ and so, it has a unique fixed w, which belongs to $\mathcal{C}_{s,\mu}$. Using (3.6), it follows that

$$|w|_\mu \leq \frac{L}{1-\tilde{a}}\left\|\mathcal{G}_s((I-\tilde{P}_B(s))u_s)-\tilde{P}_B(s)u_s\right\|_{\mathcal{C}}.$$

Consequently, we obtain from (3.4)

$$\|\bar{u}_t-u_t\|_{\mathcal{C}} = \|w_t\|_{\mathcal{C}} \leq e^{\mu r}e^{-\mu(t-s)}|w|_\mu \leq$$

$$\leq e^{\mu r}e^{-\mu(t-s)}\frac{L}{1-\tilde{a}}\big(\|\mathcal{G}_s((I-\tilde{P}_B(s))\bar{u}_s)-\mathcal{G}_s((I-\tilde{P}_B(s))u_s)\|_{\mathcal{C}}+\|\tilde{P}_B(s)(\bar{u}_s-u_s)\|_{\mathcal{C}}\big) \leq$$

$$\leq e^{\mu r}e^{-\mu(t-s)}\frac{L}{1-\tilde{a}}\left(\operatorname{Lip}(\mathcal{G}_s)(1+LK)\|u_s-\bar{u}_s\|_{\mathcal{C}}+LK\|u_s-\bar{u}_s\|\right) \leq$$

$$\leq e^{\mu r}e^{-\mu(t-s)}\frac{L}{1-\tilde{a}}\left(\operatorname{Lip}(\mathcal{G}_s)(1+LK)+LK\right)\|u_s-\bar{u}_s\|_{\mathcal{C}}$$

for all $t \geq s$.

Theorem 3.1 is proved.

4. On the existence of center-unstable manifolds. This section is devoted to investigating the existence of center-unstable manifolds for solutions of (1.1), in the case that the evolution family $\big\{U_B(t,s)\big\}_{t\geq s}$ has an exponential trichotomy on $\mathbb{R}$ and the nonlinear term f is φ-Lipschitz. In the following, we assume that

(H_5) The evolution family $\{U_B(t,s)\}_{t\geq s}$ has an exponential trichotomy with the trichotomy projections $P_{B,j}(t),\ t\in\mathbb{R},\ j=1,2,3,$ and constants $L,\ \gamma,\ \zeta>0.$

Using the trichotomy projections $P_{B,j}(t),\ t\in\mathbb{R},\ j=1,2,3,$ we define the families of projections $\{\mathbb{P}_{B,j}(t)\},\ t\in\mathbb{R},\ j=1,2,3,$ on $\mathcal{C}_A$ as follows:

$$(\mathbb{P}_{B,j}(t)\xi)(\theta)=\big[U_{B|}(t,t+\theta)\big]^{-1}(I-P_{B,j}(t))\xi(0)\quad\text{for all}\quad\theta\in[-r,0]\quad\text{and}\quad\xi\in\mathcal{C}_A. \tag{4.1}$$

Now, we give our main result of this section, which proves the existence of center-unstable manifold for solutions of (1.1).

Theorem 4.1. *Assume that* (H_1) – (H_3) *and* (H_5) *hold. Set*

$$\bar{K}:=\sup\big\{\|P_{B,j}(t)\|: t\geq 0\ \ j=2,3\big\},\qquad \bar{L}:=\max\{L,2L\bar{K}\} \tag{4.2}$$

and

$$H:=\frac{\bar{L}(1+\bar{K})e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-\tilde{\mu}}}$$

for some $\tilde{\mu}$ *fixed below. Then if* $H<\min\left\{1,\dfrac{e^{\tilde{\mu}r}}{1+\bar{L}}\right\}$ *for each fixed* $\rho>\gamma$, *there exists a center-unstable manifold* $\mathcal{CU}=\big\{(t,\mathcal{CU}_t)\big\}_{t\in\mathbb{R}}\subset\mathbb{R}\times\mathcal{C}_A$ *for solutions of* (1.1), *which is represented by a family of Lipschitz continuous mappings*

$$\mathcal{G}_t:\ \mathrm{Im}(\mathbb{P}_{B,2}(t)\oplus\mathbb{P}_{B,3}(t))\to\mathrm{Im}\,\mathbb{P}_{B,1}(t)$$

with Lipschitz constants independent of t such that $\mathcal{CU}_t$ *satisfies the following conditions:*

a) $\mathcal{CU}_t$ *is homeomorphic to* $\mathrm{Im}(\mathbb{P}_{B,2}(t)+\mathbb{P}_{B,3}(t))$ *for all* $t\in\mathbb{R}$.

b) *To each* $\xi\in\mathcal{CU}_s$, *there corresponds one and only one solution of* (1.1) *on* $(-\infty,s]$ *such that* $e^{-\tau(s+\theta)}u_s(\theta)=\xi(\theta)$ *for* $\theta\in[-r,0]$ *and* $\sup_{t\leq s}\|e^{-\tau(t+.)}u_t(.)\|_{\mathcal{C}}<\infty$, *where* $\tau=\dfrac{\rho+\gamma}{2}$. *Furthermore, for any two solutions* $u(t)$ *and* $v(t)$ *of equation* (1.1) *corresponding to different initial functions* $\xi_1,\ \xi_2\in\mathcal{CU}_s$, *there exist positive constants* ν *and* C_ν *independent of* $s\in\mathbb{R}$ *such that*

$$\|u_t-v_t\|_{\mathcal{C}}\leq C_\nu e^{(\tau-\nu)(t-s)}\big\|(\mathbb{P}_B(s)\xi_1)(0)-(\mathbb{P}_B(s)\xi_2)(0)\big\|\quad\text{for}\quad t\geq s,$$

where $\mathbb{P}_B(t)=\mathbb{P}_{B,2}(t)+\mathbb{P}_{B,3}(t)$.

c) $\mathcal{CU}$ *is invariant under equation* (1.1). *That is, if* $u(t),\ t\in\mathbb{R}$, *is a solution of* (1.1) *such that* $e^{-\tau(s+.)}u_s\in\mathcal{CU}_s$ *and* $\sup_{t\leq s}\|e^{-\tau(t+.)}u_t(.)\|_{\mathcal{C}}<\infty$, *then* $e^{-\tau(t+.)}u_t(.)\in\mathcal{CU}_t$ *for all* $t\in\mathbb{R}$.

Proof. Put $P_B(t):=P_{B,2}(t)+P_{B,3}(t)$ and $Q_B(t):=P_{B,1}(t)=Id-P_B(t)$ for $t\in\mathbb{R}$. Then $P_B(t)$ and $Q_B(t)$ are projections complemented to each other on X_0. We then define the corresponding projections $\{\mathbb{P}_{B,j}(t)\},\ t\in\mathbb{R},\ j=1,2,3,$ on $\mathcal{C}_A$ as in equality (4.1). Let $\mathbb{P}_B(t):=$ $:=\mathbb{P}_{B,2}(t)+\mathbb{P}_{B,3}(t)$ and $\mathbb{Q}_{B,t}:=\mathbb{P}_{B,1}(t)=Id-\mathbb{P}_B(t)$ for $t\in\mathbb{R}$. Hence, $\mathbb{P}_B(t)$ and $\mathbb{Q}_B(t)$ are complemented on $\mathcal{C}_A$ for each $t\in\mathbb{R}$. Now, consider the following evolution family:

$$\bar{U}_B(t,s)=e^{-\tau(t-s)}U_B(t,s)\quad\text{for all}\quad t\geq s.$$

Let us prove that the evolution family $\bar{U}_B(t,s)$ has an exponential dichotomy with the dichotomy projections $Q_B(t),\ t\in\mathbb{R}$. In fact, we have, for $t\geq s$,

$$Q_B(t)\bar{U}_B(t,s) = e^{-\tau(t-s)}P_{B,1}(t)U_B(t,s) =$$

$$= e^{-\tau(t-s)}U_B(t,s)P_{B,1}(s) = \bar{U}_B(t,s)Q_B(s).$$

Since $U_{B|}(t,s)$ is a homeomorphism from $\operatorname{Im} P_{B,j}(s)$ onto $\operatorname{Im} P_{B,j}(t),\ j = 2,3$, then it is a homeomorphism from $\operatorname{Im} P_B(s)$ onto $\operatorname{Im} P_B(t)$. As $\operatorname{Im} P_B(s) = \operatorname{Ker} Q_B(s),\ s \in \mathbb{R}$, then $\bar{U}_{B|}(t,s)$ is a homeomorphism from $\operatorname{Ker} Q_B(s)$ onto $\operatorname{Ker} Q_B(t)$. By using the definition of exponential trichotomy, we obtain

$$\|\bar{U}_B(t,s)Q_B(s)x\| \le Le^{-(\zeta+\tau)(t-s)}\|Q_B(s)x\| \quad \text{for all} \quad t \ge s,$$

besides

$$\|[\bar{U}_{B|}(t,s)]^{-1}P_B(t)x\| = e^{-\tau(t-s)}\|[U_{B|}(t,s)]^{-1}(P_{B,2}(t) + P_{B,3}(t))x\| \le$$

$$\le Le^{-\tau(t-s)}\big(e^{-\zeta(t-s)}\|P_{B,2}(t)P_B(t)x\| + e^{\gamma(t-s)}\|P_{B,3}(t)P_B(t)x\|\big)$$

for all $t \ge s$ and $x \in X_0$. By (4.2), we obtain

$$\big\|[\bar{U}_{B|}(t,s)]^{-1}P_B(t)x\big\| \le 2L\bar{K}e^{-\frac{\rho-\gamma}{2}(t-s)}\|P_B(t)x\|.$$

Consequently, $\bar{U}_B(t,s)$ has an exponential dichotomy with the dichotomy projections $P_B(t),\ t \in \mathbb{R}$, and constants $\bar{L},\ \tilde{\mu} := \dfrac{\rho-\gamma}{2}$.

Set $z(t) := e^{-\tau t}u(t)$ and let the mapping $G\colon \mathbb{R}\times\mathcal{C} \to X$ be defined by

$$G(t,\xi) = e^{-\tau t}f(t, e^{\tau(t+.)}\xi(.)) \quad \text{for} \quad (t,\xi) \in \mathbb{R}\times\mathcal{C}.$$

It is easy to show that G is also φ-Lipschitz. Hence, equation (2.2) can be rewritten as

$$z(t) = \bar{U}_B(t,s)z(s) + \lim_{\lambda\to\infty}\int_s^t \bar{U}_B(t,\sigma)\lambda R(\lambda,A)G(\sigma,z_\sigma)d\sigma \quad \text{for all} \quad t \ge s,$$

$$z_s(.) = e^{-\tau(s+.)}\Phi(.) \in \mathcal{C}_A. \tag{4.3}$$

By virtue of Theorem 2.3, we obtain that, if $H < \min\left\{1, \dfrac{e^{\tilde{\mu}r}}{1+\bar{L}}\right\}$, then there exists an unstable manifold $\mathcal{M}$ for solutions of (4.3). Returning to equation (2.2), by the relation $u(t) := e^{\tau t}z(t)$, in view of Theorems 2.2 and 2.3, it is easy to check properties of $\mathcal{CU}$. Therefore, $\mathcal{CU}$ is a center-unstable manifold for solutions of (1.1).

References

1. N. N. Bogoliubov, Yu. A. Mitropolsky, *The method of integral manifolds in nonlinear mechanics*, Contrib. Different. Equat., **2**, 123 – 196 (1963).
2. L. Boutet de Monvel, I. D. Chueshov, A. V. Rezounenko, *Inertial manifolds for retarded semilinear parabolic equations*, Nonlinear Anal., **34**, 907 – 925 (1998).
3. S. N. Chow, K. Lu, *Invariant manifolds for flows in Banach spaces*, J. Different. Equat., **74**, 285 – 317 (1988).

4. P. Constantin, C. Foias, B. Nicolaenko, R. Temam, *Integral manifolds and inertial manifolds for dissipative partial differential equations*, Springer-Verlag, New York (1989).
5. G. Da Prato, E. Sinestrari, *Differential operators with non-dense domains*, Ann. Scuola Norm. Super. Pisa Cl. Sci., **14**, 285 – 344 (1987).
6. T. V. Duoc, N. T. Huy, *Integral manifolds and their attraction property for evolution equations in admissible function spaces*, Taiwanese J. Math., **16**, 963 – 985 (2012).
7. T. V. Duoc, N. T. Huy, *Integral manifolds for partial functional differential equations in admissible spaces on a half line*, J. Math. Anal. and Appl., **411**, 816 – 828 (2014).
8. T. V. Duoc, N. T. Huy, *Unstable manifolds for partial functional differential equations in admissible spaces on the whole line*, Vietnam J. Math., **43**, 37 – 55 (2015).
9. G. Guhring, F. Rabiger, *Asymptotic properties of mild solutions for nonautonomous evolution equations with applications to retarded differential equations*, J. Abstr. and Appl. Anal., **4**, № 3, 169 – 194 (1999).
10. M. W. Hirsch, C. C. Pugh, M. Shub, *Invariant manifolds*, Springer-Verlag, Berlin; Heidelberg (1977).
11. N. T. Huy, *Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line*, J. Funct. Anal., **235**, 330 – 354 (2006).
12. N. T. Huy, *Stable manifolds for semi-linear evolution equations and admissibility of function spaces on a half-line*, J. Math. Anal. and Appl., **354**, 372 – 386 (2009).
13. N. T. Huy, *Invariant manifolds of admissible classes for semi-linear evolution equations*, J. Differen. Equat., **246**, 1822 – 1844 (2009).
14. C. Jendoubi, *Unstable manifolds of a class of delayed partial differential equations with nondense domain*, Ann. Polon. Math., 181 – 208 (2016).
15. C. Jendoubi, *Integral manifolds of a class of delayed partial differential equations with nondense domain*, Numer. Funct. Anal. and Optim., **38**, 1024 – 1044 (2017).
16. Z. Liu, P. Magal, S. Ruan, *Center-unstable manifolds for non-densely defined Cauchy problems and applications to stability of Hopf bifurcation*, Canad. Appl. Math. Quart., **20**, 135 – 178 (2012).
17. L. Maniar, *Stability of asymptotic properties of Hille – Yosida operators under perturbations and retarded differential equations*, Quaest. Math., **28**, 39 – 53 (2005).
18. N. V. Minh, J. Wu, *Invariant manifolds of partial functional differential equations*, J. Different. Equat., **198**, 381 – 421 (2004).
19. J. D. Murray, *Mathematical biology I: an introduction*, Springer-Verlag, Berlin (2002).
20. J. D. Murray, *Mathematical biology II: spatial models and biomedical applications*, Springer-Verlag, Berlin (2003).
21. A. Rhandi, *Extrapolation methods to solve non-autonomous retarded partial differential equations*, Stud. Math., **126**, № 3, 219 – 233 (1998).
22. H. R. Thieme, *Semiflows generated by Lipschitz perturbations of nondensely defined operators*, Different. Integral Equat., **3**, 1035 – 1066 (1990).
23. H. R. Thieme, *"Integrated semigroups" and integrated solutions to abstract Cauchy problems*, J. Math. Anal. and Appl., **152**, 416 – 447 (1990).

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