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ON THE COMMUTATOR OF MARCINKIEWICZ INTEGRALS WITH ROUGH KERNELS IN VARIABLE MORREY TYPE SPACES*

ПРО КОМУТАТОР ІНТЕГРАЛІВ МАРЦИНКЕВИЧА З ГРУБИМИ ЯДРАМИ У ЗМІННИХ ПРОСТОРАХ ТИПУ МОРРЕЯ

In the framework of variable exponent Morrey and Morrey – Herz spaces, we prove some boundedness results for the commutator of Marcinkiewicz integrals with rough kernels. The approach is based on the theory of variable exponent and on generalization of the BMO-norms.

У рамках змінних експонент просторів Моррея та Моррея – Герца доведено деякі результати щодо обмеженості комутатора інтегралів Марцинкевича з грубими ядрами. Цей підхід базується на теорії змінних експонент та узагальненні BMO-норм.

1. Introduction. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ with norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $n \geq 2$, equipped with the normalized Lebesgue measure $d\sigma(x')$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int\limits_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral operator μ_Ω of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int\limits_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(f)(x) = \int\limits_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

A locally integrable function b is said to be a $\mathrm{BMO}(\mathbb{R}^n)$ function, if it satisfies

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B|} \int\limits_B |b(y) - b_B|\, dy < \infty,$$

where B is ball centered at x and radius of r, $b_B = \dfrac{1}{|B|}\displaystyle\int_B b(t)dt$ and $\|b\|_*$ is the norm in $\mathrm{BMO}(\mathbb{R}^n)$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutator of the Marcinkiewicz integral operator $\mu_{\Omega,b}$, is then

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defined by

$$\mu_{\Omega,b}(f)=\left(\int\limits_0^\infty\left|\int\limits_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}.$$

It is well-known that Stein [23] first proved that if $\Omega\in\mathrm{Lip}_\gamma(\mathbb{S}^{n-1}),\ 0<\gamma\le1$, then μ_Ω is of type (p,p) for $1<p\le2$ and of weak type $(1,1)$. Afterwards, Ding, Fan and Pan [7] removed the smoothness assumed on Ω and showed that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ if $\Omega\in H^1(\mathbb{S}^{n-1})$. Here $H^1(\mathbb{S}^{n-1})$ denotes the classical Hardy space on $\mathbb{S}^{n-1}$. On the other hand, using a good-λ inequality, Torchinsky and Wang [25] established the weighted L^p-boundedness of μ_Ω and $\mu_{\Omega,b}$ when $\Omega\in\mathrm{Lip}_\gamma(\mathbb{S}^{n-1}),\ 0<\gamma\le1$. For some recent development, we refer to [2, 8, 14 – 18] and their references.

In recent years, following the fundamental work of Kováčik and Rákosník [13], function spaces with variable exponent, such as the vaiable exponent Lebesgue, Herz and Morrey spaces etc., have attracted a great attention due mainly to their useful applications in fluid dynamics, image restoration and differential equations with $p(x)$-growth (see [1, 3, 11, 30 – 32] and the references therein). In many applications, a crucial step has been to prove that the classical operators are bounded in variable exponent function spaces. Ho in [9, 10] has given some sufficient conditions for the boundedness of fractional operators and singular integral operators in variable exponent Morrey spaces $\mathcal{M}_{p(\cdot),u}$, where u is a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$ (see Definition 3.1). In 2016, Tao and Li [26] showed that if $\Omega\in\mathrm{Lip}_\gamma(\mathbb{S}^{n-1}),\ 0<\gamma\le1$, then the commutator $\mu_{\Omega,b}$ is bounded on $\mathcal{M}_{p(\cdot),u}$. On the other hand, based on the extrapolation theory and some pointwise estimates, operators with rough kernels have recently been discussed in [5, 22, 27]. These results inspire us to consider the question: whether the variable exponent Morrey spaces estimates for $\mu_{\Omega,b}$ are still true if $\Omega\in L^s(\mathbb{S}^{n-1})$, $s>1$? The first aim of this paper is to give an affirmative answer to this question.

Morrey – Herz spaces $M\dot K^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents p and α were recently studied by Lu and Zhu [20]. Under natural regularity assumptions on the exponent α and p, either at the origin or at infinity, they established the boundedness of a wide class of sublinear operators (including maximal, potential and Calderón – Zygmund operators) and their commutators on such spaces. In [29], we made a further step and generalized the main theorems in [20] to the case of rough kernels. Motivated by the work of [20] and [29], the second aim of this paper is to prove that $\mu_{\Omega,b}$ is bounded on $M\dot K^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ provided that $\lambda\ge0$ and α, p are variable exponents. This result improves the corresponding main theorem in [28], where the authors considered only the case $\lambda=0$ and α is a constant.

We usually denote cubes in $\mathbb{R}^n$ by Q, $|Q|$ is the Lebesgue measure of Q. χ_E is a characteristic function of a measurable set $E\subset\mathbb{R}^n$. Let $B_l=\{x\in\mathbb{R}^n:|x|\le2^l\},\ l\in\mathbb{Z}$, and $B:=B(x,r)=\{y\in\mathbb{R}^n:|x-y|<r\}$. f_B means the integral average of f on B, namely, $f_B=\dfrac{1}{|B|}\displaystyle\int_B f(x)dx$. $p'(\cdot)$ denotes the conjugate exponent defined by $1/p(\cdot)+1/p'(\cdot)=1$. The letter C stands for a positive constant, which may vary from line to line. The expression $f\lesssim g$ means that $f\le Cg$, and $f\approx g$ means $f\lesssim g\lesssim f$.

2. Preliminaries and lemmas. We begin with a brief and necessarily incomplete review of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see [4, 6] for more information).

Given a measurable function p, we assume that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{2}$$

where $p_- := \text{ess inf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ := \text{ess sup}_{x\in\mathbb{R}^n} p(x)$.

By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the set of all measurable functions f on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

This is a Banach space with the norm (the Luxemburg – Nakano norm)

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\mu > 0 : I_{p(\cdot)}(f/\mu) \le 1\}.$$

It is easy to see that this norm has the following property:

$$\||f|^\sigma\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^\sigma_{L^{\sigma p(\cdot)}(\mathbb{R}^n)}, \qquad \sigma \ge 1/p_-. \tag{3}$$

By $\mathcal{P}(\mathbb{R}^n)$ we denote the set of variable exponents $p(\cdot)$ satisfying (2). When $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the generalized Hölder inequality holds in the form

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \tag{4}$$

with $r_p = 1 + 1/p_- - 1/p_+$ (see [13], Theorem 2.1).

The set $\mathcal{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, where M denotes the Hardy – Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

It is well-known that if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ (see [12], Proposition 2).

A function $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is called log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $C_{\log} > 0$ such that

$$|\phi(x) - \phi(0)| \le \frac{C_{\log}}{\log(e + 1/|x|)}, \qquad x \in \mathbb{R}^n.$$

If, for some $\phi_\infty \in \mathbb{R}$ and $C_{\log} > 0$, there holds

$$|\phi(x) - \phi_\infty| \le \frac{C_{\log}}{\log(e + |x|)}, \qquad x \in \mathbb{R}^n,$$

then $\phi(\cdot)$ is called log-Hölder continuous at infinity (or has a log decay at the infinity).

Lemma 2.1. *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *then we have*

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim |B|.$$

Lemma 2.2. *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we have, for all measurable subsets $E \subset B$,*

$$\frac{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_1}, \qquad \frac{\|\chi_E\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Lemma 2.3. *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $k > j$, k, $j \in \mathbb{N}$, then we have*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|b\|_*,$$

$$\|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim (k - j)\|b\|_*\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemmas 2.1 – 2.3 are due to Izuki [12].

Lemma 2.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $q > p_+$ and $\dfrac{1}{p(x)} = \dfrac{1}{\widetilde{q}(x)} + \dfrac{1}{q}$, then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\widetilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

for all measurable functions f and g.

Lemma 2.5. *Let $r_1 > 0$. Suppose $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ is* log-*Hölder continuous both at origin and at infinity, then we have*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\dfrac{r_1}{r_2}\right)^{\alpha_+}, & 0 < r_2 \le r_1/2, \\ 1, & r_1/2 < r_2 \le 2r_1, \\ \left(\dfrac{r_1}{r_2}\right)^{\alpha_-}, & r_2 > 2r_1, \end{cases}$$

for any $x \in B(0, r_1)\backslash B(0, r_1/2)$ and $y \in B(0, r_2)\backslash B(0, r_2/2)$.

The proof of Lemmas 2.4 and 2.5 can be found in [21] and [1], respectively.

3. Boundedness on variable exponent Morrey spaces. We first recall the following definitions given by Ho in [9].

Definition 3.1. *Let $p(\cdot) \in L^\infty(\mathbb{R}^n)$. A Lebesgue measurable function $u(z, r): \mathbb{R}^n \times (0, +\infty) \to \to (0, +\infty)$ is said to be a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$ if u satisfies*

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(z, 2^{j+1}r) \lesssim u(z, r) \tag{5}$$

for any $z \in \mathbb{R}^n$ and $r > 0$.

By $\mathbb{W}_{p(\cdot)}$, we denote the class of Morrey weight functions. We note that condition (5) is also used to study the Fefferman – Stein vector-valued inequalities in weighted Morrey spaces (see [10]).

For any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, let $\mathcal{K}_{p(\cdot)}$ denote the supremum of those $q > 1$ such that $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{E}_{p(\cdot)}$ be the conjugate of $\mathcal{K}_{p'(\cdot)}$. The following result can be seen as a special case of the general result in [10] for Banach function spaces.

Proposition 3.1. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For any $1 < q < \mathcal{K}_{p(\cdot)}$ and $1 < \tau < \mathcal{K}_{p'(\cdot)}$, we have, for any $z \in \mathbb{R}^n$ and $r > 0$,*

$$2^{jn(1-\frac{1}{\tau})} \lesssim \frac{\|\chi_{B(z,2^j r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim 2^{\frac{jn}{q}} \qquad \forall j \in \mathbb{N}. \tag{6}$$

Remark **3.1.** It is easy to check that condition (5) together with (6) yields $u(z, 2r) \lesssim u(z, r)$ for any $z \in \mathbb{R}^n$ and $r > 0$.

Definition 3.2. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $u \in \mathbb{W}_{p(\cdot)}$. The variable Morrey space $\mathcal{M}_{p(\cdot),u}$ is the collection of all Lebesgue measurable functions f satisfying*

$$\|f\|_{\mathcal{M}_{p(\cdot),u}} = \sup_{z\in\mathbb{R}^n, R>0} \frac{1}{u(z,R)} \|f\chi_{B(z,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Now, let us state the main result in this section.

Theorem 3.1. *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s > (p')_+$ is homogeneous of degree zero and satisfies* (1)*. If* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and*

$$\sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(z, 2^{j+1}r) \lesssim u(z,r), \tag{7}$$

for any $z \in \mathbb{R}^n$ and $r > 0$, then we have

$$\|\mu_{\Omega,b}(f)\|_{\mathcal{M}_{p(\cdot),u}} \lesssim \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

Remark **3.2.** Clearly, in comparison with the corresponding result by Tao and Li in [26, p. 56], the smoothness condition on Ω has been removed. More precisely, our result is an improvement of Theorem 1.4 in [26].

Remark **3.3.** There do exist some functions satisfying condition (7). For instance, if, for any $0 \le \gamma < 1/\mathcal{E}_{p(\cdot)}$, a weight function u satisfies $u(z, 2r) \le 2^{n\gamma} u(z, r)$ for any $z \in \mathbb{R}^n$ and $r > 0$, then (7) holds. In fact, for any $\gamma < 1/\mathcal{E}_{p(\cdot)}$, there always exists a $\tau < 1/\mathcal{K}_{p'(\cdot)}$ such that $\gamma < 1 - 1/\tau < < 1 - 1/\mathcal{K}_{p'(\cdot)} = 1/\mathcal{E}_{p(\cdot)}$. An application of Proposition 3.1 gives

$$\sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(z, 2^{j+1}r)}{u(z,r)} \lesssim \sum_{j=0}^{\infty} (j+1) 2^{jn(\frac{1}{\tau}+\gamma-1)} \lesssim 1.$$

Proof of Theorem **3.1.** Let $f \in \mathcal{M}_{p(\cdot),u}$. For any $z \in \mathbb{R}^n$ and $r > 0$, we decompose $f = g + h$, where $g = f\chi_{B(z,2r)}$ and $h = \sum_{j=1}^{\infty} f\chi_{B(z,2^{j+1}r)\backslash B(z,2^j r)}$. Noting that $\mu_{\Omega,b}$ is a nonlinear operator, then we have

$$\frac{1}{u(z,r)} \|\chi_{B(z,r)}\mu_{\Omega,b}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \frac{1}{u(z,r)} \|\chi_{B(z,r)}\mu_{\Omega,b}(g)\|_{L^{p(\cdot)}(\mathbb{R}^n)} +$$

$$+ \frac{1}{u(z,r)} \|\chi_{B(z,r)}\mu_{\Omega,b}(h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} := I + II.$$

For I, using $u(z, 2r) \lesssim u(z, r)$ and the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of $\mu_{\Omega,b}$ (see [28, p. 262]), we obtain

$$I \lesssim \|b\|_* \frac{1}{u(z,2r)} \|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \|b\|_* \sup_{z\in\mathbb{R}^n, R>0} \frac{1}{u(z,R)} \|f\chi_{B(z,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

For II, we note that if $x \in B(z,r)$ and $y \in \widetilde{R_j} := B(z, 2^{j+1}r) \setminus B(z, 2^j r)$, then $|x-y| \approx \approx |y-z| \approx 2^j r$. The Minkowski inequality yields

$$|\mu_{\Omega,b}(h)(x)| \lesssim \sum_{j=1}^{\infty} \left\{ \int\limits_{\widetilde{R_j}} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(x)-b(y)||f(y)| \left(\int\limits_{|x-y|<t} \frac{dt}{t^3} \right)^{1/2} dy \right\} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} \int\limits_{\widetilde{R_j}} |b(x)-b(y)||\Omega(x-y)||f(y)|dy \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} |b(x) - b_{B(z,r)}| \int\limits_{\widetilde{R_j}} |\Omega(x-y)||f(y)|dy +$$

$$+ \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} |b_{B(z,2^{j+1}r)} - b_{B(z,r)}| \int\limits_{\widetilde{R_j}} |\Omega(x-y)||f(y)|dy +$$

$$+ \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} \int\limits_{\widetilde{R_j}} |b(y) - b_{B(z,2^{j+1}r)}||\Omega(x-y)||f(y)|dy :=$$

$$:= U_1 + U_2 + U_3.$$

For U_1, since $s > (p')_+$, then we can choose $p^*(\cdot) > 0$ such that $\dfrac{1}{s'} = \dfrac{1}{p(x)} + \dfrac{1}{p^*(x)}$, by the property (3) and the generalized Hölder inequality (4), we get

$$\|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}(\mathbb{R}^n)} \lesssim \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}. \tag{8}$$

In view of $\dfrac{1}{p^*(\cdot)} = \dfrac{1}{s'} - \dfrac{1}{p(\cdot)} = \dfrac{1}{p'(\cdot)} - \dfrac{1}{s}$, Lemma 2.4 in [27, p. 178] yields

$$\|\chi_{B(z,2^{j+1}r)}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)} \approx (2^j r)^{-\frac{n}{s}} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{9}$$

Thus, from (8), (9) and the Hölder inequality, it follows that

$$U_1 \lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} |b(x) - b_{B(z,r)}| \|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}(\mathbb{R}^n)} \left(\int\limits_{\widetilde{R_j}} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} |b(x) - b_{B(z,r)}| \|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}(\mathbb{R}^n)} \left(\int\limits_{|x-y|\lesssim 2^j r} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^{n(1-\frac{1}{s})}} |b(x) - b_{B(z,r)}| \|f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} |b(x) - b_{B(z,r)}| \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

For U_2, noting that $|b_{B(z,2^{j+1}r)} - b_{B(z,r)}| \lesssim (j+1)\|b\|_*$ (see [24, p. 206]), we have

$$U_2 \lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} |b_{B(z,2^{j+1}r)} - b_{B(z,r)}| \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} (j+1) \frac{1}{(2^j r)^n} \|b\|_* \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

For U_3, applying Lemma 2.3 with $B = B(z, 2^{j+1}r)$, (8) and (9), we obtain

$$U_3 \lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^{n(1-\frac{1}{s})}} \|(b - b_{B(z,2^{j+1}r)}) f\chi_{B(z,2^{j+1}r)}\|_{L^{s'}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b - b_{B(z,2^{j+1}r)})\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^n} \|b\|_* \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimate of U_1, U_2 and U_3, by Lemmas 2.3 and 2.1, we get

$$\|\chi_{B(z,r)}\mu_{\Omega,b}(h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} (j+1) \frac{1}{(2^j r)^n} \|b\|_* \|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \|b\|_* \sum_{j=1}^{\infty} (j+1) \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(z, 2^{j+1}r) \times$$

$$\times \sup_{z\in\mathbb{R}^n, R>0} \frac{1}{u(z,R)} \|f\chi_{B(z,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \|b\|_* \sum_{j=1}^{\infty} (j+1) \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(z, 2^{j+1}r) \|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

Thus, in view of the condition (7), we arrive at the desired inequality

$$II = \frac{1}{u(z,r)} \|\chi_{B(z,r)} \mu_{\Omega,b}(h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot),u}} \sum_{j=1}^{\infty} (j+1) \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(z,2^{j+1}r)}{u(z,r)} \lesssim \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

Theorem 3.1 is proved.

4. Boundedness on variable exponent Morrey – Herz spaces. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $R_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$.

Definition 4.1. *Let* $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. *The homogeneous Herz space* $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ *is defined as the class of all* $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ *such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)} := \left(\sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} < \infty$$

with the usual modification when $q = \infty$.

Definition 4.2. *Let* $0 \le \lambda < \infty$, $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. *The homogeneous Morrey – Herz space* $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ *is defined as the class of all* $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ *such that*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f \chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} < \infty$$

with the usual modification when $q = \infty$.

***Remark* 4.1.** It obviously follows that $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constants, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ coincides with the classical Morrey – Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ defined in [19].

Lu and Zhu [20] obtained the following result.

Proposition 4.1. *Let* $0 \le \lambda < \infty$, $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. *If* $\alpha(\cdot)$ *is* log-*Hölder continuous both at origin and at infinity, then*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx \max\left\{ \sup_{k_0<0, k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}, \right.$$

$$\left. \sup_{k_0 \ge 0, k_0 \in \mathbb{Z}} \left[2^{-k_0\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} + 2^{-k_0\lambda} \left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} \right] \right\}.$$

The results obtained in this section can be summarized as follows.

Theorem 4.1. *Suppose* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *and* $\Omega \in L^s(\mathbb{S}^{n-1})$ *with* $s > (p')_+$ *is homogeneous of degree zero and satisfies* (1). *Let* $\lambda > 0$, $0 < q \le \infty$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *be* log-*Hölder continuous both at the origin and at infinity such that*

$$\lambda - n\delta_1 < \alpha_- \le \alpha_+ < n\delta_2 - (n-1)/s,$$

where $0 < \delta_1, \delta_2 < 1$ *are the constants appearing in Lemma* 2.2. *Then the commutator* $\mu_{\Omega,b}$ *is bounded on* $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.

We would like to point out that Theorem 4.1 is still true in the particular case $\lambda = 0$, namely, in the framework of Herz spaces with variable exponents. By using the same method of proving Theorem 4.1, we get the following corollary.

***Corollary* 4.1.** *Suppose* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *and* $\Omega \in L^s(\mathbb{S}^{n-1})$ *with* $s > (p')_+$ *is homogeneous of degree zero and satisfies* (1). *Let* $0 < q \le \infty$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *be* log-*Hölder continuous both at the origin and at infinity such that*

$$-n\delta_1 < \alpha_- \le \alpha_+ < n\delta_2 - (n-1)/s,$$

where $0 < \delta_1,\ \delta_2 < 1$ *are the constants appearing in Lemma* 2.2. *Then the commutator* $\mu_{\Omega,b}$ *is bounded on* $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$.

***Proof of Theorem* 4.1.** Let $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. We decompose

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

The Minkowski inequality implies that

$$\|\mu_{\Omega,b}(f)\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} |\mu_{\Omega,b}(f)|\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left(\sum_{j=-\infty}^{k-2} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} +$$

$$+ \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left(\sum_{j=k-1}^{k+1} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} +$$

$$+ \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left(\sum_{j=k+2}^{\infty} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} :=$$

$$:= V_1 + V_2 + V_3.$$

For V_1, noticing that $|x-y| \approx |x| \approx 2^k$ for $x \in R_k, y \in R_j$ and $j \le k-2$, then we have

$$|\mu_{\Omega,b}(f_j)(x)| \lesssim \left(\int_0^{|x|} \left| \int_{|x-y|\le t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} +$$

$$+\left(\int\limits_{|x|}^{\infty}\left|\int\limits_{|x-y|\le t}(b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}\lesssim$$

$$\lesssim\int\limits_{R_j}\frac{|b(x)-b(y)||\Omega(x-y)||f_j(y)|}{|x-y|^{n-1}}\frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}}dy+$$

$$+\int\limits_{R_j}\frac{|b(x)-b(y)||\Omega(x-y)||f_j(y)|}{|x-y|^{n-1}}\frac{1}{|x|}dy\lesssim$$

$$\lesssim 2^{-kn}\int\limits_{R_j}|b(x)-b(y)||\Omega(x-y)||f_j(y)|dy.$$

This together with Lemma 2.5 yields

$$2^{k\alpha(x)}\sum_{j=-\infty}^{k-2}|\mu_{\Omega,b}(f_j)(x)|\chi_k(x)\lesssim$$

$$\lesssim\sum_{j=-\infty}^{k-2}2^{-kn}\int\limits_{R_j}2^{k\alpha(x)}|\Omega(x-y)||b(x)-b(y)||f_j(y)|dy\cdot\chi_k(x)\lesssim$$

$$\lesssim\sum_{j=-\infty}^{k-2}2^{-kn}2^{(k-j)\alpha_+}\int\limits_{R_j}2^{j\alpha(y)}|\Omega(x-y)||b(x)-b(y)||f_j(y)|dy\cdot\chi_k(x)\lesssim$$

$$\lesssim\sum_{j=-\infty}^{k-2}2^{-kn}2^{(k-j)\alpha_+}\Bigg(|b(x)-b_{B_j}|\int\limits_{R_j}2^{j\alpha(y)}|\Omega(x-y)||f_j(y)|dy+$$

$$+\int\limits_{R_j}2^{j\alpha(y)}|b_{B_j}-b(y)||\Omega(x-y)||f_j(y)|dy\Bigg)\cdot\chi_k(x). \tag{10}$$

Define a variable exponent $p^*(\cdot)$ by $\dfrac{1}{p'(x)}=\dfrac{1}{p^*(x)}+\dfrac{1}{s}$, since $s>(p')_+$, using the fact that $\|\chi_{B_j}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}$ (see [28, p. 258]) and Lemma 2.4, we obtain

$$\|\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\lesssim\|\Omega(x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)}\|\chi_j\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\lesssim$$

$$\lesssim\left(\int\limits_{|x|-2^j}^{|x|+2^j}\int\limits_{\mathbb{S}^{n-1}}|\Omega(y')|^s d\sigma(y')\varrho^{n-1}d\varrho\right)^{\frac{1}{s}}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}\lesssim$$

$$\lesssim 2^{(k-j)(n-1)/s}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{11}$$

An application of Lemmas 2.3, 2.4 and (11) gives

$$\|(b-b_{B_j})\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim \|(b-b_{B_j})\chi_j\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)} \lesssim$$

$$\lesssim \|b\|_*\|\chi_{B_j}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)} \lesssim 2^{(k-j)(n-1)/s}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{12}$$

Then, from (10)–(12), (4) and Lemmas 2.1–2.3, we deduce

$$\left\|2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}|\mu_{\Omega,b}(f_j)|\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn}2^{(k-j)\alpha_+}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big(\|(b-b_{B_j})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} +$$

$$+\|(b-b_{B_j})\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big) \lesssim$$

$$\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn}2^{(k-j)\alpha_+}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big((k-j)2^{(k-j)(n-1)/s}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} +$$

$$+2^{(k-j)(n-1)/s}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big) \lesssim$$

$$\lesssim \sum_{j=-\infty}^{k-2}(k-j)2^{-kn}2^{(k-j)(\alpha_+ + (n-1)/s)}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha_+ + (n-1)/s)}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \lesssim$$

$$\lesssim \sum_{j=-\infty}^{k-2}(k-j)2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Therefore, we arrive at the estimate

$$V_1 \approx \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}\left(\sum_{j=-\infty}^{k-2}|\mu_{\Omega,b}(f_j)|\right)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2}(k-j)2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q.$$

Now we can distinguish two cases as follows:

Case 1°. If $0<q\le 1$, using the well-known inequality

$$\left(\sum_{j=1}^{\infty}a_j\right)^q \le \sum_{j=1}^{\infty}a_j^q, \qquad a_j>0, \quad j=1,2,\ldots, \tag{13}$$

we obtain

$$V_1 \lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-2} (k-j)^q 2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)q} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=j+2}^{k_0} (k-j)^q 2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)q} \lesssim$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

Case 2°. If $1 < q < \infty$, the Hölder inequality implies that

$$V_1 \lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)q/2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \times$$

$$\times \left(\sum_{j=-\infty}^{k-2} (k-j)^{q'} 2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)q'/2} \right)^{q/q'} \lesssim$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=j+2}^{k_0} 2^{(j-k)(n\delta_2-\alpha_+-(n-1)/s)q/2} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We proceed now to estimate V_2. By Proposition 4.1 and the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of the commutator $\mu_{\Omega,b}$, we get

$$V_2 \approx \max\left\{ \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \left(\sum_{j=k-1}^{k+1} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \right.$$

$$\sup_{k_0\geq 0,k_0\in\mathbb{Z}} \left[2^{-k_0\lambda q} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left\| \left(\sum_{j=k-1}^{k+1} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} + \right.$$

$$\left.\left. +2^{-k_0\lambda q} \sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \left\| \left(\sum_{j=k-1}^{k+1} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right] \right\} \lesssim$$

$$\lesssim \max\left\{ \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha_\infty} |f\chi_k| \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \right.$$

$$\left. \sup_{k_0\geq 0,k_0\in\mathbb{Z}} \left[2^{-k_0\lambda q} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha_\infty} |f\chi_k| \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} + 2^{-k_0\lambda q} \sum_{k=0}^{k_0} \left\| 2^{k\alpha_\infty} |f\chi_k| \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right] \right\} \lesssim$$

$$\lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For V_3, once again by Proposition 4.1, we have

$$V_3 \approx \max\{E, F\},$$

where

$$E = \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \left(\sum_{j=k+2}^{\infty} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$F = \sup_{k_0\geq 0, k_0\in\mathbb{Z}} \left\{ 2^{-k_0\lambda q} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left\| \left(\sum_{j=k+2}^{\infty} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} + \right.$$

$$\left. + 2^{-k_0\lambda q} \sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \left\| \left(\sum_{j=k+2}^{\infty} |\mu_{\Omega,b}(f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}.$$

For E, noticing that $|x-y| \approx |y| \approx 2^j$ for $x \in R_k, y \in R_j$ and $j \geq k+2$, then we obtain

$$|\mu_{\Omega,b}(f_j)(x)| \lesssim \left(\int_0^{|y|} \left| \int_{|x-y|\leq t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} +$$

$$+ \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\leq t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \lesssim$$

$$\lesssim \int_{R_j} \frac{|b(x)-b(y)||\Omega(x-y)||f_j(y)|}{|x-y|^{n-1}} \frac{|x|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} dy +$$

$$+ \int_{R_j} \frac{|b(x)-b(y)||\Omega(x-y)||f_j(y)|}{|x-y|^{n-1}} \frac{1}{|y|} dy \lesssim$$

$$\lesssim 2^{-jn} \int_{R_j} |b(x)-b(y)||\Omega(x-y)||f_j(y)| dy \lesssim$$

$$\lesssim 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big(|b(x)-b_{B_k}| \|\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} +$$

$$+ \|\Omega(x-\cdot)(b-b_{B_k})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \Big). \tag{14}$$

Similarly to (11), we get

$$\|\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim \|\Omega(x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)}\|\chi_j\|_{L^{p^*(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \left(\int_0^{2^{j+1}}\int_{\mathbb{S}^{n-1}} |\Omega(y')|^s d\sigma(y')\varrho^{n-1}d\varrho\right)^{\frac{1}{s}} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s} \lesssim$$

$$\lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{15}$$

which in conjunction with Lemma 2.4 implies

$$\|(b-b_{B_k})\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim \|(b-b_{B_k})\chi_j\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)} \lesssim$$

$$\lesssim (j-k)\|b\|_*\|\chi_{B_j}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)} \lesssim$$

$$\lesssim (j-k)\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{16}$$

Now from (14) – (16) and Lemmas 2.1 – 2.3, we obtain

$$\|\mu_{\Omega,b}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big(\|(b-b_{B_k})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}+$$

$$+\|(b-b_{B_k})\Omega(x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big) \lesssim$$

$$\lesssim (j-k)2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim (j-k)\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim$$

$$\lesssim (j-k)2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Consequently, we have

$$E = \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\left\|\left(\sum_{j=k+2}^{\infty}|\mu_{\Omega,b}(f_j)|\right)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\left(\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q.$$

If $0<q\le 1$, by (13), we get

$$E \lesssim \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\sum_{j=k+2}^{k_0-1}(j-k)^q 2^{(k-j)n\delta_1 q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}+$$

$$+\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\sum_{j=k_0}^{\infty}(j-k)^q 2^{(k-j)n\delta_1 q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} :=$$

$$:= E_1 + E_2.$$

For E_1, in view of $n\delta_1 + \alpha(0) > n\delta_1 + \alpha_- > 0$, we obtain

$$E_1 \lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-2} (j-k)^q 2^{(k-j)(n\delta_1+\alpha(0))q} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For E_2, noting that $\alpha(0) + n\delta_1 - \lambda > \alpha_- + n\delta_1 - \lambda > 0$, we get

$$E_2 \approx \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)(n\delta_1+\alpha(0))q} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)(n\delta_1+\alpha(0))q} 2^{j\lambda q} \times$$

$$\times 2^{-j\lambda q} \sum_{l=-\infty}^{j} 2^{l\alpha(0)q} \|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\lambda q} \sum_{j=k_0}^{\infty} (j-k)^q 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q} \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda q}\right) \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

If $1 < q < \infty$, we have

$$E \lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left(\sum_{j=k+2}^{k_0-1} (j-k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q +$$

$$+ \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left(\sum_{j=k_0}^{\infty} (j-k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q :=$$

$$:= E_3 + E_4.$$

For E_3, the Hölder inequality yields

$$E_3 \approx \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left(\sum_{j=k+2}^{k_0-1} (j-k) 2^{(k-j)(n\delta_1+\alpha(0))} 2^{j\alpha(0)} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left(\sum_{j=k+2}^{k_0-1} 2^{(k-j)(n\delta_1+\alpha(0))q/2} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \times$$

$$\times \left(\sum_{j=k+2}^{k_0-1} (j-k)^{q'} 2^{(k-j)(n\delta_1+\alpha(0))q'/2} \right)^{q/q'} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha(0))q/2} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For E_4, as argued for E_2, we obtain

$$E_4 \lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left(\sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)+\lambda)q/2} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \times$$

$$\times \left(\sum_{j=k_0}^{\infty} (j-k)^{q'} 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q'/2} \right)^{q/q'} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left(\sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)+\lambda)q/2} 2^{j\lambda q} 2^{-j\lambda q} \sum_{l=-\infty}^{j} 2^{l\alpha(0)q} \|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\lambda q} \left(\sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q/2} \right) \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda q} \right) \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We omit the estimation of F since it is essentially similar to that of E.

Theorem 4.1 is proved.

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