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# VARIABLE HERZ ESTIMATES FOR FRACTIONAL INTEGRAL OPERATORS
# ЗМІННІ ОЦІНКИ ГЕРЦА ДЛЯ ДРОБОВИХ ІНТЕГРАЛЬНИХ ОПЕРАТОРІВ

In this paper, the author study the boundedness of fractional integral operators on a variable Herz-type Hardy space $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by using the atomic decomposition.

За допомогою атомарної декомпозиції вивчається обмеженість дробових інтегральних операторів у змінному просторі Гарді $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ типу Герца.

**1. Introduction.** Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors. The motivation to study such function spaces comes from applications to other fields of applied mathematics, such that fluid dynamics and image processing (see [2, 13]).

Herz spaces $K^{\alpha,p}_{q(\cdot)}$ and $\dot{K}^{\alpha,p}_{q(\cdot)}$ with variable exponent $q$ but fixed $\alpha \in \mathbb{R}$ and $p \in (0,\infty]$ were recently studied by Izuki [6, 8]. These spaces with variable exponents $\alpha(\cdot)$ and $q(\cdot)$ were studied in [1], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [9]. In [5], the authors gave a new equivalent norms of these function spaces. See [14], where new variable Herz spaces are given. For more details, we refer the reader to the reference [4].

H. Wang, L. Zongguang and F. Zunwei [16] considered variable Herz-type Hardy spaces $H\dot{K}^{\alpha,p}_{q(\cdot)}$ with variable $q$, were the boundedness of fractional integral operators and their commutators on these spaces are obtained.

H. B. Wang and Z. G. Liu [15] studied Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}$ with variables $\alpha$ and $q$, but fixed $p$, where the authors introduced the anisotropic Herz spaces and established their block decomposition, also they obtain some boundedness on the anisotropic Herz spaces with two variable exponents for a class of sublinear operators. D. Drihem and F. Seghiri in [5] introduce a new Herz-type Hardy spaces with variable exponent, where all the three parameters are variables.

This paper is organized as follows. In Section 2, we give some preliminaries where we fix some notations and recall some basic facts on function spaces with variable integrability. In Section 3, we give some key technical lemmas needed in the proofs of the main statements. Finally, in Section 4, we present main results. In particular we will prove the boundedness of fractional integral operators and their commutators on Herz-type Hardy spaces, where all the three parameters are variables.

**2. Preliminaries.** As usual, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The Euclidean scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is given by $x \cdot y = x_1 y_1 + \ldots + x_n y_n$. The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant $c$ (and nonnegative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to $x$.

For $x \in \mathbb{R}^n$ and $r > 0,$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center $x$ and radius $r.$ By supp $f$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function.

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

The variable exponents that we consider are always measurable functions on $\mathbb{R}^n$ with range in $[c,\infty[$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0(\mathbb{R}^n)$. The subset of variable exponents with range $[1,\infty)$ is denoted by $\mathcal{P}$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, we use the notation

$$p^- = \operatorname{ess}\inf_{x\in\mathbb{R}^n} p(x), \qquad p^+ = \operatorname{ess}\sup_{x\in\mathbb{R}^n} p(x).$$

Everywhere below we shall consider bounded exponents.

Let $p$ belongs to $\mathcal{P}_0(\mathbb{R}^n)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions $f$ on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f) := \int\limits_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\mu > 0\colon\ \varrho_{p(\cdot)}\left(\frac{1}{\mu}f\right) \le 1\right\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

A useful property is that $\varrho_{p(\cdot)}(f) \le 1$ if and only if $\|f\|_{p(\cdot)} \le 1$ (*unit ball property*). This property is clear for constant exponents due to the obvious relation between the norm and the modular in that case.

We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log*-Hölder continuous*, if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}}{\ln(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \le \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that $g$ is log*-Hölder continuous at the origin* (or has a log *decay at the origin*). If, for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|g(x) - g_\infty| \le \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that $g$ is log*-Hölder continuous at infinity* (or has a log *decay at infinity*).

By $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$. Obviously, we get $\mathcal{P}^{\log}(\mathbb{R}^n) \subset \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. Here, $p'$ denotes the conjugate exponent of $p$ given by $1/p(\cdot) + 1/p'(\cdot) = 1$. Note that $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}^{\log}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$ we write only $p'_\infty$ for any of these quantities.

Let $p, q \in \mathcal{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\left\{\lambda_v > 0 \colon \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \le 1\right\}.$$

The (quasi)norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\mu > 0 \colon \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \le 1\right\}.$$

Since $q^+ < \infty$, then we can replace by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if $p$ and $q$ are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. It is known that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \ge 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \ge 1$, or if $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} \le 1$ a.e. on $\mathbb{R}^n$, or if $1 \le q(x) \le p(x) < \infty$ a.e. on $\mathbb{R}^n$.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical $L^p$ spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}^{\log}$ we obtain

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{2.1}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B, \tag{2.2}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \le 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}} \tag{2.3}$$

for large balls ($|B| \ge 1$) with constants only depending on the $\log$-Hölder constant of $p$ (see, for example, [3], Section 4.5). Let $L^1_{\text{Loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$.

Recall that the space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions $f$ such that

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int\limits_Q |f(x) - f_Q|\, dx < \infty,$$

where $f_Q = \dfrac{1}{|Q|} \displaystyle\int_Q f(y) dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

We refer the reader to the recent monograph [3] (Section 4.5) for further details, historical remarks and more references on variable exponent spaces.

**3. Basic tools.** In this section, we present some results which are useful for us. The following lemma plays an important role in the proof of the main results of this paper, is given in [1], where is a generalization of (2.1), (2.2), and (2.3) to the case of dyadic annuli.

**Lemma 3.1.** *Let $p \in \mathcal{P}^{\log}_{\infty}(\mathbb{R}^n)$ and $R = B(0,r) \setminus B\Big(0, \dfrac{r}{2}\Big)$. If $|R| \geq 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

*with the implicit constants independent of $r$ and $x \in R$.*

*The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, that $p$ belongs to $\mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_{\infty}(\mathbb{R}^n)$.*

The next lemma is a Hardy-type inequality which is easy to prove.

**Lemma 3.2.** *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers such that*

$$\big\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q} = I < \infty.$$

*Then the sequences $\Big\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\Big\}_{k\in\mathbb{Z}}$ and $\Big\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\Big\}_{k\in\mathbb{Z}}$ belong to $\ell^q$, and*

$$\big\|\{\delta_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q} + \big\|\{\eta_k\}_{k\in\mathbb{Z}}\big\|_{\ell^q} \leq c\, I$$

*with $c > 0$ only depending on $a$ and $q$.*

For convenience, we set

$$B_k := B(0, 2^k), \qquad R_k := B_k \setminus B_{k-1} \text{ and } \chi_k = \chi_{R_k}, \qquad k \in \mathbb{Z}.$$

**Definition 3.1.** *Let $p$, $q$ belong to $\mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{p(\cdot)}_{\mathrm{Loc}}(\mathbb{R}^n)$ such that*

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \left\|\Big(2^{k\alpha(\cdot)} f\,\chi_k\Big)_{k\geq 1}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

*Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in$ $\in L^{p(\cdot)}_{\mathrm{Loc}}(\mathbb{R}^n \setminus \{0\})$ such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\Big(2^{k\alpha(\cdot)} f\,\chi_k\Big)_{k\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

If $\alpha$ and $p, q$ are constant, then $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$ is the classical Herz spaces $\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$.

The following proposition is very important for the proof of the main results it is from D. Drihem and F. Seghiri in [5].

**Proposition 3.1.** *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If $\alpha$ and $q$ are* log*-Hölder continuous at infinity, then*

$$K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = K^{\alpha_\infty,q_\infty}_{p(\cdot)}(\mathbb{R}^n).$$

*Additionally, if $\alpha$ and $q$ have a* log *decay at the origin, then*

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}} \approx \left(\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f\,\chi_k\|^{q(0)}_{p(\cdot)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f\,\chi_k\|^{q_\infty}_{p(\cdot)}\right)^{1/q_\infty}.$$

Let $\varphi$ belongs to $C_0^\infty(\mathbb{R}^n)$ with supp $\varphi \subseteq B_0$, $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$ and $\varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right)$ for any $t > 0$. Let $M_\varphi(f)$ be the grand maximal function of $f$ defined by

$$M_\varphi(f)(x) := \sup_{t>0} |\varphi_t * f(x)|.$$

We will give the definition of the homogeneous Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$.

**Definition 3.2.** *Let* $p$, $q$ *belong to* $\mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$. *The homogeneous Herz-type Hardy space* $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $M_\varphi(f) \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \|M_\varphi(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

It can be shown that, if $p, q$, and $\alpha$ satisfy the conditions of definition, then the quasinorm $\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}$ does not depend, up to the equivalence of quasinorms, on the choice of the function $\varphi$ and, hence, the space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined independently of the choice $\varphi$. If $p$ belongs to $\mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n - \frac{n}{p^-}$$

and $q \in \mathcal{P}_0(\mathbb{R}^n)$, then

$$H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) = \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n).$$

If $\alpha(\cdot) = 0, p(\cdot) = q(\cdot)$, then $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ coicide with $L^{p(\cdot)}(\mathbb{R}^n)$.

One recognizes immediately that if $\alpha$, $p$ and $q$ are constants, then the spaces $H\dot{K}_p^{\alpha,q}$ are just the usual Herz-type Hardy spaces were recently studied in [11, 12].

Now, we introduce the basic notation of atomic decomposition.

**Definition 3.3.** *Let* $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $s \in \mathbb{N}_0$. *A function* $a$ *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-atom, if*

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leq r\},\ r > 0,$

(ii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha(0)/n},\ 0 < r < 1,$

(iii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha_\infty/n},\ r \geq 1,$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0,\ |\beta| \leq s.$

A function $a$ on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and $\operatorname{supp} a \subset B(0,r), r \geq 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 3.3, then the corresponding central $(\alpha(\cdot), p(\cdot))$-atom is called a dyadic central $(\alpha(\cdot), p(\cdot))$-atom.

Now we establish characterizations of the spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ in terms of central atomic decompositions, which make it convenient to study the boundedness of operators on these spaces. In [5], we have the following the atomic decomposition characterization of spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

**Theorem 3.1.** *Let $\alpha$ and $q$ are be* $\log$*-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. For any $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

*where the series converges in the sense of distributions, $\lambda_k \ge 0$, each $a_k$ is a central $(\alpha(\cdot), p(\cdot))$-atom with* $\operatorname{supp} a \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \le c\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

*Conversely, if $\alpha(\cdot) \ge n\left(1 - \dfrac{1}{p^-}\right)$ and $s \ge \left[\alpha^+ + n\left(\dfrac{1}{p^-} - 1\right)\right]$, and if holds, then $f$ belongs to $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, and*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

*where the infimum is taken over all the decompositions of $f$ as above.*

The following lemma is from [9] (Lemma 2.9), see also [16] (Lemma 0.5).

**Lemma 3.3.** *Let $p$ belongs to $\mathcal{P}^{\log}(\mathbb{R}^n)$, $k$ be a positive integer and $B$ be a ball in $\mathbb{R}^n$. Then, for all $b \in BMO(\mathbb{R}^n)$and all $i, j \in \mathbb{Z}$ with $j > i$, we have*

$$\frac{1}{c}\|b\|_{BMO}^k \le \sup_B \frac{1}{\|\chi_B\|_{p(\cdot)}} \|(b - b_B)^k \chi_B\|_{p(\cdot)} \le c\|b\|_{BMO}^k,$$

$$\|(b - b_{B_i})^k \chi_{B_j}\|_{p(\cdot)} \le c(j - i)^k \|b\|_{BMO}^k \|\chi_{B_j}\|_{p(\cdot)}.$$

Given $0 < \sigma < n$, for an appropriate function $f$, the commutator with $m$-order of fractional integral operators $I_{\sigma,b}^m$, $m = 1, 2, \ldots$, is defined by

$$I_{\sigma,b}^m(f)(x) := \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x - y|^{n-\sigma}} f(y)dy.$$

We denote $I_{\sigma,b}^1$ by $[b, I_\sigma]$ and $I_{\sigma,b}^0$ by the fractional integral operator $I_\sigma$, respectively.

The next two lemmas are from [5] treat the case when $m = 0, 1$ for $I_{\sigma,b}^m$.

**Lemma 3.4.** *Suppose that $p_1$, $p_2$ belong to $\mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \dfrac{n}{\sigma}$ and $\dfrac{1}{p_1(\cdot)} - \dfrac{1}{p_2(\cdot)} = \dfrac{\sigma}{n}$. Then, for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|I_\sigma(f)\|_{p_2(\cdot)} \le c\|f\|_{p_1(\cdot)}.$$

**Lemma 3.5.** *Suppose that $p_1$, $p_2$ belong to $\mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \dfrac{n}{\sigma}$, $\dfrac{1}{p_1(\cdot)} - \dfrac{1}{p_2(\cdot)} = \dfrac{\sigma}{n}$ and $b \in BMO(\mathbb{R}^n)$. Then, for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|[b, I_\sigma](f)\|_{p_2(\cdot)} \le C\|b\|_{BMO}\|f\|_{p_1(\cdot)}.$$

**4. Boundedness of fractional integral and their commutators on variable Herz-type Hardy spaces.** In this section, we present the boundedness of fractional integral operators and their commutators on variable Herz-type Hardy spaces.

First, we treat the boundedness of $I_\sigma$ on variable Herz-type Hardy spaces.

**Theorem 4.1.** *Suppose that* $p_1,\ p_2$ *belong to* $\mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $p_1^+ < \dfrac{n}{\sigma}$ *and* $\dfrac{1}{p_1(\cdot)} - \dfrac{1}{p_2(\cdot)} = \dfrac{\sigma}{n}$, $\alpha \in L^\infty(\mathbb{R}^n),\ q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$. *If* $\alpha, q_1$ *and* $q_2$ *are* log-*Hölder continuous, both at the origin and at infinity with* $\alpha(\cdot) \geq n\Big(1 - \dfrac{1}{p_1^-}\Big), q_1(0) \leq q_2(0)$ *and* $(q_1)_\infty \leq (q_2)_\infty$, *then* $I_\sigma$ *is bounded from* $H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ *to* $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)$.

Next, we present the boundedness of $[b, I_\sigma]$ on variable Herz-type Hardy spaces.

**Theorem 4.2.** *Suppose that* $p_1,\ p_2$ *belong to* $\mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $p_1^+ < \dfrac{n}{\sigma}$ *and* $\dfrac{1}{p_1(\cdot)} - \dfrac{1}{p_2(\cdot)} = \dfrac{\sigma}{n}$, $\alpha \in L^\infty(\mathbb{R}^n),\ q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$. *If* $\alpha, q_1$ *and* $q_2$ *are* log-*Hölder continuous, both at the origin and at infinity with* $\alpha(\cdot) \geq n(1 - \dfrac{1}{p_1^-}), q_1(0) \leq q_2(0), (q_1)_\infty \leq (q_2)_\infty$ *and* $b \in BMO(\mathbb{R}^n)$, *then* $[b, I_\sigma]$ *is bounded from* $H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ *to* $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)$.

***Remark*** **4.1.** If $\alpha, q_1$ and $q_2$ are constants, then the statements corresponding to Theorems 4.1 and 4.2 can be found in [16] (Theorems 1.1 and 1.2).

Our proofs use partially some techniques already used in [16] where $\alpha, q_1$ and $q_2$ are constants.

***Proof of Theorem*** **4.1.** We must show that

$$\|I_\sigma(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)} \leq c\, \|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$. By using Theorem 3.1, we may assume that

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$$

where $\lambda_i \geq 0$ and $a_i$'s are $(\alpha(\cdot), p_1(\cdot))$-atom with $\operatorname{supp} a_i \subseteq B_i$. By using Proposition 3.1, we have

$$\|I_\sigma(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)} \approx$$

$$\approx \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \|I_\sigma(f)\chi_k\|_{p_2(\cdot)}^{q_2(0)} \right\}^{1/q_2(0)} + c \left\{ \sum_{k=0}^{+\infty} 2^{k\alpha_\infty(q_2)_\infty} \|I_\sigma(f)\chi_k\|_{p_2(\cdot)}^{(q_2)_\infty} \right\}^{1/(q_2)_\infty} \leq$$

$$\leq \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left( \sum_{i=-\infty}^{k-2} |\lambda_i| \, \|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} +$$

$$+ \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left( \sum_{i=k-1}^{\infty} |\lambda_i| \, \|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} +$$

$$+\left\{\sum_{k=0}^{+\infty}2^{k\alpha_\infty(q_2)_\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\right)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}+$$

$$+\left\{\sum_{k=0}^{+\infty}2^{k\alpha_\infty(q_2)_\infty}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\right)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}=:$$

$$=: H_1+H_2+H_3+H_4.$$

Let us estimate $H_1$. By the $s$-order vanishing moments of $a_i$ with

$$s\geq\left[\alpha^+-n\left(1-\frac{1}{p_1^-}\right)\right],$$

we can subtract the Taylor expansion of $|x-y|^{-n+\sigma}$ at $x$, we obtain

$$|I_\sigma(a_i)(x)|\leq\int\limits_{B_i}\frac{|a_i(y)|\,|y|^{s+1}}{|x|^{n-\sigma+s+1}}dy\leq$$

$$\leq c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\int\limits_{B_i}|a_i(y)|\,dy.$$

Applying Hölder inequality, we get

$$|I_\sigma(a_i)(x)|\leq c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\,\|a_i\|_{p_1(\cdot)}\,\|\chi_{B_i}\|_{p_1'(\cdot)}\,. \tag{4.1}$$

On the other hand (see [7, p. 350]), we have

$$I_\sigma(\chi_{B_k})(x)\geq\int\limits_{B_k}\frac{dy}{|x-y|^{n-\sigma}}\chi_{B_k}(x)\geq c\cdot 2^{k\sigma}\chi_{B_k}(x)\,. \tag{4.2}$$

By (4.1), (4.2) and Lemma 3.4, gives

$$\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\leq c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\,\|a_i\|_{p_1(\cdot)}\,\|\chi_{B_i}\|_{p_1'(\cdot)}\,\|\chi_k\|_{p_2(\cdot)}\leq$$

$$\leq c\cdot 2^{-k(n+s+1)+i(s+1)}\,\|a_i\|_{p_1(\cdot)}\,\|\chi_{B_i}\|_{p_1'(\cdot)}\,\|I_\sigma(\chi_{B_k})\|_{p_2(\cdot)}\leq$$

$$\leq c\cdot 2^{-k(n+s+1)+i(s+1)}\,\|a_i\|_{p_1(\cdot)}\,\|\chi_{B_i}\|_{p_1'(\cdot)}\,\|\chi_{B_k}\|_{p_1(\cdot)}\,.$$

By (2.2), we have

$$H_1=\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\leq$$

$$\leq c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,2^{(i-k)(n+s+1-(\alpha+n/p_1)(0))}\right)^{q_2(0)}\right\}^{1/q_2(0)}.$$

Since $s+1-\alpha^{+}+n\left(1-\dfrac{1}{p_1^{-}}\right)>0$, then, by Lemma 3.2, we obtain

$$H_1\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)}\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)}\leq c\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}}.$$

Let us estimate $H_2$. By Lemma 3.4, we get

$$H_2=\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\leq$$

$$\leq c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\leq$$

$$\leq c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-1}^{-1}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}+$$

$$+c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=0}^{+\infty}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\leq$$

$$\leq c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=k-1}^{-1}|\lambda_i|\,2^{(k-i)\alpha(0)}\right)^{q_2(0)}\right\}^{1/q_2(0)}+$$

$$+c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=0}^{+\infty}|\lambda_i|\,2^{(k-i)\alpha^{-}+k(\alpha(0)-\alpha^{-})+i(\alpha^{-}-\alpha_\infty)}\right)^{q_2(0)}\right\}^{1/q_2(0)}$$

for $k<0\leq i$ and since $\alpha^{-}\leq\min(\alpha(0),\alpha_\infty)$, we have $k(\alpha(0)-\alpha^{-})+i(\alpha^{-}-\alpha_\infty)\leq 0$. Then, by Lemma 3.2, we have

$$H_2\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)}\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)}\leq c\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}}.$$

We can estimate $H_3$ and $H_4$ by the same technique as in the estimation of $H_1$ and $H_2$ when replacing $\alpha(0)$ and $q_2(0)$ by $\alpha_\infty$ and $(q_2)_\infty$, respectively. A combination of estimations of $H_1$, $H_2$, $H_3$, and $H_4$ finish the proof ot Theorem 4.1.

***Proof of Theorem*** **4.2.** We must show that

$$\|[b,I_\sigma]f\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \le c\,\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. By using Theorem 3.1, we may assume that

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i,$$

where $\lambda_i \ge 0$ and $a_i$'s are $(\alpha(\cdot), p_1(\cdot))$-atom with supp$a_i \subseteq B_i$. By using Proposition 3.1, we obtain

$$\|[b,I_\sigma]f\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \approx$$

$$\approx \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}^{q_2(0)}\right\}^{1/q_2(0)} + c\left\{\sum_{k=0}^{+\infty}2^{k\alpha_\infty(q_2)_\infty}\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \le$$

$$\le \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)} +$$

$$+ \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)} +$$

$$+ \left\{\sum_{k=0}^{+\infty}2^{k\alpha_\infty(q_2)_\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,\|[b,I_\sigma]\chi_k\|_{p_2(\cdot)}\right)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} +$$

$$+ \left\{\sum_{k=0}^{+\infty}2^{k\alpha_\infty(q_2)_\infty}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}\right)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} =:$$

$$=: Q_1 + Q_2 + Q_3 + Q_4.$$

Let us estimate $Q_1$. As in $H_1$, we use the Taylor expansion of $|x-y|^{-n+\sigma}$ at $x$ and the $s$-order vanishing moments of $a_i$ with $s \ge \left[\alpha^+ - n\left(1 - \dfrac{1}{p_1^-}\right)\right]$, we get

$$|[b,I_\sigma](a_i)| \le$$

$$\le \int\limits_{B_i} |b(x)-b(y)|\,\frac{|a_i(y)|\,|y|^{s+1}}{|x|^{n-\sigma+s+1}}\,dy \le$$

$$\le c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\int\limits_{B_i}|a_i(y)|\,|b(x)-b(y)|\,dy \le$$

$$\leq c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\left(|b(x)-b_{B_i}|\int\limits_{B_i}|a_i(y)|\,dy+\int\limits_{B_i}|a_i(y)|\,|b_{B_i}-b(y)|\,dy\right).$$

We use the Hölder inequality, the last expression is bounded by

$$c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\left\|a_i\right\|_{p_1(\cdot)}\left(|b(x)-b_{B_i}|\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}+\left\||b_{B_i}-b(y)|\,\chi_{B_i}\right\|_{p_1'(\cdot)}\right).$$

By (4.2) and Lemma 3.3, we have

$$\left\|[b,I_\sigma](a_i)\,\chi_k\right\|_{p_2(\cdot)}\leq$$

$$\leq c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\left\|a_i\right\|_{p_1(\cdot)}\Big(\left\||b(x)-b_{B_i}|\,\chi_k\right\|_{p_2(\cdot)}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}+$$

$$+\left\||b_{B_i}-b(y)|\,\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|\chi_k\right\|_{p_2(\cdot)}\Big)\leq$$

$$\leq c\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\left\|a_i\right\|_{p_1(\cdot)}\Big((k-i)\left\|b\right\|_{BMO}\left\|\chi_{B_k}\right\|_{p_2(\cdot)}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}+$$

$$+\left\|b\right\|_{BMO}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|\chi_k\right\|_{p_2(\cdot)}\Big)\leq$$

$$\leq c(k-i)\cdot 2^{-k(n-\sigma+s+1)+i(s+1)}\left\|a_i\right\|_{p_1(\cdot)}\left\|b\right\|_{BMO}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|\chi_{B_k}\right\|_{p_2(\cdot)}\leq$$

$$\leq c(k-i)\cdot 2^{-k(n+s+1)+i(s+1)}\left\|a_i\right\|_{p_1(\cdot)}\left\|b\right\|_{BMO}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|I_\sigma(a_i)\chi_k\right\|_{p_2(\cdot)}\leq$$

$$\leq c(k-i)\cdot 2^{-k(n+s+1)+i(s+1)}\left\|a_i\right\|_{p_1(\cdot)}\left\|b\right\|_{BMO}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|\chi_k\right\|_{p_1(\cdot)}.$$

By (2.2) and Lemma 3.1, we obtain

$$Q_1=\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\left\|[b,I_\sigma](a_i)\,\chi_k\right\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\leq$$

$$\leq c\left\|b\right\|_{BMO}\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,(k-i)\cdot 2^{(i-k)(s+1+n-(\alpha+n/p_1)(0))}\right)^{q_2(0)}\right\}^{1/q_2(0)}\leq$$

$$\leq c\left\|b\right\|_{BMO}\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,(k-i)\cdot 2^{(i-k)(s+1+n-(\alpha+n/p_1)(0))}\right)^{q_2(0)}\right\}^{1/q_2(0)}. \tag{4.3}$$

Since $s+1-\alpha^++n\left(1-\dfrac{1}{p_1^-}\right)>0$, then, by Lemma 3.2, we get that (4.3) is bounded by

$$Q_1\leq c\left\|b\right\|_{BMO}\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)}\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)}\leq c\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}}.$$

Let us estimate $Q_2$. By Lemma 3.3, we get

$$Q_2 = \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left( \sum_{i=k-1}^{+\infty} |\lambda_i| \left\| [b, I_\sigma] (a_i) \chi_k \right\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \leq$$

$$\leq c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left( \sum_{i=k-1}^{+\infty} |\lambda_i| \|a_i\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \leq$$

$$\leq c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left( \sum_{i=k-1}^{-1} |\lambda_i| \|a_i\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} +$$

$$+c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left( \sum_{i=0}^{+\infty} |\lambda_i| \|a_i\|_{p_2(\cdot)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \leq$$

$$\leq c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{-1} \left( \sum_{i=k-1}^{-1} |\lambda_i| 2^{(k-i)\alpha(0)} \right)^{q_2(0)} \right\}^{1/q_2(0)} +$$

$$+c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{-1} \left( \sum_{i=0}^{+\infty} |\lambda_i| 2^{(k-i)\alpha^- + k(\alpha(0)-\alpha^-)+i(\alpha^- - \alpha_\infty)} \right)^{q_2(0)} \right\}^{1/q_2(0)}$$

for $k < 0 \leq i$. Since $\alpha^- \leq \min(\alpha(0), \alpha_\infty)$, we obtain $k(\alpha(0) - \alpha^-) + i(\alpha^- - \alpha_\infty) \leq 0$. Then, by Lemma 3.2, we have

$$Q_2 \leq c \|b\|_{BMO} \left( \sum_{k=-\infty}^{-1} |\lambda_k|^{q_2(0)} \right)^{1/q_2(0)} \leq c \left( \sum_{k=-\infty}^{-1} |\lambda_k|^{q_1(0)} \right)^{1/q_1(0)} \leq c\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}}.$$

We can estimate $Q_3$ and $Q_4$ by the same technique as in the estimation of $Q_1$ and $Q_2$, when replacing $\alpha(0)$ and $q_2(0)$ by $\alpha_\infty$ and $(q_2)_\infty$, respectively.

A combination of estimations of $Q_1$, $Q_2$, $Q_3$, and $Q_4$ finish the proof ot Theorem 4.2.

## References


1. A. Almeida, D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. and Appl., **394**, № 2, 781 – 795 (2012).
2. Y. Chen, S. Levine, R. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., **66**, 1383 – 1406 (2006).
3. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lect. Notes Math., **2017**, Springer-Verlag, Berlin (2011).
4. D. Drihem, R. Heraiz, *Herz-type Besov spaces of variable smoothness and integrability*, Kodai Math. J., **40**, № 1, 31 – 57 (2017).
5. D. Drihem, F. Seghiri, *Notes on the Herz-type Hardy spaces of variable smoothness and integrability*, Math. Inequal. and Appl., **19**, 145 – 165 (2016).

6. M. Izuki, *Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system*, East J. Approxim., **15**, 87 – 109 (2009).
7. M. Izuki, *Fractional integrals on Herz – Morrey spaces with variable exponent*, Hiroshima Math. J., **40**, 343 – 355 (2010).
8. M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math., **36**, 33 – 50 (2010).
9. M. Izuki, T. Noi, *Boundedness of some integral operators and commutators on generalized Herz spaces with variable exponents*, http://www.sci.osaka-cu.ac.jp/math/OCAMI/preprint/2011/11-15.pdf.
10. X. Li, D. Yang, *Boundedness of some sublinear operators on Herz spaces*, Illinois J. Math., **40**, 484 – 501 (1996).
11. S. Lu, D. Yang, *Some characterizations of weighted Herz-type Hardy spaces and their applications*, Acta Math. Sinica (N.S.), **13**, 45 – 58 (1997).
12. A. Miyachi, *Remarks on Herz-type Hardy spaces*, Acta Math. Sinica (Engl. Ser.), **17**, 339 – 360 (2001).
13. M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lect. Notes Math., **1748**, Springer-Verlag, Berlin (2000).
14. S. Samko, *Variable exponent Herz spaces*, Mediterr. J. Math., **10**, 2007 – 2025 (2013).
15. H. B. Wang, Z. G. Liu, *The Herz-type Hardy spaces with variable exponent and their applications*, Taiwanese J. Math., **16**, 1363 – 1389 (2012).
16. H. Wang, L. Zongguang, F. Zunwei, *Boundedness of fractional integrals on Herz-type Hardy spaces with variable exponent*, Adv. Math. (China), **46**, № 2, 252 – 260 (2017).


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