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PARTIAL ORDERS BASED ON THE CS DECOMPOSITION *

ЧАСТКОВІ ВПОРЯДКУВАННЯ НА ОСНОВІ CS-РОЗКЛАДУ

A new decomposition for square matrices is given by using two known matrix decompositions, a new characterization of the core-EP order is obtained by using this new matrix decomposition. Also, we will use a matrix decomposition to investigate the minus, star, sharp and core partial orders in the setting of complex matrices.

За допомогою двох відомих розкладів матриць отримано новий розклад для квадратних матриць, а за допомогою цього нового розкладу отримано нову характеристику core-EP-впорядкування. Також використано матричний розклад для вивчення часткового порядку типу “minus”, “star”, “sharp” та “core” для комплексних матриць.

1. Introduction. Let $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ complex matrices. Let A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\mathrm{rk}(A)$ denote the conjugate transpose, column space, null space and rank of $A\in\mathbb{C}^{m\times n}$, respectively. For $A\in\mathbb{C}^{m\times n}$, if $X\in\mathbb{C}^{n\times m}$ satisfies

$$AXA=A,\qquad XAX=X,\qquad (AX)^*=AX\qquad\text{and}\qquad (XA)^*=XA,$$

then X is called a *Moore–Penrose inverse* of A. If such a matrix X exists, then it is unique and denoted by $A^{\dagger}$. Many existence criteria and properties of the Moore–Penrose inverse can be found in [1, 6, 8, 9]. If $AXA=A$ holds, then X (and denoted by A^-) is called an *inner inverse* of A and the set of all inner inverses of A is denoted by $A\{1\}$.

Let $A\in\mathbb{C}^{n\times n}$. It can be easily proved that the set of elements $X\in\mathbb{C}^{n\times n}$ such that

$$AXA=A,\qquad XAX=X\qquad\text{and}\qquad AX=XA$$

is empty or a singleton. If this set is a singleton, its unique element is called the *group inverse* of A and denoted by $A^{\#}$.

The core inverse for a complex matrix was introduced by Baksalary and Trenkler [4]. Let $A\in\mathbb{C}^{n\times n}$. A matrix $X\in\mathbb{C}^{n\times n}$ is called a *core inverse* of A, if it satisfies $AX=P_A$ and $\mathcal{R}(X)\subseteq\mathcal{R}(A)$, here P_A denotes the orthogonal projector onto $\mathcal{R}(A)$. If such a matrix exists, then it is unique and denoted by $A^{\#\!\!\!\bigcirc}$. For a square complex matrix A, one has that A is core invertible, A is group invertible, and $\mathrm{rk}(A)=\mathrm{rk}(A^2)$ are three equivalent conditions (see [1]). We denote $\mathbb{C}_n^{CM}=\{A\in\mathbb{C}^{n\times n}\mid \mathrm{rk}(A)=\mathrm{rk}(A^2)\}$. The core partial order for a complex matrix was also introduced in [4]. For $A\in\mathbb{C}_n^{CM}$ and $B\in\mathbb{C}^{n\times n}$, the binary relation $A\overset{\#\!\!\!\bigcirc}{\leq}B$ is defined as follows:

$$A\overset{\#\!\!\!\bigcirc}{\leq}B\ \Leftrightarrow\ A^{\#\!\!\!\bigcirc}A=A^{\#\!\!\!\bigcirc}B\ \text{ and }\ AA^{\#\!\!\!\bigcirc}=BA^{\#\!\!\!\bigcirc}.$$

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In [4] (Theorem 6), it is proved that the core partial order is a matrix partial order. Baksalary and Trenkler [4] gave several characterizations and various relationships between the matrix core partial order and other matrix partial orders by using the decomposition of Hartwig and Spindelböck [9]. Let us recall some other well-known partial orders in $\mathbb{C}^{n\times n}$. For $A, B \in \mathbb{C}^{n\times n}$,

the left star partial order $A \ast\leq B$: $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ [3];

the star partial order $A \overset{*}{\leq} B$: $A^*A = A^*B$ and $AA^* = BA^*$ [7];

the minus partial order $A \overset{-}{\leq} B$: $A^-A = A^-B$ and $AA^- = BA^-$, where A^- denotes some inner inverse of A [8];

the sharp partial order $A \overset{\#}{\leq} B$: $A^\# A = A^\# B$ and $AA^\# = BA^\#$ [12].

In addition, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ column vectors all of whose components are 1 and 0, respectively. $0_{m\times n}$ (abbr., 0) denotes the zero matrix of size $m \times n$. If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$, then $P_{\mathcal{S}}$ stands for the *orthogonal projector* onto the subspace $\mathcal{S}$. A matrix $A \in \mathbb{C}^{n\times n}$ is called an *EP* matrix if $\mathcal{R}(A) = \mathcal{R}(A^*)$, A is called *projection* if $A^* = A = A^2$ and A is *unitary* if $AA^* = I_n$, where I_n denotes the *identity matrix* of size n. Let $A \in \mathbb{C}^{n\times n}$, the smallest integer k such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$ is called the *index* of A and denoted by $\mathrm{ind}(A) = k$.

2. Preliminaries. A related decomposition of the matrix decomposition of Hartwig and Spindelböck [9] was given in [1] (Theorem 2.1) by Benítez. In [2] it can be found a simpler proof of this decomposition. Let us start this section with the concept of principal angles.

Definition 2.1 [18]. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two nontrivial subspaces of $\mathbb{C}^n$. We define the principal angles $\theta_1, \ldots, \theta_r \in [0, \pi/2]$ between $\mathcal{S}_1$ and $\mathcal{S}_2$ by*

$$\cos\theta_i = \sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}),$$

for $i = 1, \ldots, r$, where $r = \min\{\dim \mathcal{S}_1, \dim \mathcal{S}_2\}$. The real numbers $\sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}) \geq 0$ are the singular values of $P_{\mathcal{S}_1} P_{\mathcal{S}_2}$.

The following theorem can be found in [1] (Theorem 2.1).

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $r = \mathrm{rk}(A)$, and let $\theta_1, \ldots, \theta_p$ be the principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ belonging to $]0, \pi/2[$. Denote by x and y the multiplicities of the angles 0 and $\pi/2$ as a canonical angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. There exists a unitary matrix $Y \in \mathbb{C}^{n\times n}$ such that*

$$A = Y \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} Y^*, \tag{2.1}$$

where $M \in \mathbb{C}^{r\times r}$ is nonsingular,

$$C = \mathrm{diag}(\mathbf{0}_y, \cos\theta_1, \ldots, \cos\theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \mathrm{diag}(\mathbf{1}_y, \sin\theta_1, \ldots, \sin\theta_p) & 0_{p+y, n-(r+p+y)} \\ 0_{x,p+y} & 0_{x,n-(r+p+y)} \end{bmatrix},$$

and $r = y + p + x$. Furthermore, x and $y + n - r$ are the multiplicities of the singular values 1 and 0 in $P_{\mathcal{R}(A)} P_{\mathcal{R}(A^*)}$, respectively. We call (2.1) as the CS decomposition of A.

In this decomposition, one has $C^2 + SS^* = I_r$. Recall that $A^\dagger$ always exists. We have that $A^\#$ exists if and only if C is nonsingular [1] (Theorem 3.7). The following equalities hold:

$$A^\dagger = Y \begin{bmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{bmatrix} Y^*, \qquad A^\# = Y \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*.$$

From $A^{\tiny\textcircled{\#}} = A^\# A A^\dagger$, we obtain $A^{\tiny\textcircled{\#}} = Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*$, $AA^{\tiny\textcircled{\#}} = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^*$ and $A^{\tiny\textcircled{\#}}A = Y \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*$. We have

$$AA^{\tiny\textcircled{\#}} - A^{\tiny\textcircled{\#}}A = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^* - Y \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & -C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*. \quad (2.2)$$

Thus,

$$|AA^{\tiny\textcircled{\#}} - A^{\tiny\textcircled{\#}}A| = \left| Y \begin{bmatrix} 0 & -C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* \right| = 0.$$

Therefore, $AA^{\tiny\textcircled{\#}} - A^{\tiny\textcircled{\#}}A$ is always singular and $\mathrm{rk}(AA^{\tiny\textcircled{\#}} - A^{\tiny\textcircled{\#}}A) = \mathrm{rk}(C^{-1}S) = \mathrm{rk}(S) < n$. From (2.2), we have that A is an EP matrix if and only if $S = 0$, that is all the canonical angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are 0. This result also can be found in [1] (Theorem 3.7).

Proposition 2.1. *If $A \in \mathbb{C}^{n\times n}$ is core invertible and A has the form* (2.1)*, then $AA^{\tiny\textcircled{\#}} - A^{\tiny\textcircled{\#}}A$ is always singular with* $\mathrm{rk}(AA^{\tiny\textcircled{\#}} - A^{\tiny\textcircled{\#}}A) = \mathrm{rk}(S) < n$.

In [19] (Theorem 3.1), the authors proved the following lemma for an element in a ring with involution.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$. Then A is core invertible with $A^{\tiny\textcircled{\#}} = X$ if and only if $(AX)^* = AX$, $XA^2 = A$ and $AX^2 = X$.*

Proposition 2.2. *Let A, B, $U \in \mathbb{C}^{n\times n}$ with $A = UBU^*$, where B is core invertible and U is unitary. Then A is core invertible. In this case, one has $A^{\tiny\textcircled{\#}} = UB^{\tiny\textcircled{\#}}U^*$.*

Proof. Let $X = UB^{\tiny\textcircled{\#}}U^*$, we have

$$AX = AUB^{\tiny\textcircled{\#}}U^* = UBU^*UB^{\tiny\textcircled{\#}}U^* = UBB^{\tiny\textcircled{\#}}U^* \text{ is Hermitian,}$$

$$XA^2 = UB^{\tiny\textcircled{\#}}U^*(UBU^*)^2 = UB^{\tiny\textcircled{\#}}(B)^2U^* = UBU^* = A,$$

$$AX^2 = UAU^*(UB^{\tiny\textcircled{\#}}U^*)^2 = UB(B^{\tiny\textcircled{\#}})^2U^* = UB^{\tiny\textcircled{\#}}U^* = X.$$

Thus, $A^{\tiny\textcircled{\#}} = UB^{\tiny\textcircled{\#}}U^*$ in view of Lemma 2.1.

Recently, Wang introduced a new decomposition for square matrices, named the Core-EP decomposition in [17] (Theorem 2.1).

Lemma 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then A can be written as*

$$A = A_1 + A_2, \quad (2.3)$$

in which

(1) $A_1 \in \mathbb{C}_n^{CM}$;

(2) $A_2^k = 0$;

(3) $A_1^* A_2 = A_2 A_1 = 0$.
We call the equality (2.3) *as the Core-EP decomposition of* A.

Definition 2.2 ([14], Definition 3.1)**.** *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *A matrix* $X \in \mathbb{C}^{n\times n}$ *is called a core-EP inverse of* A *if* X *is an outer inverse of* A *and satisfying*

$$\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k).$$

If such X *exists, then it is unique and denoted by* $A^{\textcircled{\dagger}}$.

3. A matrix decomposition related the CS decomposition and the core-EP decomposition.

Theorem 3.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$ *and* $r = \operatorname{rk}(A)$. *Then there exists a unitary matrix* $U \in \mathbb{C}^{n\times n}$ *such that*

$$A = U \begin{bmatrix} MC & MS \\ 0 & D_4 \end{bmatrix} U^*, \tag{3.1}$$

where M *and* C *are both nonsingular,* D_4 *is nilpotent,* $C^2 + SS^* = I_r$ *and matrices* C *and* S *have the form after equality* (2.1).

Proof. From Lemma 2.2, we have

$$A = A_1 + A_2,$$

in which $A_1 \in \mathbb{C}_n^{CM}$, $A_2^k = 0$ and $A_1^* A_2 = A_2 A_1 = 0$. Now, utilizing the decomposition in Theorem 2.1 to A_1, there exists a unitary matrix U such that

$$A_1 = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*,$$

in which M is nonsingular. We also have that C is nonsingular in view of $A_1 \in \mathbb{C}_n^{CM}$ and [1] (Theorem 3.7). Have in mind C is Hermitian. If we let $A_2 = U \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} U^*$, where $D_1 \in \mathbb{C}^{r\times r}$, then

$$A_1^* A_2 = U \begin{bmatrix} CM^*D_1 & CM^*D_2 \\ S^*M^*D_1 & S^*M^*D_2 \end{bmatrix} U^* \quad \text{and} \quad A_2 A_1 = U \begin{bmatrix} D_1MC & D_1MS \\ D_3MC & D_3MS \end{bmatrix} U^*. \tag{3.2}$$

From (3.2) and $A_1^* A_2 = A_2 A_1 = 0$, we get $CM^*D_1 = 0$, $CM^*D_2 = 0$ and $D_3MC = 0$. The nonsingularity of C and M implies that D_1, D_2 and D_3 are zero matrices. Thus,

$$A = A_1 + A_2 = U \begin{bmatrix} MC & MS \\ 0 & D_4 \end{bmatrix} U^*.$$

The equality $A_2^k = 0$ implies that D_4 is nilpotent.

Theorem 3.1 is proved.

Note that the decomposition in Theorem 3.1 has the same form as Schur form, but the decomposition in Theorem 3.1 seems easier to handle. Have in mind that M and C are both nonsingular, C is diagonal and real, D_4 is nilpotent and $C^2 + SS^* = I_r$ by Theorem 3.1.

Since C is nonsingular, we obtain

$$A_1^{\#} = U \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} U^*$$

by [1] (Theorem 3.7). It is evident that $A_1^{\textcircled{\#}} = U \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$. From [17] (Theorem 3.2), we get $A^{\textcircled{\dagger}} = A_1^{\textcircled{\#}} = U \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$.

In the following theorem, we will use the matrix decomposition in Theorem 3.1 to investigate the core-EP order, which was introduced by Wang in [17], defined as follows: for matrices $A, B \in \mathbb{C}^{n\times n}$

$$A \overset{\textcircled{\dagger}}{\leq} B \quad \Leftrightarrow \quad A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B \quad \text{and} \quad AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}.$$

Theorem 3.2. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form* (3.1). *Then $A \overset{\textcircled{\dagger}}{\leq} B$ if and only if $B - A$ can be written as*

$$B - A = U \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} U^* \text{ for some } B_4 \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{3.3}$$

Proof. Let $B - A = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^*$, where $B_1 \in \mathbb{C}^{r\times r}$. Note that $A \overset{\textcircled{\dagger}}{\leq} B$ if and only if $A^{\textcircled{\dagger}}(B - A) = (B - A)A^{\textcircled{\dagger}} = 0$, where

$$A^{\textcircled{\dagger}}(B - A) = U \begin{bmatrix} C^{-1}M^{-1}B_1 & C^{-1}M^{-1}B_2 \\ 0 & 0 \end{bmatrix} U^*, \tag{3.4}$$

$$(B - A)A^{\textcircled{\dagger}} = U \begin{bmatrix} B_1C^{-1}M^{-1} & 0 \\ B_3C^{-1}M^{-1} & 0 \end{bmatrix} U^*. \tag{3.5}$$

Assume $A \overset{\textcircled{\dagger}}{\leq} B$. From (3.4) and (3.5), we get $C^{-1}M^{-1}B_1 = 0$, $C^{-1}M^{-1}B_2 = 0$ and $B_3C^{-1}M^{-1} = 0$, thus from the nonsingularity of C and M, we obtain that B_1, B_2 and B_3 are zero matrices. Therefore, we get (3.3). To prove the opposite implication, it is easy to check that $(B - A)A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}(B - A) = 0$, that is $A \overset{\textcircled{\dagger}}{\leq} B$.

Theorem 3.2 is proved.

4. Core, star, group and minus partial order. In this section, we will consider the core, star, group and minus partial orders by using Theorem 2.1.

Lemma 4.1 ([12], Lemma 2.2)**.** *Let $A \in \mathbb{C}^{n\times n}$ be group invertible. Then $A \overset{\#}{\leq} B$ if and only if $A^2 = AB = BA$.*

Lemma 4.2 ([20], [Theorem 3.2)**.** *Let $A, B \in \mathbb{C}^{n\times n}$ be two core invertible matrices. Then $A \overset{\textcircled{\#}}{\leq} B$ if and only if $A \ast\!\leq B$ and $B^{\textcircled{\#}}AA^{\textcircled{\#}} = A^{\textcircled{\#}}$.*

An equivalent form of the minus partial order is the following statement: for complex matrix case can be found in [5, 15] and for element in rings case can be found in [11].

Lemma 4.3. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

(1) $B \overset{-}{\leq} A$;

(2) $B = AA^-B = BA^-A = BA^-B$ *for some* $A^- \in A\{1\}$;

(3) $B = AA^-B = BA^-A = BA^-B$ *for all* $A^- \in A\{1\}$.

The following lemma was proved in the more general setting of rings with involution in [16] (Theorem 4.10).

Lemma 4.4. *Let $A, B \in \mathbb{C}^{n\times n}$. If A, B are both core invertible and $B \overset{-}{\leq} A$, then $B \overset{\#}{\leq} A$ if and only if $A^{\#}BA^{\#} = B^{\#}$.*

Theorem 4.1. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form* (2.1). *If A is core invertible, then the following statements are equivalent:*

(1) $A \overset{\#}{\leq} B$;

(2) *$B - A$ can be written as*

$$B - A = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}; \tag{4.1}$$

(3) $AA^{\#}(B - A) = 0$ *and* $(B - A)AA^{\#} = 0$;

(4) $B = A + (I_n - AA^{\#})X(I_n - AA^{\#})$ *for some matrix* $X \in \mathbb{C}^{n\times n}$.

Proof. (1) $\Leftrightarrow$ (2). Since A is core invertible and core invertibility of a matrix gives the group invertibility of such matrix, hence C is nonsingular. Note that

$$A^{\#} = A^{\#}AA^{\dagger} = Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*.$$

If we let $B - A = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$, where $B_1 \in \mathbb{C}^{r\times r}$, and suppose that $A \overset{\#}{\leq} B$, then $A \overset{\#}{\leq} B$ if and only of $A^{\#}(B - A) = (B - A)A^{\#} = 0$ and

$$A^{\#}(B - A) = Y \begin{bmatrix} C^{-1}M^{-1}B_1 & C^{-1}M^{-1}B_2 \\ 0 & 0 \end{bmatrix} Y^*, \tag{4.2}$$

$$(B - A)A^{\#} = Y \begin{bmatrix} B_1C^{-1}M^{-1} & 0 \\ B_3C^{-1}M^{-1} & 0 \end{bmatrix} Y^*. \tag{4.3}$$

From (4.2) and (4.3), we get $C^{-1}M^{-1}B_1 = 0$, $C^{-1}M^{-1}B_2 = 0$ and $B_3C^{-1}M^{-1} = 0$. Thus, from the nonsingularity of C and M, we obtain that B_1, B_2 and B_3 are zero matrices. Conversely, if we have (4.1), it is easy to check that $AA^{\#}B = A$, which is equivalent to $A^{\#}A = A^{\#}B$. And we have $AA^{\#} = BA^{\#}$ in a similar way.

(2) $\Rightarrow$ (3). It is easy to check that $AA^{\#}(B - A) = 0$ and $(B - A)AA^{\#} = 0$ by $AA^{\#} = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^*$.

(3) $\Rightarrow$ (2). If we let $B - A = Y \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} Y^*$, then $AA^{\#}(B - A) = 0$ implies that X_1 and X_2 are zero matrices and $(B - A)AA^{\#} = 0$ implies that X_1 and X_3 are zero matrices. Thus we have the form in (4.1).

(2) $\Rightarrow$ (4). Note that (4.1) can be written as

$$B - A = Y \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Y^* =$$

$$= Y\left\{\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)\right\}Y^*.$$

Therefore, $B - A = (I_n - AA^{\circledast})X(I_n - AA^{\circledast})$ for some matrix $X \in \mathbb{C}^{n\times n}$.

$(4) \Rightarrow (3)$ is trivial.

Theorem 4.1 is proved.

***Remark* 4.1.** If $A \in \mathbb{C}^{n\times n}$ is core invertible, then $A^{\circledast} = A^{\oplus}$ by [14] (Theorem 3.8). Thus, the equivalence between (1) and (2) in Theorem 4.1 also can be got by Theorem 3.2.

Theorem 4.2. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form* (2.1)*. If A is group invertible, then we have the following two parts.*

Part (I). *The following statements are equivalent:*

(1) $A \overset{*}{\leq} B$;

(2) *$B - A$ can be written as*

$$B - A = Y\begin{bmatrix} 0 & 0 \\ -B_4S^*C^{-1} & B_4 \end{bmatrix}Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}; \tag{4.4}$$

(3) $AA^{\dagger}(B - A) = 0$ *and* $(B - A)AA^{\dagger} = (B - A)[AA^{\dagger} - (AA^{\#})^*]$;

(4) $B = A + (I_n - AA^{\dagger})X(I_n - AA^{\dagger})(I_n - A^{\#}A)^*$ *for some matrix* $X \in \mathbb{C}^{n\times n}$.

Part (II). *The following statements are equivalent:*

(1) $A \overset{\#}{\leq} B$;

(2) *$B - A$ can be written as*

$$B - A = Y\begin{bmatrix} 0 & -C^{-1}SB_4 \\ 0 & B_4 \end{bmatrix}Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}; \tag{4.5}$$

(3) $AA^{\#}(B - A) = 0$ *and* $(B - A)AA^{\#} = 0$;

(4) *there exists a projection Q such that $QA = 0$ and $B = A + (I_n - A^{\#}A)QXQ$ for some matrix $X \in \mathbb{C}^{n\times n}$.*

Proof. Part (I).

$(1) \Leftrightarrow (2)$. Since A is group invertible, we get that C is nonsingular. Let

$$B - A = Y\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}Y^*$$

and suppose $A \overset{*}{\leq} B$. We marked with $\bigstar$, the entries that we are not interest in. Since $A \overset{*}{\leq} B$ if and only if $A^*(B - A) = (B - A)A^* = 0$ and

$$A^*(B - A) = Y\begin{bmatrix} CM^*B_1 & CM^*B_2 \\ \bigstar & \bigstar \end{bmatrix}Y^*, \tag{4.6}$$

$$(B - A)A^* = Y\begin{bmatrix} \bigstar & 0 \\ B_3CM^* + B_4S^*M^* & 0 \end{bmatrix}Y^*, \tag{4.7}$$

thus, from (4.6) and (4.7), we get $CM^*B_1 = 0$, $CM^*B_2 = 0$ and $B_3CM^* + B_4S^*M^* = 0$. The nonsingularity of C and M leads to B_1 and B_2 are zero matrices and $B_3 = -B_4S^*C^{-1}$.

Conversely, by $B - A = Y \begin{bmatrix} 0 & 0 \\ -B_4 S^* C^{-1} & B_4 \end{bmatrix} Y^*$ we have

$$(B - A)A^* = Y \begin{bmatrix} 0 & 0 \\ -B_4 S^* C^{-1} & B_4 \end{bmatrix} \begin{bmatrix} CM^* & 0 \\ S^* M^* & 0 \end{bmatrix} Y^* = 0, \tag{4.8}$$

$$A^*(B - A) = Y \begin{bmatrix} CM^* & 0 \\ S^* M^* & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -B_4 S^* C^{-1} & B_4 \end{bmatrix} Y^* = 0. \tag{4.9}$$

From (4.8) and (4.9), we get $BA^* = AA^*$ and $A^*B = A^*A$. That is $A \overset{*}{\leq} B$.

(2) $\Rightarrow$ (3). Since we have $A^\dagger = Y \begin{bmatrix} CM^{-1} & 0 \\ S^* M^{-1} & 0 \end{bmatrix} Y^*$, so, $AA^\dagger = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^*$. Observe that

$$(AA^\#)^* = \left(Y \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* \right)^* = Y \begin{bmatrix} I_r & 0 \\ S^* C^{-1} & 0 \end{bmatrix} Y^*,$$

thus,

$$AA^\dagger - (AA^\#)^* = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^* - Y \begin{bmatrix} I_r & 0 \\ S^* C^{-1} & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & 0 \\ -S^* C^{-1} & 0 \end{bmatrix} Y^*.$$

Therefore, it is easy to check that $AA^\dagger(B - A) = 0$ and $(B - A)AA^\dagger = (B - A)[AA^\dagger - (AA^\#)^*]$.

(3) $\Rightarrow$ (2). Let $B - A = Y \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} Y^*$, then $AA^\dagger(B - A) = 0$ implies that X_1 and X_2 are zeros and $(B - A)AA^\dagger = (B - A)Y \begin{bmatrix} 0 & 0 \\ -S^* C^{-1} & 0 \end{bmatrix} Y^*$ implies $X_3 = -X_4 S^* C^{-1}$. Thus, we have the form in (4.4).

(2) $\Rightarrow$ (4). Note that (4.4) can be written as

$$B - A = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -S^* C^{-1} & I_{n-r} \end{bmatrix} Y^* =$$

$$= Y \left\{ \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} \left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} \right)^* \right\} Y^* =$$

$$= Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^* \left(I_n - A^\# A \right)^*.$$

Therefore, $B - A = (I_n - AA^\dagger)X(I_n - AA^\dagger)\left(I_n - A^\# A\right)^*$ for some matrix $X \in \mathbb{C}^{n\times n}$.

(4) $\Rightarrow$ (1). Since $B - A = (I_n - AA^\dagger)X(I_n - AA^\dagger)(I_n - A^\# A)^*$ for some matrix $X \in \mathbb{C}^{n\times n}$, thus, it is not difficult to verify that $A^*(B - A) = 0 = (B - A)A^*$ by $A^*(I_n - AA^\dagger) = 0$ and $(I_n - AA^\#)^* A^* = 0$.

Part (II).

(1) $\Rightarrow$ (2). Since A is group invertible, we get that C is nonsingular. If we let $B - A = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$, where $B_1 \in \mathbb{C}^{r\times r}$, and suppose $A \overset{\#}{\leq} B$, then $AB = A^2 = BA$ by

Lemma 4.1. It is obvious that $AB = A^2 = BA$ if and only if $(B - A)A = 0 = A(B - A)$. We marked with ★ the entries that we are not interest in

$$(B-A)A = Y\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix}Y^* = Y\begin{bmatrix} B_1MC & \bigstar \\ B_3MC & \bigstar \end{bmatrix}Y^*, \tag{4.10}$$

$$A(B-A) = Y\begin{bmatrix} \bigstar & MCB_2 + MSB_4 \\ 0 & 0 \end{bmatrix}Y^*. \tag{4.11}$$

From (4.10) and (4.11), we get $B_3MC = 0$, $B_1MC = 0$ and $0 = MCB_2 + MSB_4$. Thus, from the nonsingularity of C and M, we obtain that B_1 and B_3 are zero matrices and $B_2 = -C^{-1}SB_4$.

$(2) \Rightarrow (3)$. Since we have

$$A^{\#} = Y\begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix}Y^*, \quad \text{so} \quad AA^{\#} = Y\begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix}Y^*.$$

It is easy to check that $AA^{\#}(B - A) = 0$ and $(B - A)AA^{\#} = 0$.

$(3) \Rightarrow (1)$. Since $AA^{\#}(B - A) = 0$ and $(B - A)AA^{\#} = 0$ are equivalent to $AA^{\#}B = A$ and $BAA^{\#} = A$, respectively, we get $A^{\#}B = A^{\#}A$ and $AA^{\#} = BA^{\#}$ by multiplying $A^{\#}$ on the left-hand side of $AA^{\#}B = A$ and multiplying $A^{\#}$ on the right-hand side of $BAA^{\#} = A$. That is $A \overset{\#}{\leq} B$ by the definition of the sharp star partial order.

$(2) \Rightarrow (4)$. Note that (4.5) can be written as

$$B - A = Y\begin{bmatrix} 0 & -C^{-1}S \\ 0 & I_{n-r} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}Y^* =$$

$$= Y\left\{\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\right\}Y^*.$$

Therefore, $B - A = (I_n - A^{\#}A)(I_n - AA^{\dagger})X(I_n - AA^{\dagger})$ for some matrix $X \in \mathbb{C}^{n\times n}$.

$(4) \Rightarrow (1)$. Multiplying by A on the left-hand side of $B - A = (I_n - A^{\#}A)QXQ$, we have $A^2 = AB$, and multiplying by A on the right-hand side of $B - A = (I_n - A^{\#}A)QXQ$, we obtain $A^2 = BA$. Thus, $A \overset{\#}{\leq} B$ by Lemma 4.1.

Theorem 4.2 is proved.

Let $A, B \in \mathbb{C}^{n\times n}$ and let A be a group invertible matrix. If A has the form (2.1) and let $B - A = Y\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}Y^*$. From part (I) in Theorem 4.2, we get $A \overset{*}{\leq} B$ if and only if $B - A$ can be written as

$$B - A = Y\begin{bmatrix} 0 & 0 \\ -B_4S^*C^{-1} & B_4 \end{bmatrix}Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}.$$

Similarly, from part (II) in Theorem 4.2, we have $A \overset{\#}{\leq} B$ if and only if $B - A$ can be written as

$$B - A = Y\begin{bmatrix} 0 & -C^{-1}SB_4 \\ 0 & B_4 \end{bmatrix}Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}.$$

Thus $B_4S^* = 0$ and $SB_4 = 0$ implies $A \overset{\#}{\leq} B$ if and only if $A \overset{*}{\leq} B$. It is obvious that $A^{\#} - A^{\textcircled{\#}} = Y \begin{bmatrix} 0 & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*$. Therefore, A is an EP matrix if and only if $S = 0$, that is all the canonical angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are 0. This result can be found in [1] (Theorem 3.7). Thus if all the canonical angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are 0, then $A \overset{\#}{\leq} B$ if and only if $A \overset{*}{\leq} B$. But in general, the condition, $B_4S^* = 0$ and $SB_4 = 0$ is weaker than A is an EP matrix.

Theorem 4.3. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A is group invertible.*

(1) *If $(AA^\dagger - AA^\#)B(I_n - AA^\dagger) = 0$ and $(AA^\dagger - AA^\#)B^*(I_n - AA^\dagger) = 0$, then $A \overset{\#}{\leq} B$ if and only if $A \overset{*}{\leq} B$.*

(2) *If $A \overset{\#}{\leq} B$ and $A \overset{*}{\leq} B$, then $(AA^\dagger - AA^\#)B(I_n - AA^\dagger) = 0$ and $(AA^\dagger - AA^\#)B^*(I_n - AA^\dagger) = 0$.*

Proof. Let us write A as in Theorem 2.1 and $B - A = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$. Since A is group invertible, matrix C is nonsingular.

(1). The equality $(AA^\dagger - AA^\#)B(I_n - AA^\dagger) = 0$ can be rewritten as $(AA^\dagger - AA^\#)(B - A)(I_n - AA^\dagger) = 0$. Now,

$$0=(AA^\dagger - AA^\#)(B - A)(I_n - AA^\dagger)=Y \begin{bmatrix} 0 & -C^{-1}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Y^* =$$

$$= Y \begin{bmatrix} 0 & -C^{-1}SB_4 \\ 0 & 0 \end{bmatrix} Y^*$$

implies $SB_4 = 0$.

Now, $(I_n - AA^\dagger)B(AA^\dagger - AA^\#)^* = 0$ is equivalent to $(I_n - AA^\dagger)(B - A)(AA^\dagger - AA^\#)^* = 0$. Hence,

$$0=(I_n - AA^\dagger)(B - A)(AA^\dagger - AA^\#)^* =Y \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -S^*C^{-1} & 0 \end{bmatrix} Y^* =$$

$$= Y \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}\begin{bmatrix} -B_2S^*C^{-1} & 0 \\ -B_4S^*C^{-1} & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & 0 \\ -B_4S^*C^{-1} & 0 \end{bmatrix} Y^*$$

implies $B_4S^* = 0$. By Theorem 4.2, we obtain

$$A \overset{*}{\leq} B \iff B_1 = 0, B_2 = 0, B_3 = B_4S^*C^{-1} \iff$$

$$\iff B_1 = 0, B_3 = 0, B_2 = -C^{-1}SB_4 \iff A \overset{\#}{\leq} B.$$

(2). By Theorem 4.2 we have $B_1 = 0$, $B_2 = 0$, $B_3 = 0$, $0 = B_4S^*$, and $SB_4 = 0$, and the computations made in the previous item show that $(AA^\dagger - AA^\#)B(I_n - AA^\dagger) = 0$ and $(AA^\dagger - AA^\#)B^*(I_n - AA^\dagger) = 0$.

Theorem 4.3 is proved.

***Corollary* 4.1.** *If A is an EP matrix, then $A \overset{\#}{\leq} B$ if and only if $A \overset{*}{\leq} B$.*

Note that Corollary 4.1 is a consequence of Theorem 4.3. There exists a another method to prove this result as follows. In [4] (Theorem 7), Baksalary and Trenker proved that for complex matrices A and B, if A is an EP matrix, then $A \overset{\textcircled{\#}}{\le} B$ if and only if $A \overset{*}{\le} B$. In [13] (Theorem 3.3), Mailk proved that for complex matrices A and B, if A is an EP matrix, then $A \overset{\textcircled{\#}}{\le} B$ if and only if $A \overset{\#}{\le} B$. Thus, if A is an EP matrix, then $A \overset{\#}{\le} B$ if and only if $A \overset{*}{\le} B$.

Theorem 4.4. *Let $A, B \in \mathbb{C}^{n\times n}$ are core invertible. Then $A \overset{\textcircled{\#}}{\le} B$ if and only if $A \ast\!\le B$ and $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$.*

Proof. Assume that A has the form (2.1). Since A is core invertible and a core invertible matrix is group invertible, we get C is nonsingular. Let $B^{\textcircled{\#}} = Y \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} Y^*$. It is enough to prove that $B^{\textcircled{\#}} A A^{\textcircled{\#}} = A^{\textcircled{\#}}$ if and only if $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$ by Lemma 4.2. If $B^{\textcircled{\#}} A A^{\textcircled{\#}} = A^{\textcircled{\#}}$, then $(B^{\textcircled{\#}} - A^{\textcircled{\#}})(I_n - AA^{\textcircled{\#}}) = B^{\textcircled{\#}} - A^{\textcircled{\#}}$, and, thus, exists $X \in \mathbb{C}^{n\times n}$ such that $B^{\textcircled{\#}} - A^{\textcircled{\#}} = X(I_n - AA^{\textcircled{\#}})$, and $B^{\textcircled{\#}} - A^{\textcircled{\#}} = X(I_n - AA^{\textcircled{\#}})$ implies $(B^{\textcircled{\#}} - A^{\textcircled{\#}})^* = (I_n - AA^{\textcircled{\#}})X^*$. Hence, $\mathcal{R}[(B^{\textcircled{\#}} - A^{\textcircled{\#}})^*] \subseteq \mathcal{R}(I_n - AA^{\textcircled{\#}})$. But $\mathcal{R}[(B^{\textcircled{\#}} - A^{\textcircled{\#}})^*] = [\mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})]^{\perp}$ and, by using that $AA^{\textcircled{\#}}$ is the orthogonal projector onto $\mathcal{R}(A)$, we have $\mathcal{R}(I_n - AA^{\textcircled{\#}}) = \mathcal{R}(AA^{\textcircled{\#}})^{\perp} = \mathcal{R}(A)^{\perp}$. Therefore, $[\mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})]^{\perp} \subseteq \mathcal{R}(A)^{\perp}$, hence $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$.

Conversely, if $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$, then $\mathcal{R}[(B^{\textcircled{\#}} - A^{\textcircled{\#}})^*] = [\mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})]^{\perp} \subseteq [\mathcal{R}(A)]^{\perp} = \mathcal{R}(AA^{\textcircled{\#}})^{\perp} = \mathcal{R}(I_n - AA^{\textcircled{\#}})$, hence, $B^{\textcircled{\#}} - A^{\textcircled{\#}} = X'(I_n - AA^{\textcircled{\#}})$ for some matrix X'. Therefore, $B^{\textcircled{\#}} A A^{\textcircled{\#}} = [A^{\textcircled{\#}} + X'(I_n - AA^{\textcircled{\#}})]AA^{\textcircled{\#}} = A^{\textcircled{\#}} A A^{\textcircled{\#}} = A^{\textcircled{\#}}$.

Theorem 4.4 is proved.

Let $A, B \in \mathbb{C}^{n\times n}$. To study a partial order between A and B, we have two directions. One is to use the matrix decomposition of A; another is to use the matrix decomposition of B.

Theorem 4.5. *Let $A, B \in \mathbb{C}^{n\times n}$ and A be group invertible. Assume that A has the form* (2.1). *Then $B \overset{-}{\le} A$ if and only if B can be written as*

$$B = Y \begin{bmatrix} B_1 & B_1 C^{-1} S \\ 0 & 0 \end{bmatrix} Y^*, \quad C^{-1} M^{-1} \in B_1\{1\}. \tag{4.12}$$

Moreover, if $B \overset{-}{\le} A$, then B_1 is core invertible and $B^{\textcircled{\#}} = Y \begin{bmatrix} B_1^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix} Y^*$.

Proof. Since A is group invertible, we have that C is nonsingular. Let $B = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$ with $B_1 \in \mathbb{C}^{r\times r}$. If $B \overset{-}{\le} A$, then $B = AA^{\textcircled{\#}} B = BA^{\textcircled{\#}} A = BA^{\textcircled{\#}} B$ by Lemma 4.3. From $AA^{\textcircled{\#}} B = Y \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} Y^*$ and $B = AA^{\textcircled{\#}} B$, we get that B_3 and B_4 are zero matrices. From $BA^{\textcircled{\#}} A = Y \begin{bmatrix} B_1 & B_1 C^{-1} S \\ 0 & 0 \end{bmatrix} Y^*$ and $B = BA^{\textcircled{\#}} A$, we get $B_2 = B_1 C^{-1} S$. From

$$BA^{\textcircled{\#}} B = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} C^{-1} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^* =$$

$$= Y \begin{bmatrix} B_1 C^{-1} M^{-1} B_1 & B_1 C^{-1} M^{-1} B_2 \\ 0 & 0 \end{bmatrix} Y^*$$

and $BA^{\#}B = B$, we get $B_1C^{-1}M^{-1}B_1 = B_1$. Thus B has the form in (4.12).

For the opposite implication, it is easy to check that $B = AA^{\#}B = BA^{\#}A = BA^{\#}B$, which gives $B \overset{-}{\leq} A$ by Lemma 4.3.

Since B can be written as

$$B = Y \begin{bmatrix} B_1 & B_1C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*.$$

The group invertibility of B leads to the core invertibility of B and B_1 by [10] (Theorem 1). It is easy to verify that $B^{\#} = Y \begin{bmatrix} B_1^{\#} & 0 \\ 0 & 0 \end{bmatrix} Y^*$ by Proposition 2.1 and Lemma 2.1.

Theorem 4.5 is proved.

Theorem 4.5 will be useful in the next theorem. In the following, we will answer the question, when the minus partial order is the core partial order.

Theorem 4.6. *Let $A, B \in \mathbb{C}^{n\times n}$ be core invertible. Assume that A has the form* (2.1). *Then $B \overset{\#}{\leq} A$ if and only if B can be written as*

$$B = Y \begin{bmatrix} B_1 & B_1C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*, \quad C^{-1}M^{-1} \in B_1\{1\} \tag{4.13}$$

and $B_1 = MCB_1^{\#}MC$.

Proof. Since A is core invertible and the core invertibility is equivalent to the group invertibility, we get that C is nonsingular. Let $B = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$, where $B_1 \in \mathbb{C}^{r\times r}$. Suppose $B \overset{\#}{\leq} A$. Since $B \overset{\#}{\leq} A$ implies $B \overset{-}{\leq} A$, Theorem 4.5 and Lemma 4.4 imply $B = Y \begin{bmatrix} B_1 & B_1C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*$, $C^{-1}M^{-1} \in B_1\{1\}$ and $B^{\#} = Y \begin{bmatrix} B_1^{\#} & 0 \\ 0 & 0 \end{bmatrix} Y^*$. Since $B \overset{\#}{\leq} A$ implies $A^{\#}BA^{\#} = B^{\#}$ by Lemma 4.4,

$$A^{\#}BA^{\#} = Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^* =$$

$$= Y \begin{bmatrix} C^{-1}M^{-1}B_1C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*. \tag{4.14}$$

From (4.14) and $A^{\#}BA^{\#} = B^{\#}$ (by Lemma 4.4), we get

$$C^{-1}M^{-1}B_1C^{-1}M^{-1} = B_1^{\#}.$$

That is $B_1 = MCB_1^{\#}MC$.

The opposite implication is trivial by Lemma 4.4 and Theorem 4.5.

Theorem 4.6 is proved.

5. Core invertibility under the core partial order. In [12] (Theorem 2.2), Mitra has shown that for matrices $A, B \in \mathbb{C}^{n\times n}$, if $A \overset{*}{\leq} B$, then $B^{\dagger} - A^{\dagger} = (B - A)^{\dagger}$. A natural question is that if A and $B - A$ are core invertible and $A \overset{\#}{\leq} B$, is B core invertible? Moreover, if B is core invertible, do we have $B^{\#} - A^{\#} = (B - A)^{\#}$? In the following theorem, we will answer this question.

Theorem 5.1. *Let $A, B \in \mathbb{C}^{n\times n}$. If A and $B - A$ are both core invertible and $A \overset{\#}{\leq} B$, then B is core invertible. In this case*

$$B^{\#} = A^{\#} + (B-A)^{\#} - A^{\#}A(B-A)^{\#}.$$

Moreover, if $(AA^{\dagger} - AA^{\#})B(I_n - AA^{\dagger}) = 0$, then

(1) $(B-A)^{\#} = B^{\#} - A^{\#}$;

(2) $(B-A) \overset{\#}{\leq} B$.

Proof. Let us write A as in (2.1). From Theorem 4.1, we have

$$B - A = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^*. \tag{5.1}$$

Since A and $B - A$ are core invertible, we get that C is nonsingular and B_4 is core invertible in view of the Proposition 2.1. The equality (5.1) gives that $B = Y\begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix} Y^*$.

Let

$$X = Y\begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\#} \\ 0 & B_4^{\#} \end{bmatrix} Y^*,$$

then we have

$$BX = Y\begin{bmatrix} I_r & 0 \\ 0 & B_4B_4^{\#} \end{bmatrix} Y^* \quad \text{is Hermitian,}$$

$$XB^2 = Y\begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\#} \\ 0 & B_4^{\#} \end{bmatrix}\begin{bmatrix} (MC)^2 & MCMS + MSB_4 \\ 0 & B_4^2 \end{bmatrix} Y^* = B,$$

$$BX^2 = Y\begin{bmatrix} I_r & 0 \\ 0 & B_4B_4^{\#} \end{bmatrix}\begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\#} \\ 0 & B_4^{\#} \end{bmatrix} Y^* = X.$$

Thus, $B^{\#} = X$ in view of Lemma 2.1.

That is, we have $B^{\#} = Y\begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\#} \\ 0 & B_4^{\#} \end{bmatrix} Y^*$. The equality (5.1) gives that $(B-A)^{\#} = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^{\#} \end{bmatrix} Y^*$ in view of the Proposition 2.1. Thus, $B^{\#} = A^{\#} + (B-A)^{\#} + {} + Y\begin{bmatrix} 0 & -C^{-1}SB_4^{\#} \\ 0 & 0 \end{bmatrix} Y^*$.

Having in mind $A^{\#} = Y\begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*$. Finally, since we have $A^{\#}A = AA^{\#}$ and

$$Y\begin{bmatrix} 0 & C^{-1}SB_4^{\#} \\ 0 & 0 \end{bmatrix} Y^* = Y\begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4^{\#} \end{bmatrix} Y^* = AA^{\#}(B-A)^{\#}.$$

Thus, $B^{\#} = A^{\#} + (B-A)^{\#} - A^{\#}A(B-A)^{\#}$.

Theorem 5.1 is proved.

***Corollary* 5.1.** *Let $A, B \in \mathbb{C}^{n\times n}$. If A and $B - A$ are both core invertible, $A \overset{\#}{\leq} B$, then $A^2 = AB$ if and only if $B^{\#} - A^{\#} = (B-A)^{\#}$.*

Proof. Assume that A has the form (2.1). Since $B^{\text{\textcircled{\#}}} - A^{\text{\textcircled{\#}}} = (B-A)^{\text{\textcircled{\#}}}$ if and only if $A^{\text{\textcircled{\#}}}A(B-A)^{\text{\textcircled{\#}}} = 0$ by Theorem 5.1 and $A^{\text{\textcircled{\#}}}A(B-A)^{\text{\textcircled{\#}}} = Y\begin{bmatrix} 0 & C^{-1}SB_4^{\text{\textcircled{\#}}} \\ 0 & 0 \end{bmatrix}Y^*$, thus it is sufficient to prove that $A^2 = AB$ if and only if $Y\begin{bmatrix} 0 & C^{-1}SB_4^{\text{\textcircled{\#}}} \\ 0 & 0 \end{bmatrix}Y^* = 0$. It is equivalent to show that $AA^{\#}(B-A)^{\text{\textcircled{\#}}} = 0$ if and only if $A^2 = AB$. Since we get $B - A = (B-A)^{\text{\textcircled{\#}}}(B-A)^2$ and $(B-A)^{\text{\textcircled{\#}}} = (B-A)((B-A)^{\text{\textcircled{\#}}})^2$, thus,

$$AA^{\#}(B-A)^{\text{\textcircled{\#}}} = 0 \;\;\Leftrightarrow\;\; A(B-A)^{\text{\textcircled{\#}}} = 0 \;\;\Leftrightarrow\;\; A(B-A) = 0.$$

Let us write A as in Theorem 2.1 and $B - A = Y\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}Y^*$. We have that B_1, B_2 and B_3 are zero matrices by Theorem 4.1. From the proof of Theorem 4.3, the condition $(AA^\dagger - AA^{\#})B(I_n - AA^\dagger) = 0$ implies $SB_4 = 0$.

The part (1) is obvious by

$$A(B-A) = Y\begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}Y^* = 0.$$

To prove the part (2). It is sufficient to show that $(B-A)^*(B-A) = (B-A)^*B$ and $(B-A)^2 = B(B-A)$ by [16] (Theorem 2.4):

$$(B-A)^*(B-A) = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^* \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}Y^* = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^*B_4 \end{bmatrix}Y^*, \tag{5.2}$$

$$(B-A)^*B = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^* \end{bmatrix}\begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix}Y^* = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^*B_4 \end{bmatrix}Y^*, \tag{5.3}$$

$$(B-A)^2 = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}Y^* = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^2 \end{bmatrix}Y^*, \tag{5.4}$$

$$B(B-A) = Y\begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}Y^* = Y\begin{bmatrix} 0 & 0 \\ 0 & B_4^2 \end{bmatrix}Y^*. \tag{5.5}$$

From (5.2), (5.3), (5.4) and (5.5) we get $(B-A) \overset{\text{\textcircled{\#}}}{\leq} B$.

Corollary 5.1 is proved.

In [1] (Theorem 3.7), the author proved that if A is an EP matrix, then $S = 0$.

***Corollary* 5.2.** *Let $A, B \in \mathbb{C}^{n\times n}$. If A and $B - A$ are core invertible, $A \overset{\text{\textcircled{\#}}}{\leq} B$ and A is an EP matrix, then $B^{\text{\textcircled{\#}}} - A^{\text{\textcircled{\#}}} = (B-A)^{\text{\textcircled{\#}}}$.*

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