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A. Azari, A. Khojali, N. Zamani (Univ. Mohaghegh Ardabili, Ardabil, Iran)

VANISHING AND ARTINIANNESS OF GRADED GENERALIZED LOCAL COHOMOLOGY

ЗНИКНЕННЯ ТА АРТІНОВІСТЬ ГРАДУЙОВАНОЇ УЗАГАЛЬНЕНОЇ ЛОКАЛЬНОЇ КОГОМОЛОГІЇ

Let $R = \oplus_{j\geq 0} R_j$ be a homogeneous Noetherian ring with semilocal base ring R_0. Let $R_+ = \oplus_{j\geq 1} R_j$ be the irrelevant ideal of R. For two finitely generated graded R-modules M and N, several results on the vanishing, Artiniannes and tameness property of the graded R-modules $H^i_{R_+}(M,N)$ will be investigated.

Нехай $R = \oplus_{j\geq 0} R_j$ — однорідне ньотерове кільце з напівлокальним базовим кільцем R_0. Нехай також $R_+ = \oplus_{j\geq 1} R_j$ є іррелевантним ідеалом R. Для двох скінченнопороджених градуйованих R-модулів M і N наведено деякі результати щодо властивостей зникнення, артіновості та приборкання градуйованих R-модулів $H^i_{R_+}(M,N)$.

1. Introduction. Throughout this paper $R = \oplus_{n\geq 0} R_n$ is a homogeneous graded (Noetherian) ring with semilocal base ring R_0, so that R_0 is a Noetherian ring and R, as an R_0-algebra is generated by finitely many homogeneous elements of degree one. Let $R_+ = \oplus_{n>0} R_n$ be the irrelevant ideal of R and $\mathfrak{m}_0^{(1)}, \ldots, \mathfrak{m}_0^{(t)}$ be the maximal ideals of R_0. Assume that $M = \oplus_{n\in\mathbb{Z}} M_n$ and $N = \oplus_{n\in\mathbb{Z}} N_n$ are two finitely generated $\mathbb{Z}$-graded R-modules. For any graded ideal I of R, the ith generalized local cohomology module $H^i_I(M,N)$ has a natural graded structure, such that the long exact sequences induced from suitable short exact sequences (in both component) in the category of finitely generated graded R-modules and homogeneous homomorphisms is an exact sequence in this category. Furthermore, with $I = R_+$, it is well-known that the R_0-module $H^i_{R_+}(M,N)_n$ is finitely generated for all $n \in \mathbb{Z}$ and is zero for all $n \gg 0$ (see [12]). For more results on the graded modules $H^i_{R_+}(M,N)$ see [13].

In this paper we shall study the vanishing, Artinianness and tameness behavior of the graded R-modules $H^i_{R_+}(M,N)$, in case that R_0 is a semilocal ring and the projective dimension of M (denoted by $\mathrm{pd}(M)$) is finite. In Section 2, among some preliminaries, a vanishing theorem on these modules will be proved which improves [11] (Lemma 3.1) and [13] (Theorem 3.2) in this graded case. More precisely, it will be shown that $H^i_{R_+}(M,N) = 0$ for all $i > \mathrm{pd}(M) + \dim(N/\Gamma_{J_0R}(N))$, where $J_0 = \cap_{i=1}^t \mathfrak{m}_0^{(i)}$ is the Jacobson radical of R_0, $\Gamma_{J_0R}(N) = \{x \in N \mid \exists n \in \mathbb{N} \text{ such that } J_0^n x = 0\}$ is the J_0R-torsion submodule of N and $\dim$ stands for the Krull dimension of an R-module. Section

3 deals with Artinianness and tameness properties of the modules $H^i_{R_+}(M,N)$. One of the results in this section states as follows: Let R_0 be a semilocal ring with Jacobson radical J_0. Let M,N be two finitely generated graded R-modules with $p=\mathrm{pd}(M)<\infty$. Set $s=\dim(N/J_0N+\Gamma_{J_0R}(N))$. Then $H^i_{R_+}(M,N)$ is Artinian for $i>p+s$ and is tame for $i=p+s$. It is well-known that over a complete semilocal ring any Artinian module is Matlis reflexive. So, it is natural to ask that when generalized local cohomology modules are Matlis reflexive. Concerning this question we refer to [8].

2. Vanishing theorem. Our aim in this section is to prove a theorem on vanishing of the graded modules $H^i_{R_+}(M,N)$. Recall that these modules was defined in [6], as the direct limits of some Ext-modules; that is, for two R-modules M,N,

$$H^i_{R_+}(M,N)=\varinjlim_{n\in\mathbb{N}}\mathrm{Ext}^i_R(M/(R_+)^nM,N).$$

One can observe that each element of $H^i_{R_+}(M,N)$ is annihilated by a power of R_+ and so $H^i_{R_+}(M,N)$ is an R_+-torsion module. Other approaches of these modules can be found in [11] and [1]. To name one of them in a special case for which the first component is finitely generated, we have

$$H^i_{R_+}(M,N)\cong H^i(\Gamma_{R_+}(\mathrm{Hom}_R(M,\mathbf{I}^N)))\cong H^i(\mathrm{Hom}_R(M,\Gamma_{R_+}(\mathbf{I}^N))), \qquad (2.1)$$

where $\mathbf{I}^N$ is an injective resolution of N. From this fact and using [3] (Corollary 2.1.6), it is concluded that whenever M is finitely generated and $\Gamma_{R_+}(N)=N$. Then $H^i_{R_+}(M,N)=\mathrm{Ext}^i_R(M,N)$, and if in addition $p=\mathrm{pd}(M)<\infty$, then $H^i_{R_+}(M,N)=0$ for all $i>p$. This fact will be used several times in this paper.

We continue with the following key lemma. This lemma, appeared in [2] in the case that R_0 is a local ring, has been proved using a theorem of Kirby [7]. Here we give another proof, whenever R_0 is semilocal.

Lemma 2.1. *Let R be a homogeneous Noetherian ring with semilocal base ring R_0 and $N=\oplus_{i\in\mathbb{Z}}N_i$ be a finitely generated graded R-module. Let J_0 be the Jacobson radical of R_0 and $d=\dim(N/J_0N)$. Then $\Gamma_{R_+}(N)=N$ if and only if $d\leq 0$.*

Proof. One direction is clear. If $\Gamma_{R_+}(N)=N$, then $N_n=0$ for all $n\gg 0$. This gives that $N_n/J_0N_n=0$ for all $n\gg 0$ and so $\dim(N/J_0N)\leq 0$ as desired.

Now let $d\leq 0$. As in the introduction we assume that $\mathfrak{m}_0^{(1)},\ldots,\mathfrak{m}_0^{(t)}$ are the maximal ideals of R_0. If $d<0$ there is nothing to prove. So assume that $d=0$. In this case the only minimal prime ideals of N/J_0N are among the graded maximal ideals $\mathfrak{m}_0^{(1)}+R_+,\ldots,\mathfrak{m}_0^{(t)}+R_+$ and so there exists $n\in\mathbb{N}$ such that $(\cap_{j=1}^t(\mathfrak{m}_0^{(j)}+R_+))^n\subseteq(0:_R N/J_0N)$. This, in turn, gives that $R_+^m\subseteq(0:_R N/J_0N)$ and so $R_mN\subseteq J_0N$ for $m\geq n$. Therefore, we conclude that $\oplus_{i\geq d_1}N_{i+m}\subseteq\oplus_{i\geq d_1}J_0N_i$ for $m\geq n$, where $d_1=\min\{i\in\mathbb{Z}|N_i\neq 0\}$ is the beginning of N. From this, using NAK lemma, we obtain that $N_m=0$ for $m\geq n+d_1$ and, so, $\Gamma_{R_+}(N)=N$ as desired.

Lemma 2.1 is proved.

The next theorem improves [11] (Lemma 3.1) and [13] (Theorem 3.2).

Theorem 2.1. *Let R be a homogeneous Noetherian ring with semilocal base ring R_0. Let M,N be two finitely generated graded R-modules such that $p=\mathrm{pd}(M)$ is finite. Assume that $d=\dim(N/J_0N)$. Then $H^i_{R_+}(M,N)=0$ for all $i>p+d$.*

Proof. We prove this by induction on d. If $d \leq 0$, then by Lemma 2.1, $\Gamma_{R_+}(N) = N$ and so $H^i_{R_+}(M,N) = \mathrm{Ext}^i_R(M,N) = 0$ for all $i > p$.

So, assume that $d > 0$ and the result has been proved for $d-1$. Put $\bar{N} = N/\Gamma_{R_+}(N)$. Since $H^i_{R_+}(M,\Gamma_{R_+}(N)) = \mathrm{Ext}^i_R(M,\Gamma_{R_+}(N)) = 0$ for all $i > p$, the short exact sequence

$$0 \to \Gamma_{R_+}(N) \to N \to \bar{N} \to 0$$

gives rise to the isomorphism $H^i_{R_+}(M,N) \cong H^i_{R_+}(M,N/\Gamma_{R_+}(N))$ in the category of graded R-modules and R-morphisms (i.e., homogeneous R-homomorphisms) for all $i > p$. Since $\Gamma_{R_+}(N)$ has only finitely many non-zero components and since $d > 0$, then $\dim(\bar{N}/J_0\bar{N}) = \dim(N/J_0N)$. Therefore, we can replace N by $\bar{N}$ and may assume that $\Gamma_{R_+}(N) = 0$. So, by [3] (Lemma 2.1.1(ii)), $R_+ \nsubseteq Z_R(N) = \cup_{\mathfrak{p}\in \mathrm{Ass}_R(N)}\mathfrak{p}$, where $Z_R(N)$ denotes the set of all zero divisors of N in R. On the other hand, since $d > 0$, we see that for each minimal member $\mathfrak{p}$ of the set $\mathrm{Ass}_R(N/J_0N)$, $R_+ \nsubseteq \mathfrak{p}$. So,

$$R_+ \nsubseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(N)} \mathfrak{p} \cup \bigcup_{\mathfrak{p}\in \mathrm{MinAss}_R(N/J_0N)} \mathfrak{p}$$

and by [2] (Lemma 15.10), there exists a homogeneous element $x \in R_+$ which is a non-zero divisor on N and at the same time

$$\dim((N/xN)/J_0(N/xN)) = \dim((N/J_0N)/x(N/J_0N)) = d-1.$$

Considering the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ and using the induction hypothesis we get the isomorphisms

$$H^i_{R_+}(M,N) \overset{x}{\cong} H^i_{R_+}(M,N)$$

for all $i > p+d$. Now, as $H^i_{R_+}(M,N)$ is R_+-torsion we conclude that $H^i_{R_+}(M,N) = 0$ for each $i > p+d$.

Theorem 2.1 is proved.

The top non-vanishing problem of generalized local cohomology seems to be more subtle. While there is a partial answer for this problem in some special cases in [11], until now we were not able to formulate ordinary local cohomology non-vanishing counterparts in generalized local cohomology.

3. Artinian and tame properties. In this section, we will draw several results concerning the Artinian property and tameness of the modules $H^i_{R_+}(M,N)$. Following [2], a graded R-module T is said to be tame if there exists $m \in \mathbb{Z}$ such that $T_n = 0$ for all $n \leq m$ or $T_n \neq 0$ for all $n \leq m$. For ease in access we collect some known facts on generalized local cohomology in the frame of the following theorem.

Theorem 3.1. *Let $\mathfrak{a}$ be an (not necessarily graded) ideal of R and let X and Y be two finitely generated R-modules.*

(i) *If $R/\mathfrak{a}$ is Artinian, then for each $i \in \mathbb{N}_0$ the R-module $H^i_{\mathfrak{a}}(X,Y)$ is Artinian* [13] (Theorem 2.2).

(ii) $H^i_{\mathfrak{a}}(X,Y) \cong H^i_{\sqrt{\mathfrak{a}}}(X,Y)$, *for each $i \in \mathbb{N}_0$* [4] (Lemma 2.1 (i)).

(iii) *Let $x \in R$. Then there is a natural long exact sequence*

$$\ldots \to H^i_{\mathfrak{a}+(x)}(X,Y) \to H^i_{\mathfrak{a}}(X,Y) \to H^i_{\mathfrak{a}R_x}(X,Y) \to H^{i+1}_{\mathfrak{a}+(x)}(X,Y) \to \ldots$$

of generalized local cohomology modules. Furthermore, if R, X, Y and $\mathfrak{a}$ are graded and x is a homogeneous element of R, then all the maps in this exact sequence are homogeneous, so that for each $n \in \mathbb{Z}$, there exists the long exact sequence

$$\ldots \to H^i_{\mathfrak{a}+(x)}(X,Y)_n \to H^i_{\mathfrak{a}}(X,Y)_n \to H^i_{\mathfrak{a}R_x}(X,Y)_n \to H^{i+1}_{\mathfrak{a}+(x)}(X,Y)_n \to \ldots$$

of R_0-modules [5] (Lemma 3.1).

(iv) *If R' is another commutative Noetherian ring and $f\colon R \to R'$ is a flat ring homomorphism, then, for each ideal $\mathfrak{a}$ of R,*

$$H^i_{\mathfrak{a}}(X,Y) \otimes_R R' \cong H^i_{\mathfrak{a}R'}(X \otimes_R R', Y \otimes_R R').$$

Thus for a multiplicatively closed subset S of R,

$$S^{-1}H^i_{\mathfrak{a}}(X,Y) \cong H^i_{S^{-1}\mathfrak{a}}(S^{-1}X, S^{-1}Y).$$

If R, X, Y and $\mathfrak{a}$ are graded and $S \subseteq R_0$, then, for each $n \in \mathbb{Z}$,

$$S^{-1}(H^i_{\mathfrak{a}}(X,Y)_n) \cong [H^i_{S^{-1}\mathfrak{a}}(S^{-1}X, S^{-1}Y)]_n,$$

as R_0-modules. In particular, for each $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$ and each $n \in \mathbb{Z}$,

$$(H^i_{\mathfrak{a}}(X,Y)_n)_{\mathfrak{p}_0} \cong H^i_{\mathfrak{a}R_{\mathfrak{p}_0}}(X_{\mathfrak{p}_0}, Y_{\mathfrak{p}_0})_n.$$

Theorem 3.2. *Let R be a homogeneous Noetherian ring with semilocal base ring R_0 and J_0 be the Jacobson radical of R_0. Let M, N be two finitely generated graded R-modules with $p = \operatorname{pd}(\mathrm{M}) < \infty$. Put $d = \dim(N/J_0N)$. Then:*

(1) *The R-module $Q = R_0/J_0 \otimes_{R_0} H^{p+d}_{R_+}(M,N)$ is Artinian* (*see* [13], *Theorem* 3.3).

(2) *For each $i \geq 0$, the R-module $H^i_{R_+}(M, \Gamma_{J_0R}(N))$ is Artinian* (*see* [13], *Lemma* 3.5).

(3) *If $\dim(R_0) \leq 1$, then $\Gamma_{J_0R}(H^i_{R_+}(M,N))$, $H^1_{J_0R}(H^i_{R_+}(M,N))$ and $(0 :_{H^i_{R_+}(M,N)} J_0)$ are Artinian.*

(4) *The R-module $H^i_{R_+}(M,N)$ is Artinian for $i > p+s$ and is tame for $i = p+s$, where $s = \dim(N/J_0N + \Gamma_{J_0R}(N))$.*

(5) *For each $i \in \mathbb{N}_0$, if $R_0/J_0 \otimes_{R_0} H^i_{R_+}(M, N/\Gamma_{J_0R}(N))$ is Artinian, then $R_0/J_0 \otimes_{R_0} H^i_{R_+}(M,N)$ is Artinian too.*

Proof. (1) We prove this by induction on d. If $d \leq 0$, then, by using Lemma 2.1, we see that $H^p_{R_+}(M,N) = \operatorname{Ext}^p_R(M,N)$ vanishes by a power of R_+. Thus, $\operatorname{Supp}_R(Q) \subseteq \{\mathfrak{m}_0^{(1)} + R_+, \ldots, \mathfrak{m}_0^{(t)} + {} + R_+\}$ where as usual $\mathfrak{m}_0^{(1)}, \ldots, \mathfrak{m}_0^{(t)}$ are the maximal ideals of R_0. So, we deduce that Q is Artinian.

For $d > 0$ as in the proof of Theorem 2.1 we can find a homogeneous element $x \in R_+$ which is a non-zero divisor on N and $\dim((N/xN)/J_0(N/xN)) = d-1$. Therefore, by using Theorem 2.1, we can obtain the exact sequence

$$H^{p+d-1}_{R_+}(M, N/xN) \xrightarrow{\Delta} H^{p+d}_{R_+}(M,N) \xrightarrow{x} H^{p+d}_{R_+}(M,N) \to 0.$$

By induction hypothesis the R-module $R_0/J_0 \otimes_{R_0} H^{p+d-1}_{R_+}(M, N/xN)$ is Artinian. Thus $R_0/J_0 \otimes_{R_0} \operatorname{Im}(\Delta)$ is Artinian too. Now, considering the exact sequence

$$R_0/J_0 \otimes_{R_0} \operatorname{Im}(\Delta) \to R_0/J_0 \otimes_{R_0} H^{p+d}_{R_+}(M,N) \xrightarrow{x} R_0/J_0 \otimes_{R_0} H^{p+d}_{R_+}(M,N) \to 0$$

gives that $(0 :_{R_0/J_0 \otimes_{R_0} H^{p+d}_{R_+}(M,N)} x)$ as a homomorphic image of $R_0/J_0 \otimes_{R_0} \operatorname{Im}(\Delta)$ is Artinian. The result now follows by [10] (Theorem 1.3).

(2) By [3] (Corollary 2.1.6), there exists an injective resolution $\mathbf{I}$ of $\Gamma_{J_0R}(N)$ at which each term is a J_0R-torsion R-module. Let I^i be its ith term. Hence there exists a family $(\mathfrak{p}_\lambda)$ of prime ideals of R such that $J_0R \subseteq \mathfrak{p}_\lambda$ for each λ and

$$I^i = \oplus_\lambda E(R/\mathfrak{p}_\lambda)^{\mu_i},$$

where $E_R(-)$ stands for the injective hull and $\mu_i = \mu_i(\mathfrak{p}_\lambda, \Gamma_{J_0R}(N))$ is the ith Bass number of $\Gamma_{J_0R}(N)$ with respect to $\mathfrak{p}_\lambda$. We conclude that for each λ there exists $0 \le j \le t$ such that $\mathfrak{m}_0^{(j)} \subseteq \mathfrak{p}_\lambda$ and $\Gamma_{R_+}(E(R/\mathfrak{p}))$ would be $E(R/\mathfrak{m}_0^{(j)} + R_+)$ if $R_+ \subseteq \mathfrak{p}_\lambda$ and it is zero if $R_+ \nsubseteq \mathfrak{p}_\lambda$. Therefore, since $E(R/\mathfrak{m}_0^{(j)} + R_+)$ is an Artinian R-module, the module $\operatorname{Hom}_R(M, I^i))$ which is μ_i copies of $\operatorname{Hom}_R(M, E(R/\mathfrak{m}_0^{(j)} + R_+)$ will be Artinian. Now by (2.1) we see that $H^i_{R_+}(M, \Gamma_{J_0R}(N)) = \operatorname{Ext}^i_R(M, \Gamma_{R_+}(N))$ as a subquotient of an Artinian module is Artinian.

(3) When $\dim(R_0) = 0$, by Theorem 3.1(i), $H^i_{R_+}(M,N)$ is Artinian and the claim holds in this case. So, assume that $\dim(R_0) = 1$. By the proof of [9] (Theorem 13.6), there exists $a_0 \in J_0$ such that $\sqrt{a_0R_0} = J_0$. Thus, by using Theorem 3.1(iii),(iv), there exists an exact sequence

$$H^{i-1}_{R_+}(M,N) \xrightarrow{f^{i-1}_{a_0}} H^{i-1}_{R_+}(M,N)_{a_0} \to H^i_{(R_+,a_0)}(M,N) \to H^i_{R_+}(M,N) \xrightarrow{f^i_{a_0}} H^i_{R_+}(M,N)_{a_0}$$

of graded generalized local cohomology modules at which $f^{i-1}_{a_0}$ and $f^i_{a_0}$ are natural homomorphisms. By [3] (Corollary 2.2.18), we have

$$\operatorname{Coker}(f^{i-1}_{a_0}) \cong H^1_{a_0R}(H^{i-1}_{R_+}(M,N)) = H^1_{J_0R}(H^{i-1}_{R_+}(M,N))$$

and

$$\operatorname{Ker}(f^i_{a_0}) = \Gamma_{a_0R}(H^i_{R_+}(M,N)) = \Gamma_{J_0R}(H^i_{R_+}(M,N)),$$

which gives the short exact sequence

$$0 \to H^1_{J_0R}(H^{i-1}_{R_+}(M,N)) \to H^i_{(R_+,a_0)}(M,N) \to \Gamma_{J_0R}(H^i_{R_+}(M,N)) \to 0.$$

Now, by Theorem 3.1(ii), we get $H^i_{(a_0R,R_+)}(M,N) = H^i_{J_0R+R_+}(M,N)$ and $H^i_{(a_0R,R_+)}(M,N)$ is Artinian by Theorem 3.1(i). This proves the claim if i runs through $\mathbb{N}_0$. Finally, the R-module $(0 :_{H^i_{R_+}(M,N)} J_0)$ as a submodule of the Artinian module $\Gamma_{J_0R}(H^i_{R_+}(M,N))$ is Artinian.

(4) Consider the short exact sequence

$$0 \to \Gamma_{J_0R}(N) \to N \to N/\Gamma_{J_0R}(N) \to 0$$

to obtain the exact sequence

$$H^i_{R_+}(M, \Gamma_{J_0R}(N)) \xrightarrow{u_i} H^i_{R_+}(M,N) \to H^i_{R_+}(M, N/\Gamma_{J_0R}(N)) \xrightarrow{\Delta_i} H^{i+1}_{R_+}(M, \Gamma_{J_0R}(N))$$

of generalized local cohomology modules. By part (2), the left- and right-hand sides of this long exact sequence are Artinian for each $i \ge 0$. Hence, for each $i \ge 0$, we get the exact sequence

$$0 \to U_i \to H^i_{R_+}(M,N) \to H^i_{R_+}(M, N/\Gamma_{J_0R}(N)) \xrightarrow{\Delta_i} V_i \to 0, \qquad (3.1)$$

where $U_i = \ker(u_i)$ and $V_i = \mathrm{Im}(\Delta_i)$ are Artinian. Now, by Theorem 2.1,

$$H^i_{R_+}(M, N/\Gamma_{J_0R}(N)) = 0$$

for all $i > p+s$ and, hence, $H^i_{R_+}(M,N) \cong U_i$ is Artinian for all $i > p+s$.

For tameness at $p+s$, using the exact sequence (3.1), we get the exact sequence

$$0 \to K = \ker(\Delta_{p+s}) \to H^{p+s}_{R_+}(M, N/\Gamma_{J_0R}(N)) \to V_{p+s} \to 0.$$

This gives us the exact sequence

$$\mathrm{Tor}^R_1(R/J_0R, V_{p+s}) \to R/J_0R \otimes_R K \to R/J_0R \otimes_R H^{p+s}_{R_+}(M, N/\Gamma_{J_0R}(N))$$

of graded R-modules. Since V_{p+s} is Artinian, the left-hand side module in this exact sequence is Artinian, while

$$R/J_0R \otimes_R H^{p+s}_{R_+}(M, N/\Gamma_{J_0R}(N)) \cong R_0/J_0 \otimes_{R_0} H^{p+s}_{R_+}(M, N/\Gamma_{J_0R}(N))$$

is Artinian by (1). So $R/J_0R \otimes_R K$ is Artinian. Now, from the short exact sequence

$$0 \to U_{p+s} \to H^{p+s}_{R_+}(M,N) \to K \to 0$$

we obtain the exact sequence

$$R/J_0R \otimes_R U_{p+s} \to R/J_0R \otimes H^{p+s}_{R_+}(M,N) \to R/J_0R \otimes_R K$$

at which the left and right most modules are Artinian. So $R/J_0R \otimes H^{p+s}_{R_+}(M,N)$ as an R-Artinian module is tame and, hence, $H^{p+s}_{R_+}(M,N)$ is tame.

(5) Let $\check{N} = N/\Gamma_{J_0R}(N)$. The short exact sequence

$$0 \to \Gamma_{J_0R}(N) \to N \to \check{N} \to 0,$$

gives rise to the exact sequence

$$H^i_{R_+}(M, \Gamma_{J_0R}(N)) \xrightarrow{u_i} H^i_{R_+}(M,N) \xrightarrow{v_i} H^i_{R_+}(M, \check{N}) \xrightarrow{\Delta_i} H^{i+1}_{R_+}(M, \Gamma_{J_0R}(N)),$$

which in turn gives the following two exact sequences:

$$H^i_{R_+}(M, \Gamma_{J_0R}(N)) \xrightarrow{u_i} H^i_{R_+}(M,N) \to H^i_{R_+}(M,N)/\mathrm{Im}(u_i) \to 0 \qquad (3.2)$$

and

$$0 \to H^i_{R_+}(M,N)/\mathrm{Im}(u_i) \to H^i_{R_+}(M, \hat{N}) \to H^i_{R_+}(M, \check{N})/\mathrm{Im}(v_i) \to 0, \qquad (3.3)$$

and the monomorphism

$$0 \to H^i_{R_+}(M, \check{N})/\mathrm{Im}(v_i) \xrightarrow{\bar{\Delta}_i} H^{i+1}_{R_+}(M, \Gamma_{J_0R}(N)).$$

Set $Y := H^i_{R_+}(M, \check{N})$, $U := H^i_{R_+}(M, N)/\mathrm{Im}(u_i)$ and $V := H^i_{R_+}(M, \check{N})/\mathrm{Im}(v_i)$. We note that, by our assumption, $R_0/J_0 \otimes_{R_0} Y$ is Artinian. Also, as a submodule of $H^{i+1}_{R_+}(M, \Gamma_{J_0R}(N))$ the R-module V and, hence, $\mathrm{Tor}^{R_0}_1(R_0/J_0, V)$ is Artinian. Therefore, the exact sequence

$$\mathrm{Tor}^{R_0}_1(R_0/J_0, V) \to R_0 \otimes_{R_0} U \to R_0/J_0 \otimes_{R_0} Y,$$

which we obtain from (3.3), gives that the R-module $R_0/J_0 \otimes_{R_0} U$ is Artinian. On the other hand, from (3.2) we deduce the exact sequence

$$R_0/J_0 \otimes_{R_0} H^i_{R_+}(M, \Gamma_{J_0R}(N)) \to R_0/J_0 \otimes_{R_0} H^i_{R_+}(M, N) \to R_0/J_0 \otimes_{R_0} U.$$

Now, using part (2) the result follows.

Theorem 3.2 is proved.

In the next theorem our aim is to improve [14] (Theorem 2.8). The proof is almost the same, but we present its proof for the reader's convenience. To do so, we need the following notation. Let R_0 be a semilocal ring and let X be an R-module. We put

$$\mathrm{cd}(X) = \sup\{\dim_R(X/\mathfrak{n}_0X) | \mathfrak{n}_0 \in \mathrm{Max}(R_0)\}$$

and

$$\mathfrak{N}_0 = \prod\{\mathfrak{n}_0 | \mathfrak{n}_0 \in \mathrm{Max}(R_0) \text{ and } \dim_R(N/\mathfrak{n}_0N) = \mathrm{cd}(N)\},$$

where $\mathrm{Max}(R_0)$ is the set of all maximal ideals of R_0.

Theorem 3.3. *Let R_0 be a semilocal ring and M, N be two finitely generated graded R-modules with* $\mathrm{pd(M)} < \infty$. *Set $k = \mathrm{pd}(M) + \mathrm{cd}(N/\mathfrak{N}_0N + \Gamma_{\mathfrak{N}_0R}(N))$. Then, for each $i > k$, the R-module $H^i_{R_+}(M, N)$ is Artinian.*

Proof. We set $\overline{N} := N/\Gamma_{\mathfrak{N}_0R}(N)$ and

$$\mathcal{C} = \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) | \dim(N_{\mathfrak{p}_0}/\mathfrak{p}_0N_{\mathfrak{p}_0}) = \mathrm{cd}(N)\}.$$

Note that, if $\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \setminus \mathcal{C}$, then $\overline{N}_{\mathfrak{p}_0} \cong N_{\mathfrak{p}_0}$ and by Theorem 3.1(iv), for each $i \geq 0$, we have the isomorphism

$$H^i_{R_+}(M, N)_{\mathfrak{p}_0} \cong H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}, \overline{N}_{\mathfrak{p}_0}), \tag{3.4}$$

of graded $R_{\mathfrak{p}_0}$-modules. Since, by Theorem 2.1, the right-hand side of (3.4) is zero for all $i >$ $> \mathrm{pd}(M_{\mathfrak{p}_0}) + \dim(\overline{N}_{\mathfrak{p}_0}/J_0\overline{N}_{\mathfrak{p}_0}) =: \ell$ and $\ell \leq k$, we see that $\mathrm{Supp}_{R_0}(H^i_{R_+}(M, N)) \subseteq \mathcal{C}$ for all $i > k$. Now let $\mathfrak{m}_0 \in \mathcal{C}$. Since $N_{\mathfrak{m}_0}/\Gamma_{J_0R_{\mathfrak{m}_0}}(N_{\mathfrak{m}_0}) \cong \overline{N}_{\mathfrak{m}_0}$, by applying Theorem 3.2(4) for the graded $R_{\mathfrak{m}_0}$-modules $M_{\mathfrak{m}_0}$ and $N_{\mathfrak{m}_0}$ we conclude that $H^i_{R_+}(M, N)_{\mathfrak{m}_0} \cong H^i_{(R_{\mathfrak{m}_0})_+}(M_{\mathfrak{m}_0}, N_{\mathfrak{m}_0})$ is Artinian for $i > k$. Since the set of maximal ideals in $\mathrm{Supp}_{R_0}(H^i_{R_+}(M, N))$ is finite, this gives that $H^i_{R_+}(M, N)$ is Artinian for $i > s$.

Theorem 3.3 is proved.

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