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SOME REFINEMENTS OF NUMERICAL RADIUS INEQUALITIES

ДЕЯКІ УТОЧНЕННЯ НЕРІВНОСТЕЙ ДЛЯ ЧИСЛОВИХ РАДІУСІВ

In this paper, we give some refinements for the second inequality in $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$, where $A \in B(H)$. In particular, if A is hyponormal by refining the Young inequality with the Kantorovich constant $K(\cdot,\cdot)$, we show that $w(A) \leq \frac{1}{2\inf_{\|x\|=1}\zeta(x)}\||A|+|A^*|\| \leq \frac{1}{2}\||A|+|A^*|\|$, where $\zeta(x) = K\left(\frac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle},2\right)^r$, $r=\min\{\lambda,1-\lambda\}$ and $0 \leq \lambda \leq 1$. We also give a reverse for the classical numerical radius power inequality $w(A^n) \leq w^n(A)$ for any operator $A \in B(H)$ in the case when $n=2$.

Запропоновано деякі уточнення другої нерівності у $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$, де $A \in B(H)$. Зокрема, якщо A є гіпонормальним, то за допомогою нерівності Юнга з константою Канторовича $K(\cdot,\cdot)$ доведено, що $w(A) \leq \frac{1}{2\inf_{\|x\|=1}\zeta(x)}\||A|+|A^*|\| \leq \frac{1}{2}\||A|+|A^*|\|$, де $\zeta(x) = K\left(\frac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle},2\right)^r$, $r=\min\{\lambda,1-\lambda\}$ і $0 \leq \lambda \leq 1$. Також доведено нерівність для числових радіусів, що є оберненою до класичної степеневої нерівності $w(A^n) \leq w^n(A)$ для будь-якого оператора $A \in B(H)$ у випадку $n=2$.

1. Introduction. Suppose that $(H,\langle\cdot,\cdot\rangle)$ is a complex Hilbert space and $B(H)$ denotes the C^*-algebra of all bounded linear operators on H. For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well-known that $w(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.1}$$

An important inequality for $w(A)$ is the power inequality stating that

$$w(A^n) \leq w^n(A) \tag{1.2}$$

for each $n \in \mathbb{N}$. Many authors have investigated several inequalities involving numerical radius inequalities (see, e.g., [1, 5, 6, 8, 13, 14]). If $x,\ y \in H$ are arbitrary, then the angle between x and y is defined by

$$\cos\phi_{x,y} = \frac{\mathrm{Re}\langle x,y\rangle}{\|x\|\|y\|}$$

or by

$$\cos\psi_{x,y} = \frac{|\langle x,y\rangle|}{\|x\|\|y\|}.$$

The following inequality for angles between two vectors was obtained by Krein [11]

$$\phi_{x,z} \leq \phi_{x,y} + \phi_{y,z} \tag{1.3}$$

for any nonzero elements $x,\ y,\ z \in H$. By using the representation

$$\psi_{x,y} = \inf_{\lambda,\mu\in\mathbb{C}-\{0\}} \phi_{\lambda x,\mu y} = \inf_{\lambda\in\mathbb{C}-\{0\}} \phi_{\lambda x,y} = \inf_{\mu\in\mathbb{C}-\{0\}} \phi_{x,\mu y}$$

and inequality (1.3), he showed that the following triangle inequality is valid:

$$\psi_{x,y} \le \psi_{x,z} + \psi_{y,z} \tag{1.4}$$

for any nonzero elements $x, y, z \in H$.

In Section 2, we first introduce some new refinements of numerical radius inequality (1.1) by applying the Krein – Lin triangle inequality (1.3) and obtain a reverse of inequality (1.2) in the case when $n = 2$. In Section 3, we obtain some refinements of inequality (1.1) by applying a refinement of the Young inequality.

2. Some refinements of inequality (1.1) by Krein – Lin triangle inequality. In order to achieve our goals, we need the following lemmas. The first lemma is a simple consequence of the classical Jensen and Young inequalities.

Lemma 2.1 ([12], Lemma 2.1)**.** *Let* $a, b \ge 0$ *and* $0 \le \lambda \le 1$. *Then*

$$a^{\lambda} b^{1-\lambda} \le \lambda a + (1-\lambda) b \le [\lambda a^r + (1-\lambda) b^r]^{\frac{1}{r}}$$

for any $r \ge 1$.

The second lemma is a simple consequence of the classical Jensen inequality for convex function $f(t) = t^r$, where $r \ge 1$.

Lemma 2.2. *If* a *and* b *are nonnegative real numbers, then*

$$(a+b)^r \le 2^{r-1}(a^r + b^r)$$

for any $r \ge 1$.

Lemma 2.3 ([4], Lemma 2.4)**.** *Suppose that* $x, y \in H$ *with* $\|y\| = 1$. *Then*

$$\|x\|^2 - |\langle x, y\rangle|^2 = \inf_{\lambda\in\mathbb{C}} \|x - \lambda y\|^2.$$

The following lemma is known as a generalized mixed Schwarz inequality.

Lemma 2.4 ([12], Lemma 2.3)**.** *Let* $A \in B(H)$ *and* $x,\ y \in H$ *be two vectors.*

(i) *If* $0 \le \lambda \le 1$, *then*

$$|\langle Ax, y\rangle|^2 \le \langle |A|^{2\lambda} x, x\rangle \langle |A^*|^{2(1-\lambda)} y, y\rangle.$$

(ii) *If* f *and* g *are nonnegative continuous functions on* $[0,\infty)$ *satisfying* $f(t)g(t) = t$, *then*

$$|\langle Ax, y\rangle| \le \|f(|A|)x\| \|g(|A^*|)y\|.$$

In the next result, we use some ideas of [3].

Theorem 2.1. *Let* $A \in B(H)$ *and* $f,\ g$ *be nonnegative continuous functions on* $[0,\infty)$ *satisfying* $f(t)g(t) = t$. *Then, for* $r \ge 1$,

$$w^{2r}(A) \le \frac{1}{2^r}\left(\|f^2(|A^2|) + g^2(|(A^2)^*|)\|^r + 2^r \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|^{2r} \right). \tag{2.1}$$

Proof. By (1.4), we get the inequality (9) of [2] as follows:

$$\frac{|\langle x,z\rangle|}{\|x\|\|z\|}\frac{|\langle y,z\rangle|}{\|y\|\|z\|}\le\frac{|\langle x,y\rangle|}{\|x\|\|y\|}+\sqrt{1-\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}}\sqrt{1-\frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}} \tag{2.2}$$

for any $x,\ y,\ z\in H\setminus\{0\}$.

If we multiply (2.2) by $\|x\|\|y\|\|z\|^2$, then we deduce

$$|\langle x,z\rangle||\langle y,z\rangle|\le|\langle x,y\rangle|\|z\|^2+\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2}. \tag{2.3}$$

Applying Lemma 2.3 for any $x,y,z\in H$ with $\|z\|=1$, we obtain

$$|\langle x,z\rangle||\langle y,z\rangle|\le|\langle x,y\rangle|+\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|\inf_{\mu\in\mathbb{C}}\|y-\mu z\|. \tag{2.4}$$

Put $x=Az, y=A^*z$ in (2.4) to get

$$|\langle Az,z\rangle|^2\le|\langle A^2z,z\rangle|+\inf_{\lambda\in\mathbb{C}}\|Az-\lambda z\|\inf_{\mu\in\mathbb{C}}\|A^*z-\mu z\|\le$$

$$\le|\langle A^2z,z\rangle|+\|Az-\lambda z\|\|A^*z-\mu z\| \tag{2.5}$$

for any $z\in H$ with $\|z\|=1$ and $\lambda,\mu\in\mathbb{C}$.

On the other hand, by applying Lemma 2.4 and the AM-GM inequality, we have

$$|\langle A^2z,z\rangle|\le\|f(|A^2|)z\|\|g(|(A^2)^*|)z\|=$$

$$=\sqrt{\langle f^2(|A^2|)z,z\rangle\langle g^2(|(A^2)^*|)z,z\rangle}\le$$

$$\le\frac{1}{2}\langle(f^2(|A^2|)+g^2(|(A^2)^*|))z,z\rangle. \tag{2.6}$$

Applying again the AM-GM inequality, we get

$$\|Az-\lambda z\|\|A^*z-\mu z\|\le\frac{\|Az-\lambda z\|^2+\|A^*z-\mu z\|^2}{2}. \tag{2.7}$$

By combining inequalities (2.5), (2.6) and (2.7), we obtain

$$|\langle Az,z\rangle|^2\le\frac{1}{2}\left(\langle(f^2(|A^2|)+g^2(|(A^2)^*|))z,z\rangle+\|Az-\lambda z\|^2+\|A^*z-\mu z\|^2\right)\le$$

$$\le\frac{1}{2}\left(\langle(f^2(|A^2|)+g^2(|(A^2)^*|))z,z\rangle^r+(\|Az-\lambda z\|^2+\|A^*z-\mu z\|^2)^r\right)^{\frac{1}{r}}\quad\text{(by Lemma 2.1)}\le$$

$$\le\frac{1}{2}\left(\langle(f^2(|A^2|)+g^2(|(A^2)^*|))z,z\rangle^r+2^{r-1}(\|Az-\lambda z\|^{2r}+\|A^*z-\mu z\|^{2r})\right)^{\frac{1}{r}}\ \text{(by Lemma 2.2)}.$$

Hence

$$|\langle Az,z\rangle|^{2r}\le\frac{1}{2^r}\left(\langle(f^2(|A^2|)+g^2(|(A^2)^*|))z,z\rangle^r+2^{r-1}(\|Az-\lambda z\|^{2r}+\|A^*z-\mu z\|^{2r})\right).$$

By taking the supremum over $z\in H$ with $\|z\|=1$, we deduce

$$w^{2r}(A) \leq \frac{1}{2^r}\left(\|f^2(|A^2|)+g^2(|(A^2)^*|)\|^r + 2^{r-1}(\|A-\lambda I\|^{2r}+\|A^*-\mu I\|^{2r})\right)$$

for any $\lambda, \mu \in \mathbb{C}$.

Finally, taking the infimum over $\lambda,\ \mu \in \mathbb{C}$ in the inequality above and utilizing

$$\inf_{\mu\in\mathbb{C}} \|A^*-\mu I\| = \inf_{\mu\in\mathbb{C}} \|A-\overline{\mu} I\| = \inf_{\lambda\in\mathbb{C}} \|A-\lambda I\|$$

we obtain the result (2.1).

Theorem 2.1 is proved.

***Remark* 2.1.** In Theorem 2.1 if we choose $r=1,\ f(t)=g(t)=\sqrt{t},$ we get

$$w^2(A) \leq \frac{1}{2}\left(\||A^2|+|(A^2)^*|\| + 2\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\|^2\right).$$

Now, suppose that $s>0$ such that $s \leq \sqrt{\|A\|^2 - \frac{1}{2}\||A^2|+|(A^2)^*|\|}$, if there is $\lambda_0 \in \mathbb{C}$ in which $\|A-\lambda_0 I\| \leq s$, then $w(A) \leq \sqrt{\frac{1}{2}\||A^2|+|(A^2)^*|\| + s^2} \leq \|A\|$, that is an improvement of inequality (1.1) for nonnormal operators.

Recall that if $A \in M_2(\mathbb{R})$, then $\|A\| = \max_{1\leq i\leq n}\sigma_i$, where $\sigma_i' s$ are the square root of eigenvalues of A^*A, which are called the singular values of A, and $w(A)$ for matrix of the form $A=\begin{bmatrix} a_1 & b \\ 0 & a_2\end{bmatrix}$ or $A=\begin{bmatrix} a_1 & 0 \\ b & a_2\end{bmatrix}$ is defined by

$$w(A)=\frac{1}{2}|a_1+a_2|+\frac{1}{2}\sqrt{|a_1-a_2|^2+|b|^2},$$

where $a_1,\ a_2,\ b\in\mathbb{R}$.

***Example* 2.1.** By taking $A=\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1\end{bmatrix}$ and $\lambda_0=\frac{1}{2}$ in Remark 2.1, we have $w^2(A)\simeq 1.5625$, $\|A\|^2 \simeq 3.2822$, $\|A-\lambda_0 I\| \simeq 0.5201$ and $\frac{1}{2}\||A^2|+|(A^2)^*|\| \simeq 1.5652$. If $s^2 \leq \|A\|^2 - \frac{1}{2}\||A^2|+ +|(A^*)^2|\| \simeq 1.7170$, then $s \leq 1.3103$. Hence, inequality $w(A) \leq \sqrt{\frac{1}{2}\||A^2|+|(A^2)^*|\| + s^2} \leq \|A\|$ provides an improvement of inequality (1.1).

***Remark* 2.2.** If there exists $\lambda_0\in\mathbb{C}$ in which $\|A-\lambda_0 I\| \leq s$, then by putting $\lambda=\mu=\lambda_0$ and by taking supremum over $z\in H$ with $\|z\|=1$ in (2.5), we deduce

$$w^2(A)-w(A^2) \leq \|A-\lambda_0 I\|\|A^*-\lambda_0 I\|.$$

Therefore

$$w^2(A)-w(A^2)\leq s^2.$$

Now, if $\|A-\lambda_0 I\| \leq s \leq \sqrt{\|A\|^2-w(A^2)}$, we have $w(A)\leq\sqrt{w(A^2)+s^2}\leq\|A\|$, that is an improvement of inequality (1.1).

***Example* 2.2.** By taking $A=\begin{bmatrix} 2 & -1 \\ 0 & 3\end{bmatrix}$ and $\lambda_0=2.5$ in Remark 2.2, we have $w(A^2)\simeq \simeq 6.4142$, $\|A\|^2\simeq 10.6054$, $\|A-\lambda_0 I\|\simeq 0.955$ for $s\leq\sqrt{\|A\|^2-w(A^2)}\simeq 2.0472$. Hence, inequality $w(A)\leq\sqrt{w(A^2)+s^2}\leq\|A\|$ provides an improvement of inequality (1.1).

Recall that the vector $x \in H$ is orthogonal to $y \in H$ (denote by $x \perp y$), if $\langle x, y\rangle = 0$. Now, an argument similar to the proof of Theorem 2.1 with the aid of Lemmas 2.1 and 2.3 gives the following proposition.

Proposition 2.1. *Let $x,\ y,\ z \in H$ with $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$, $\quad a, b > 0$, and $r \geq 1$ such that*

$$\|x - \lambda z\| \leq a, \quad \|y - \mu z\| \leq b.$$

Then

$$(|\langle x, z\rangle||\langle y, z\rangle| - |\langle x, y\rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2}. \tag{2.8}$$

In particular, if $x \perp y$, then

$$(|\langle x, z\rangle||\langle y, z\rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2} \tag{2.9}$$

for any $r \geq 1$.

Proof. Since z is a unit vector, from (2.3) we have

$$|\langle x, z\rangle||\langle y, z\rangle| - |\langle x, y\rangle| \leq \sqrt{\|x\|^2 - |\langle x, z\rangle|^2}\sqrt{\|y\|^2 - |\langle y, z\rangle|^2} \leq$$

$$\leq \frac{1}{2}(\|x\|^2 - |\langle x, z\rangle|^2 + \|y\|^2 - |\langle y, z\rangle|^2) \quad \text{(by AM-GM inequality)} =$$

$$= \frac{1}{2}\left(\inf_{\lambda\in\mathbb{C}} \|x - \lambda z\|^2 + \inf_{\mu\in\mathbb{C}} \|y - \mu z\|^2\right) \quad \text{(by Lemma 2.3)} \leq$$

$$\leq \frac{1}{2}\left(\|x - \lambda z\|^2 + \|y - \mu z\|^2\right) \leq$$

$$\leq \frac{a^2 + b^2}{2} \leq \left(\frac{a^{2r} + b^{2r}}{2}\right)^{\frac{1}{r}} \quad \text{(by Lemma 2.1).}$$

Hence

$$(|\langle x, z\rangle||\langle y, z\rangle| - |\langle x, y\rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2}.$$

Proposition 2.1 is proved.

***Corollary* 2.1.** *Let $A \in B(H)$ and B be a nonzero self-adjoin element in $B(H)$, under assumptions of Proposition* 2.1, *if we choose $x = Az$ and $y = Bz$ with $\|z\| = 1$ in* (2.9) *yields*

$$(|\langle Az, z\rangle||\langle Bz, z\rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2}.$$

By taking supremum over $z \in H$ with $\|z\| = 1$, we get

$$w^r(A) \leq \frac{a^{2r} + b^{2r}}{2}\|B\|^{-r}$$

provided $\|A - \lambda I\| \leq a,\ \|B - \mu I\| \leq b$ and for any $r \geq 1$ and $a,\ b > 0$.

Proposition 2.1 induces several inequalities as special cases, but here we only focus on the case $r = 1$, i.e.,

$$|\langle x, z\rangle||\langle y, z\rangle| \leq \frac{a^2 + b^2}{2} + |\langle x, y\rangle|, \tag{2.10}$$

whenever $\|x - \lambda z\| \leq a, \quad \|y - \mu z\| \leq b$ with $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$.

***Remark* 2.3.** Suppose that the assumptions of Proposition 2.1 are still valid.

As an application of inequality (2.10) the following reverse of inequality (1.2) for $n = 2$, i.e., an upper bound for $w^2(A) - w(A^2)$ can be obtained. In fact, by choosing $x = Az$ and $y = A^*z$ with $\|z\| = 1$ and taking supremum over $z \in H$ with $\|z\| = 1$, we get

$$w^2(A) - w(A^2) \leq \frac{a^2 + b^2}{2}$$

provided $\|A - \lambda I\| \leq a, \quad \|A^* - \mu I\| \leq b.$

By choosing $x = Az$ and $y = A^{-1}z$ with $\|z\| = 1$, in inequality (2.10) and taking supremum over $z \in H$ with $\|z\| = 1$, we have

$$K(A; z) - 1 \leq \frac{a^2 + b^2}{2}$$

provided $\|A - \lambda I\| \leq a, \quad \|A^{-1} - \mu I\| \leq b$, where $K(A; z) = \langle Az, z\rangle\langle A^{-1}z, z\rangle$ is the Kantorovich functional.

3. Some refinements of inequality (1.1) by using Young's inequality. In this section, we obtain some refinements of inequality (1.1) by applying refinements of the Young inequality. The next lemma is an additive refinement of the scalar Young inequality.

Lemma 3.1 ([9], Theorem 2.1)**.** *If* $a,\ b \geq 0$ *and* $0 \leq \lambda \leq 1$, *then*

$$a^\lambda b^{1-\lambda} + r(\sqrt{a} - \sqrt{b})^2 \leq \lambda a + (1 - \lambda)b,$$

where $r = \min\{\lambda, 1 - \lambda\}$.

The main result of this section reads as follows.

Theorem 3.1. *If* $A \in B(H), \quad r = \min\{\lambda, 1 - \lambda\}$, *where* $0 \leq \lambda \leq 1$, *then*

$$w(A) \leq \frac{1 - 2r}{2}\||A| + |A^*|\| + 2r\|A\|.$$

Proof. Let $x \in H$ be a unit vector. Then we have

$$|\langle Ax, x\rangle| \leq \sqrt{\langle |A|x, x\rangle\langle |A^*|x, x\rangle} \quad \text{(by Lemma 2.4)} =$$

$$= \left(\langle |A|x, x\rangle^{1-\lambda}\langle |A^*|x, x\rangle^\lambda\right)^{\frac{1}{2}} \left(\langle |A^*|x, x\rangle^{1-\lambda}\langle |A|x, x\rangle^\lambda\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2}\left(\langle |A|x, x\rangle^{1-\lambda}\langle |A^*|x, x\rangle^\lambda + \langle |A^*|x, x\rangle^{1-\lambda}\langle |A|x, x\rangle^\lambda\right) \quad \text{(by AM-GM inequality)} \leq$$

$$\leq \frac{1}{2}\Big((1 - \lambda)\langle |A|x, x\rangle + \lambda\langle |A^*|x, x\rangle - r(\sqrt{\langle |A|x, x\rangle} - \sqrt{\langle |A^*|x, x\rangle})^2 +$$

$$+(1-\lambda)\langle |A^*|x,x\rangle + \lambda\langle |A|x,x\rangle - r\left(\sqrt{\langle |A|x,x\rangle} - \sqrt{\langle |A^*|x,x\rangle})^2\right) \quad \text{(by Lemma 3.1)} =$$

$$= \frac{1}{2}\left(\langle (|A|+|A^*|)x,x\rangle - 2r\langle (|A|+|A^*|)x,x\rangle) + 4r\sqrt{\langle |A|x,x\rangle\langle |A^*|x,x\rangle}\right),$$

so

$$|\langle Ax,x\rangle| + r\langle (|A|+|A^*|)x,x\rangle \le \frac{1}{2}(\langle (|A|+|A^*|)x,x\rangle + 4r\sqrt{\langle |A|x,x\rangle\langle |A^*|x,x\rangle}).$$

By taking supremum over $x \in H$ with $\|x\| = 1$, we deduce

$$w(A) \le \frac{1-2r}{2}\||A|+|A^*|\| + 2r\|A\|,$$

which is an improvement of inequality (1.1).

Theorem 3.1 is proved.

***Example* 3.1.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ be as in Theorem 3.1 and $r = 0.1$. Then by straightforward computation, we get $w(A) \simeq 2.2071$, $\quad \|A\| \simeq 2.2882$ and $\frac{1}{2}\||A|+|A^*|\| \simeq 2.2518$. Hence

$$w(A) \le \frac{1-2r}{2}\||A|+|A^*|\| + 2r\|A\| \le \|A\|,$$

provides an improvement of inequality (1.1). In fact, $2.2071 \le 2.2590 \le 2.2882$.

The following lemma is a multiplicative refinement of the Young inequality with the Kantorovich constant.

Lemma 3.2 ([7], Corollary 3)**.** *Let* $a,\ b > 0$. *Then*

$$(1-\lambda)a + \lambda b \ge k(h,2)^r a^{1-\lambda} b^\lambda,$$

where $0 \le \lambda \le 1$, $\ r = \min\{\lambda, 1-\lambda\}$, $\ h = \dfrac{b}{a}$ *such that* $K(h,2) = \dfrac{(h+1)^2}{4h}$ *for* $h > 0$, *which has properties* $K(h,2) = K\Big(\dfrac{1}{h}, 2\Big) \ge 1$ $(h > 0)$ *and* $K(h,2)$ *is increasing on* $[1,\infty)$ *and is decreasing on* $(0,1)$.

In [10], Kittaneh obtained the inequality

$$w(A) \le \frac{1}{2}\||A|+|A^*|\|. \tag{3.1}$$

In the following theorem, we improve inequality (3.1) for hyponormal operators. Before proceeding recall that the operator $A \in B(H)$ is said to be hyponormal if $A^*A - AA^* \ge 0$.

Theorem 3.2. *If* $A \in B(H)$ *is hyponormal,* $r = \min\{\lambda, 1-\lambda\}$, *where* $0 \le \lambda \le 1$, *then*

$$w(A) \le \frac{1}{\inf_{\|x\|=1}\zeta(x)}\frac{\||A|+|A^*|\|}{2},$$

where $\zeta(x) = K\left(\dfrac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle}, 2\right)^r$ *is a refinement of inequality* (1.1).

Proof. Let $x \in H$ be a unit vector.

$$|\langle Ax, x\rangle| \leq \sqrt{\langle |A|x, x\rangle \langle |A^*|x, x\rangle} \quad \text{(by Lemma 2.4)} =$$

$$= \left(\langle |A^*|x, x\rangle^{1-\lambda} \langle |A|x, x\rangle^{\lambda}\right)^{\frac{1}{2}} \left(\langle |A|x, x\rangle^{1-\lambda} \langle |A^*|x, x\rangle^{\lambda}\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2}\left(\left(\langle |A^*|x, x\rangle^{1-\lambda}\langle |A|x, x\rangle^{\lambda}\right) + \left(\langle |A|x, x\rangle^{1-\lambda}\langle |A^*|x, x\rangle^{\lambda}\right)\right) \leq \text{(by AM-GM inequality)} \leq$$

$$\leq \frac{1}{2}\left(\frac{1}{K\left(\dfrac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}, 2\right)^r}\big((1-\lambda)\langle |A^*|x, x\rangle + \lambda\langle |A|x, x\rangle\big) + \right.$$

$$\left. + \frac{1}{K\left(\dfrac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}, 2\right)^r}\big((1-\lambda)\langle |A|x, x\rangle + \lambda\langle |A^*|x, x\rangle\big)\right) \quad \text{(by Lemma 3.2)} =$$

$$= \frac{1}{2}\left(\frac{1}{K\left(\dfrac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}, 2\right)^r}\big(\langle |A^*|x, x\rangle + \langle |A|x, x\rangle\big)\right).$$

Taking supremum over $x \in H$ with $\|x\| = 1$, we have

$$w(A) \leq \frac{1}{\inf\limits_{\|x\|=1} \zeta(x)} \frac{\||A| + |A^*|\|}{2},$$

where $\zeta(x) = K\left(\dfrac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}, 2\right)^r$.

Note that $2\langle |A|x, x\rangle\langle |A^*|x, x\rangle \leq \langle |A|x, x\rangle^2 + \langle |A^*|x, x\rangle^2$, so

$$\big(\langle |A|x, x\rangle + \langle |A^*|x, x\rangle\big)^2 \geq 4\langle |A|x, x\rangle\langle |A^*|x, x\rangle.$$

Hence

$$\frac{\big(\langle |A|x, x\rangle + \langle |A^*|x, x\rangle\big)^2}{4\langle |A|x, x\rangle\langle |A^*|x, x\rangle} \geq 1.$$

Therefore, $K\left(\dfrac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}, 2\right) \geq 1$.

Theorem 3.2 is proved.

References

1. M. Boumazgour, A. H. Nabwey, *A note concerning the numerical range of a basic elementary operator*, Ann. Funct. Anal., **7**, № 3, 434 – 441 (2016).

2. S. S. Dragomir, *A note on numerical radius and the Krein – Lin inequality*, RGMIA Res. Rep. Collect., **18**, Article 113 (2015).
3. S. S. Dragomir, *A note on new refinements and reverses of Young's inequality*, Transylv. J. Math. and Mech., **8**, № 1, 46 – 49 (2016).
4. S. S. Dragomir, *Some Grüss type inequalities in inner product spaces*, J. Inequal. Pure and Appl. Math., **4**, № 2, Article 42 (2003), 10 p.
5. S. S. Dragomir, *Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Tamkang J. Math., **39**, № 1, 1 – 7 (2008).
6. R. Golla, *On the numerical radius of a quaternionic normal operator*, Adv. Oper. Theory, **2**, № 1, 78 – 86 (2017).
7. M. Fuji, H. Zuo, G. Shi, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **4**, № 4, 551 – 556 (2011).
8. F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. and Appl., **361**, № 1, 262 – 269 (2010).
9. F. Kittaneh, Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Linear and Multilinear Algebra, **59**, 1031 – 1037 (2011).
10. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Stud. Math., **158**, № 1, 11 – 17 (2003).
11. M. G. Krein, *The angular localization of the spectrum of a multiplicative integral in Hilbert space* (in Russian), Funkcional. Anal. i Prilozhen., **3**, 89 – 90 (1969).
12. M. Satari, M. S. Moslehian, T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra and Appl., **470**, 216 – 227 (2015).
13. A. Sheikhhosseini, M. S. Moslehian, K. Shebrawi, *Inequalities for generalized Euclidean operator radius via Young's inequality*, J. Math. Anal. and Appl., **445**, № 2, 1516 – 1529 (2017).
14. A. Zamani, *Some lower bounds for the numerical radius of Hilbert space operators*, Adv. Oper. Theory, **2**, № 2, 98 – 107 (2017).

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