DOI: 10.37863/umzh.v73i8.603

UDC 517.5

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SOME NEW BOUNDS OF GAUSS – JACOBI AND HERMITE – HADAMARD TYPE INTEGRAL INEQUALITIES

НОВІ ГРАНИЦІ ДЛЯ ІНТЕГРАЛЬНИХ НЕРІВНОСТЕЙ ТИПУ ГАУССА – ЯКОБІ ТА ЕРМІТА – АДАМАРА

In this paper, authors discover two interesting identities regarding Gauss – Jacobi and Hermite – Hadamard type integral inequalities. By using the first lemma as an auxiliary result, some new bounds with respect to Gauss – Jacobi type integral inequalities are established. Also, using the second lemma, some new estimates with respect to Hermite – Hadamard type integral inequalities via general fractional integrals are obtained. It is pointed out that some new special cases can be deduced from main results. Some applications to special means for different positive real numbers and new error estimates for the trapezoidal are provided as well. These results give us the generalizations, refinement and significant improvements of the new and previous known results. The ideas and techniques of this paper may stimulate further research.

Знайдено дві цікаві тотожності для інтегральних нерівностей типу Гаусса – Якобі та Ерміта – Адамара. З використанням першої леми як допоміжного результату встановлено деякі нові границі інтегральних нерівностей типу Гаусса – Якобі. Далі, за допомогою другої леми та загальних дробових інтегралів отримано деякі нові границі інтегральних нерівностей типу Ерміта – Адамара. Зазначено, що з основних результатів можна отримати деякі нові випадки. Також запропоновано деякі застосування до спеціальних середніх для різних додатних дійсних чисел та нові оцінки похибок для методу трапеції. Ці результати є узагальненням, уточненням та значним покращенням нових та раніше відомих результатів. Ідеї та методи цієї статті мають стимулювати подальші дослідження.

1. Introduction. The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a_1, a_2]$ is denoted by $L[a_1, a_2]$.

The following inequality, named Hermite – Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a_1, a_2 \in I$ with $a_1 < a_2$. Then the following inequality holds:*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int\limits_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1.1}$$

This inequality (1.1) *is also known as trapezium inequality.*

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1.1) through various classes of convex functions interested readers are referred to [1 – 33, 35, 37, 38]

The Gauss – Jacobi type quadrature formula has the following:

$$\int\limits_{a_1}^{a_2} (x - a_1)^p (a_2 - x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1.2}$$

for certain $B_{m,k}, \gamma_k$ and rest $R^\star_m|f|$ (see [34]).

Recently in [20], Liu obtained several integral inequalities for the left-hand side of (1.2). Also in [28], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.2) via some kinds of convexity.

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.1. *For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by*

$$\Gamma_k(x) = \lim_{n\longrightarrow\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}.$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int\limits_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{1.3}$$

One can note that

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha).$$

For $k=1$, (1.3) gives integral representation of gamma function.

Definition 1.2 [24]. *Let $f \in L[a_1,a_2]$. Then k-fractional integrals of order α, $k>0$ with $a_1 \geq 0$ are defined as*

$$I^{\alpha,k}_{a_1^+}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_{a_1}^{x}(x-t)^{\frac{\alpha}{k}-1}f(t)dt, \qquad x>a_1,$$

and

$$I^{\alpha,k}_{a_2^-}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_{x}^{a_2}(t-x)^{\frac{\alpha}{k}-1}f(t)dt, \qquad a_2>x.$$

For $k=1$, k-fractional integrals give Riemann – Liouville integrals.

Definition 1.3 [36]. *A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta\colon S\times S \longrightarrow \mathbb{R}^n$, if $x+t\eta(y,x)\in S$ for every $x,y\in S$ and $t\in[0,1]$.*

The invex set S is also termed an η-connected set.

Definition 1.4. *Let $S\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta\colon S\times S\longrightarrow\mathbb{R}^n$. A function $f\colon S\longrightarrow[0,+\infty)$ is said to be preinvex with respect to η, if, for every x, $y\in S$ and $t\in[0,1]$,*

$$f\big(x+t\eta(y,x)\big) \leq (1-t)f(x)+tf(y).$$

Also, define a function $\varphi:[0,+\infty)\longrightarrow[0,+\infty)$ satisfying the following conditions:

$$\int\limits_0^1 \frac{\varphi(t)}{t}dt < +\infty, \tag{1.4}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \qquad \text{for} \qquad \frac{1}{2}\leq\frac{s}{r}\leq 2, \tag{1.5}$$

$$\frac{\varphi(r)}{r^2} \leq B\frac{\varphi(s)}{s^2} \qquad \text{for} \qquad s \leq r, \tag{1.6}$$

$$\left|\frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2}\right| \leq C|r-s|\frac{\varphi(r)}{r^2} \qquad \text{for} \qquad \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{1.7}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\dfrac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then φ satisfies (1.4) – (1.7) (see [31]). Therefore, we define the following left- and right-hand sided generalized fractional integral operators, respectively, as follows:

$${}_{a_1^+}I_{\varphi}f(x) = \int\limits_{a_1}^{x} \frac{\varphi(x-t)}{x-t} f(t)dt, \qquad x > a_1,$$

$${}_{a_2^-}I_{\varphi}f(x) = \int\limits_{x}^{a_2} \frac{\varphi(t-x)}{t-x} f(t)dt, \qquad x < a_2.$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann – Liouville fractional integral, k-Riemann – Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals etc. (see [30]).

Motivated by the above literatures, the main objective of this paper is to discover in Sections 2 and 3, two interesting identities and to established some new bounds regarding Gauss – Jacobi and Hermite – Hadamard type integral inequalities. By using in Section 2 the first lemma as an auxiliary result, some new bounds with respect to Gauss – Jacobi type integral inequalities will be given. Also, by using in Section 3 the second lemma, some new estimates with respect to Hermite – Hadamard type integral inequalities via general fractional integrals will be obtained. It is pointed out that some new special cases will be deduced from main results. In Section 4, some applications to special means for different positive real numbers and new error estimates for the trapezoidal will be given. These results will give us the generalizations, refinement and significant improvements of the new and previous known results. The ideas and techniques of this paper may stimulate further research.

2. Some new bounds of the quadrature formula of Gauss – Jacobi type. Throughout this study, for brevity, we define

$$\Lambda^*(t) = \int\limits_0^t \frac{\varphi(\eta(a_2, a_1)x)}{x} dx < +\infty, \qquad \eta(a_2, a_1) > 0.$$

For establishing some new bounds integral inequalities for Gauss – Jacobi type, we need the following lemma.

Lemma 2.1. *Let $P = [a_1, a_1 + \eta(a_2, a_1)] \subseteq \mathbb{R}$ be an open invex subset. Assume that $f\colon P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\eta\colon P \times P \longrightarrow \mathbb{R}$ for $\eta(a_2, a_1) > 0$. Then, for any fixed $p,\ q > 0$, we have*

$$\int\limits_{a_1}^{a_1+\eta(a_2,a_1)} \left[\Lambda^*\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)\right]^p \left[\Lambda^*\left(\frac{a_1+\eta(a_2,a_1)-x}{\eta(a_2,a_1)}\right)\right]^q f(x)dx =$$

$$= \eta(a_2,a_1) \int_0^1 \left[\Lambda^*(t)\right]^p \left[\Lambda^*(1-t)\right]^q f(a_1 + t\eta(a_2,a_1))dt. \tag{2.1}$$

We denote

$$T^{p,q}_{f,\Lambda^*}(a_1,a_2) = \eta(a_2,a_1) \int_0^1 \left[\Lambda^*(t)\right]^p \left[\Lambda^*(1-t)\right]^q f(a_1 + t\eta(a_2,a_1))dt. \tag{2.2}$$

Proof. By using (2.2) and changing the variable $x = a_1 + t\eta(a_2,a_1)$, we have

$$T^{p,q}_{f,\Lambda^*}(a_1,a_2) = \eta(a_2,a_1) \times$$

$$\times \int_{a_1}^{a_1+\eta(a_2,a_1)} \left[\Lambda^*\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)\right]^p \left[\Lambda^*\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)\right]^q f(x)\frac{dx}{\eta(a_2,a_1)} =$$

$$= \int_{a_1}^{a_1+\eta(a_2,a_1)} \left[\Lambda^*\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)\right]^p \left[\Lambda^*\left(\frac{a_1+\eta(a_2,a_1)-x}{\eta(a_2,a_1)}\right)\right]^q f(x)dx.$$

Lemma 2.1 is proved.

***Corollary* 2.1.** *Taking $\eta(a_2,a_1) = a_2 - a_1$ and $\varphi(x) = x$, in Lemma* 2.1, *we get the following identity*:

$$\int_{a_1}^{a_2} (x-a_1)^p (a_2-x)^q f(x)dx = (a_2-a_1)^{p+q+1} \int_0^1 t^p(1-t)^q f(a_1 + t(a_2-a_1))dt.$$

With the help of Lemma 2.1, we have the following results.

Theorem 2.1. *Let $P = [a_1, a_1 + \eta(a_2,a_1)] \subseteq \mathbb{R}$ be an open invex subset. Assume that $f\colon P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\eta\colon P \times P \longrightarrow \mathbb{R}$ for $\eta(a_2,a_1) > 0$. If $|f|^{\frac{k}{k-1}}$ is preinvex mapping on P for $k > 1$, then, for any fixed p, $q > 0$, we have*

$$\left|T^{p,q}_{f,\Lambda^*}(a_1,a_2)\right| \leq \eta(a_2,a_1) \sqrt[k]{A^{p,q}_{\Lambda^*}(k)} \left[\frac{|f(a_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where

$$A^{p,q}_{\Lambda^*}(k) = \int_0^1 \left[\Lambda^*(t)\right]^{kp} \left[\Lambda^*(1-t)\right]^{kq} dt.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is preinvex mapping on P, combining with Lemma 2.1, Hölder's inequality and properties of the modulus, we get

$$\left|T^{p,q}_{f,\Lambda^*}(a_1,a_2)\right| \leq \eta(a_2,a_1) \int_0^1 \left[\Lambda^*(t)\right]^p \left[\Lambda^*(1-t)\right]^q |f(a_1 + t\eta(a_2,a_1))|dt \leq$$

$$\leq \eta(a_2,a_1)\left[\int_0^1 \left[\Lambda^*(t)\right]^{kp}\left[\Lambda^*(1-t)\right]^{kq}dt\right]^{\frac{1}{k}}\left[\int_0^1 |f(a_1+t\eta(a_2,a_1))|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}} \leq$$

$$\leq \eta(a_2,a_1)\sqrt[k]{A^{p,q}_{\Lambda^*}(k)}\left[\int_0^1 \left((1-t)|f(a_1)|^{\frac{k}{k-1}}+t|f(a_2)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}} =$$

$$= \eta(a_2,a_1)\sqrt[k]{A^{p,q}_{\Lambda^*}(k)}\left[\frac{|f(a_1)|^{\frac{k}{k-1}}+|f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}}.$$

Theorem 2.1 is proved.

We point out some special cases of Theorem 2.1.

***Corollary* 2.2.** *Under the assumption of Theorem* 2.1 *with* $\varphi(t)=t$, *we get*

$$\left|T^{p,q}_{f,\Lambda_1^*}(a_1,a_2)\right| \leq \eta^{p+q+1}(a_2,a_1)\sqrt[k]{\beta(kp+1,kq+1)}\left[\frac{|f(a_1)|^{\frac{k}{k-1}}+|f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where $\Lambda_1^*=\eta(a_2,a_1)t$.

***Corollary* 2.3.** *Under the assumption of Theorem* 2.1 *with* $\varphi(t)=\dfrac{t^\alpha}{\Gamma(\alpha)}$, *we have*

$$\left|T^{p,q}_{f,\Lambda_2^*}(a_1,a_2)\right| \leq \frac{\eta^{\alpha(p+q)+1}(a_2,a_1)}{\Gamma^{p+q}(\alpha+1)}\sqrt[k]{\beta(\alpha kp+1,\alpha kq+1)}\times$$

$$\times\left[\frac{|f(a_1)|^{\frac{k}{k-1}}+|f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where $\Lambda_2^*=\dfrac{\eta^\alpha(a_2,a_1)}{\Gamma(\alpha+1)}t^\alpha$.

***Corollary* 2.4.** *Under the assumption of Theorem* 2.1 *with* $\varphi(t)=\dfrac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$, *we obtain*

$$\left|T^{p,q}_{f,\Lambda_3^*}(a_1,a_2)\right| \leq \frac{\eta^{\frac{\alpha}{k_1}(p+q)+1}(a_2,a_1)}{\left[k_1\Gamma_{k_1}(\alpha+k_1)\right]^{p+q}}\sqrt[k]{\beta\left(\frac{\alpha kp}{k_1}+1,\frac{\alpha kq}{k_1}+1\right)}\times$$

$$\times\left[\frac{|f(a_1)|^{\frac{k}{k-1}}+|f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where $\Lambda_3^*=\dfrac{\eta^{\frac{\alpha}{k_1}}(a_2,a_1)}{k_1\Gamma_{k_1}(\alpha+k_1)}t^{\frac{\alpha}{k_1}}$.

***Corollary* 2.5.** *Under the assumption of Theorem* 2.1 *with* $\varphi(t) = t(a_1 + \eta(a_2, a_1) - t)^{\alpha-1}$ *and* $f(x)$ *is symmetric to* $x = a_1 + \dfrac{\eta(a_2, a_1)}{2}$, *we get*

$$\left|T^{p,q}_{f,\Lambda_4^*}(a_1, a_2)\right| \le \frac{\eta^{\frac{k-1}{k}(p+q)+1}(a_2, a_1)}{\alpha^{p+q}} \sqrt[k]{C^{p,q}(\alpha, k)} \left[\frac{|f(a_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where

$$C^{p,q}(\alpha, k) =$$

$$= \int\limits_{a_1}^{a_1+\eta(a_2,a_1)} \left[(a_1 + \eta(a_2, a_1))^\alpha - t^\alpha\right]^{kp} \left[(a_1 + \eta(a_2, a_1))^\alpha - (2a_1 + \eta(a_2, a_1) - t)^\alpha\right]^{kq} dt$$

and

$$\Lambda_4^* = \frac{(a_1 + \eta(a_2, a_1))^\alpha - (a_1 + (1-t)\eta(a_2, a_1))^\alpha}{\alpha}.$$

Theorem 2.2. *Let* $P = [a_1, a_1 + \eta(a_2, a_1)] \subseteq \mathbb{R}$ *be an open invex subset. Assume that* $f : P \longrightarrow \mathbb{R}$ *be a continuous mapping on* P° *with respect to* $\eta : P \times P \longrightarrow \mathbb{R}$ *for* $\eta(a_2, a_1) > 0$. *If* $|f|^l$ *is preinvex mapping on* P *for* $l \ge 1$, *then, for any fixed* p, $q > 0$, *we have*

$$\left|T^{p,q}_{f,\Lambda^*}(a_1, a_2)\right| \le \eta(a_2, a_1)\left[A^{p,q}_{\Lambda^*}(1)\right]^{\frac{l-1}{l}} \sqrt[l]{B^{p,q}_{\Lambda^*}|f(a_1)|^l + B^{q,p}_{\Lambda^*}|f(a_2)|^l},$$

where

$$B^{p,q}_{\Lambda^*} = \int\limits_0^1 (1-t)\left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q dt$$

and $A^{p,q}_{\Lambda^*}(1)$ *is defined as in Theorem* 2.1.

Proof. Since $|f|^l$ is preinvex mapping on P, combining with Lemma 2.1, the well-known power mean inequality and properties of the modulus, we get

$$\left|T^{p,q}_{f,\Lambda^*}(a_1, a_2)\right| \le \eta(a_2, a_1) \int\limits_0^1 \left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q |f(a_1 + t\eta(a_2, a_1))| dt \le$$

$$\le \eta(a_2, a_1)\left[\int\limits_0^1 \left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q dt\right]^{\frac{l-1}{l}} \left[\int\limits_0^1 \left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q |f(a_1 + t\eta(a_2, a_1))|^l dt\right]^{\frac{1}{l}} \le$$

$$\le \eta(a_2, a_1)\left[A^{p,q}_{\Lambda^*}(1)\right]^{\frac{l-1}{l}} \left[\int\limits_0^1 \left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q \Big((1-t)|f(a_1)|^l + t|f(a_2)|^l\Big) dt\right]^{\frac{1}{l}} =$$

$$= \eta(a_2, a_1)\left[A^{p,q}_{\Lambda^*}(1)\right]^{\frac{l-1}{l}} \sqrt[l]{B^{p,q}_{\Lambda^*}|f(a_1)|^l + B^{q,p}_{\Lambda^*}|f(a_2)|^l}.$$

Theorem 2.2 is proved.

We point out some special cases of Theorem 2.2.

***Corollary* 2.6.** *Under the assumption of Theorem* 2.2 *with* $\varphi(t) = t$, *we get*

$$\left|T^{p,q}_{f,\Lambda_1^*}(a_1,a_2)\right| \leq \eta^{p+q+1}(a_2,a_1)\,\beta^{\frac{l-1}{l}}(p+1,q+1)\times$$
$$\times\sqrt[l]{\beta(p+1,q+2)|f(a_1)|^l+\beta(q+1,p+2)|f(a_2)|^l}.$$

***Corollary* 2.7.** *Under the assumption of Theorem* 2.2 *with* $\varphi(t) = \dfrac{t^\alpha}{\Gamma(\alpha)}$, *we have*

$$\left|T^{p,q}_{f,\Lambda_2^*}(a_1,a_2)\right| \leq \frac{\eta^{\alpha(p+q)+1}(a_2,a_1)}{\Gamma^{p+q}(\alpha+1)}\,\beta^{\frac{l-1}{l}}(\alpha p+1,\alpha q+1)\times$$
$$\times\sqrt[l]{\beta(\alpha p+1,\alpha q+2)|f(a_1)|^l+\beta(\alpha q+1,\alpha p+2)|f(a_2)|^l}.$$

***Corollary* 2.8.** *Under the assumption of Theorem* 2.2 *with* $\varphi(t) = \dfrac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$, *we obtain*

$$\left|T^{p,q}_{f,\Lambda_3^*}(a_1,a_2)\right| \leq \frac{\eta^{\frac{\alpha}{k_1}(p+q)+1}(a_2,a_1)}{\Big[k_1\Gamma_{k_1}(\alpha+k_1)\Big]^{p+q}}\,\beta^{\frac{l-1}{l}}\left(\frac{p\alpha}{k_1}+1,\frac{q\alpha}{k_1}+1\right)\times$$
$$\times\sqrt[l]{\beta\left(\frac{p\alpha}{k_1}+1,\frac{q\alpha}{k_1}+2\right)|f(a_1)|^l+\beta\left(\frac{q\alpha}{k_1}+1,\frac{p\alpha}{k_1}+2\right)|f(a_2)|^l}.$$

***Corollary* 2.9.** *Under the assumption of Theorem* 2.2 *with* $\varphi(t) = t(a_1+\eta(a_2,a_1)-t)^{\alpha-1}$ *and* $f(x)$ *is symmetric to* $x = a_1+\dfrac{\eta(a_2,a_1)}{2}$, *we get*

$$\left|T^{p,q}_{f,\Lambda_4^*}(a_1,a_2)\right| \leq \eta(a_2,a_1)\left[\frac{C^{p,q}(\alpha,1)}{\alpha^{p+q}}\right]^{\frac{l-1}{l}}\sqrt[l]{D^{p,q}|f(a_1)|^l+D^{q,p}|f(a_2)|^l},$$

where

$$D^{p,q} = \frac{1}{\alpha^{p+q}\eta^2(a_2,a_1)}\int\limits_{a_1}^{a_1+\eta(a_2,a_1)}(t-a_1)\big[(a_1+\eta(a_2,a_1))^\alpha-t^\alpha\big]^p\times$$
$$\times\big[(a_1+\eta(a_2,a_1))^\alpha-(2a_1+\eta(a_2,a_1)-t)^\alpha\big]^q dt.$$

3. Some new bounds of Hermite – Hadamard type via general fractional integral inequalities.

Theorem 3.1. *Let* $f\colon P = [a_1, a_1+\eta(a_2,a_1)] \longrightarrow \mathbb{R}$ *be a preinvex function on* P *with* $\eta(a_2,a_1)>0$. *Then the following inequalities for generalized fractional integral hold*:

$$f\left(a_1+\frac{\eta(a_2,a_1)}{2}\right) \leq \frac{1}{2\Lambda^*(1)}\Big[{}_{a_1^+}I_\varphi f(a_1+\eta(a_2,a_1)) + {}_{(a_1+\eta(a_2,a_1))^-}I_\varphi f(a_1)\Big] \leq$$
$$\leq \frac{f(a_1)+f(a_2)}{2}. \tag{3.1}$$

Proof. For $t \in [0,1]$, let $x = a_1 + t\eta(a_2,a_1)$ and $y = a_1 + (1-t)\eta(a_2,a_1)$. From preinvexity of f, we get

$$f\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right) = f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2},$$

i.e.,

$$2f\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right) \le f(a_1 + t\eta(a_2,a_1)) + f(a_1 + (1-t)\eta(a_2,a_1)). \tag{3.2}$$

Multiplying both sides of (3.2) by $\dfrac{\varphi(\eta(a_2,a_1)t)}{t}$ and integrating the resulting inequality with respect to t over $(0,1]$, we obtain

$$2f\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right) \int\limits_0^1 \frac{\varphi(\eta(a_2,a_1)t)}{t} dt \le$$

$$\le \int\limits_0^1 \frac{\varphi(\eta(a_2,a_1)t)}{t} f(a_1 + t\eta(a_2,a_1))dt + \int\limits_0^1 \frac{\varphi(\eta(a_2,a_1)t)}{t} f(a_1 + (1-t)\eta(a_2,a_1))dt.$$

Hence,

$$2f\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right) \int\limits_0^1 \frac{\varphi(\eta(a_2,a_1)t)}{t} dt \le \Big[{}_{a_1^+}I_\varphi f(a_1 + \eta(a_2,a_1)) + {}_{(a_1+\eta(a_2,a_1))^-}I_\varphi f(a_1)\Big].$$

So, the first inequality is proved.

To prove the other half of the inequality in (3.1), since f is preinvex, we have

$$f(a_1 + t\eta(a_2,a_1)) + f(a_1 + (1-t)\eta(a_2,a_1)) \le f(a_1) + f(a_2). \tag{3.3}$$

Multiplying both sides of (3.3) by $\dfrac{\varphi(\eta(a_2,a_1)t)}{t}$ and integrating the resulting inequality with respect to t over $(0,1]$, we obtain

$$\Big[{}_{a_1^+}I_\varphi f(a_1 + \eta(a_2,a_1)) + {}_{(a_1+\eta(a_2,a_1))^-}I_\varphi f(a_1)\Big] \le \big[f(a_1) + f(a_2)\big] \int\limits_0^1 \frac{\varphi(\eta(a_2,a_1)t)}{t} dt.$$

Therefore, the second inequality is proved.

Theorem 3.1 is proved.

We point out some special cases of Theorem 3.1.

Corollary **3.1.** *Taking* $\eta(a_2,a_1) = a_2 - a_1$ *in Theorem* 3.1, *we get Theorem* 5 *of* [3].

Corollary **3.2.** *If in Theorem* 3.1 *we take* $\varphi(t) = t$, *then the inequalities* (3.1) *become the inequalities*

$$f\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right) \le \frac{1}{2\eta(a_2,a_1)} \Big[I_{a_1^+} f(a_1 + \eta(a_2,a_1)) + I_{(a_1+\eta(a_2,a_1))^-} f(a_1)\Big] \le$$

$$\leq \frac{f(a_1)+f(a_2)}{2},$$

where $I_{a_1^+}f$ and $I_{a_2^-}f$ are the classical Riemann integrals.

***Corollary* 3.3.** *If in Theorem* 3.1 *we choose $\varphi(t)=\dfrac{t^\alpha}{\Gamma(\alpha)}$, then the inequalities* (3.1) *become the inequalities*

$$f\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)\leq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(a_2,a_1)}\left[J^\alpha_{a_1^+}f(a_1+\eta(a_2,a_1))+J^\alpha_{(a_1+\eta(a_2,a_1))^-}f(a_1)\right]\leq$$

$$\leq \frac{f(a_1)+f(a_2)}{2},$$

where $J^\alpha_{a_1^+}f$ and $J^\alpha_{a_2^-}f$ are the fractional Riemann integrals.

***Corollary* 3.4.** *If in Theorem* 3.1 *we take $\varphi(t)=\dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the inequalities* (3.1) *become the inequalities*

$$f\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)\leq \frac{\Gamma_k(\alpha+k)}{2\eta^{\frac{\alpha}{k}}(a_2,a_1)}\left[I^{\alpha,k}_{a_1^+}f(a_1+\eta(a_2,a_1))+I^{\alpha,k}_{(a_1+\eta(a_2,a_1))^-}f(a_1)\right]\leq$$

$$\leq \frac{f(a_1)+f(a_2)}{2}.$$

***Corollary* 3.5.** *If in Theorem* 3.1 *we choose $\varphi(t)=t(a_1+\eta(a_2,a_1)-t)^{\alpha-1}$ and $f(x)$ is symmetric to $x=a_1+\dfrac{\eta(a_2,a_1)}{2}$, then the inequalities* (3.1) *become the inequalities*

$$f\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)\leq \frac{\alpha}{(a_1+\eta(a_2,a_1))^\alpha-a_1^\alpha}\int\limits_{a_1}^{a_1+\eta(a_2,a_1)}f(t)d_\alpha t\leq \frac{f(a_1)+f(a_2)}{2}.$$

***Corollary* 3.6.** *If in Theorem* 3.1 *we take $\varphi(t)=\dfrac{t}{\alpha}\exp\left[\left(-\dfrac{1-\alpha}{\alpha}\right)t\right],\ \alpha\in(0,1)$, then the inequalities* (3.1) *become the inequalities*

$$f\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)\leq \frac{1-\alpha}{2(1-\exp(-D))}\left[\mathcal{I}^\alpha_{a_1^+}f(a_1+\eta(a_2,a_1))+\mathcal{I}^\alpha_{(a_1+\eta(a_2,a_1))^-}f(a_1)\right]\leq$$

$$\leq \frac{f(a_1)+f(a_2)}{2},$$

where $\mathcal{I}^\alpha_{a_1^+}f$ and $\mathcal{I}^\alpha_{a_2^-}f$ are the right- and left-hand sided fractional integral operators with exponential kernel and $D=\left(\dfrac{1-\alpha}{\alpha}\right)\eta(a_2,a_1)$.

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 3.1. *Let* $f\colon P=[a_1,a_1+\eta(a_2,a_1)]\longrightarrow\mathbb{R}$ *be a differentiable mapping on* $(a_1,a_1+ +\eta(a_2,a_1))$ *with* $\eta(a_2,a_1)>0$. *If* $f'\in L(P)$, *then the following identity for generalized fractional integrals holds*:

$$\frac{f(a_1)+f(a_1+\eta(a_2,a_1))}{2}-\frac{1}{2\Lambda^*(1)}\Big[{}_{a_1^+}I_\varphi f(a_1+\eta(a_2,a_1))+{}_{(a_1+\eta(a_2,a_1))^-}I_\varphi f(a_1)\Big]=$$

$$=\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\int\limits_0^1\Big[\Lambda^*(1-t)-\Lambda^*(t)\Big]f'(a_1+(1-t)\eta(a_2,a_1))dt.$$

We denote

$$H_{f,\Lambda^*}(a_1,a_2)=\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\int\limits_0^1\Big[\Lambda^*(1-t)-\Lambda^*(t)\Big]f'(a_1+(1-t)\eta(a_2,a_1))dt. \tag{3.4}$$

Proof. Integrating by parts (3.4) and changing the variable of integration, we have

$$H_{f,\Lambda^*}(a_1,a_2)=\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\times$$

$$\times\left\{\int\limits_0^1\Lambda^*(1-t)f'(a_1+(1-t)\eta(a_2,a_1))dt-\int\limits_0^1\Lambda^*(t)f'(a_1+(1-t)\eta(a_2,a_1))dt\right\}=$$

$$=\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\left\{-\left.\frac{\Lambda^*(1-t)f(a_1+(1-t)\eta(a_2,a_1))}{\eta(a_2,a_1)}\right|_0^1-\right.$$

$$-\frac{1}{\eta(a_2,a_1)}\int\limits_0^1\frac{\varphi(\eta(a_2,a_1)(1-t))}{1-t}f(a_1+(1-t)\eta(a_2,a_1))dt+$$

$$\left.+\left.\frac{\Lambda^*(t)f(a_1+(1-t)\eta(a_2,a_1))}{\eta(a_2,a_1)}\right|_0^1-\frac{1}{\eta(a_2,a_1)}\int\limits_0^1\frac{\varphi(\eta(a_2,a_1)t)}{t}f(a_1+(1-t)\eta(a_2,a_1))dt\right\}=$$

$$=\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\left\{\frac{\Lambda^*(1)f(a_1+\eta(a_2,a_1))}{\eta(a_2,a_1)}-\frac{1}{\eta(a_2,a_1)}\,{}_{(a_1+\eta(a_2,a_1))^-}I_\varphi f(a_1)+\right.$$

$$\left.+\frac{\Lambda^*(1)f(a_1)}{\eta(a_2,a_1)}-\frac{1}{\eta(a_2,a_1)}\,{}_{a_1^+}I_\varphi f(a_1+\eta(a_2,a_1))\right\}=$$

$$=\frac{f(a_1)+f(a_1+\eta(a_2,a_1))}{2}-\frac{1}{2\Lambda^*(1)}\Big[{}_{a_1^+}I_\varphi f(a_1+\eta(a_2,a_1))+{}_{(a_1+\eta(a_2,a_1))^-}I_\varphi f(a_1)\Big].$$

Lemma 3.1 is proved.

Remark **3.1.** Taking $\eta(a_2,a_1)=a_2-a_1$ in Lemma 3.1, we get Lemma 5 of [30].

Theorem 3.2. *Let* $f: P = [a_1, a_1 + \eta(a_2, a_1)] \longrightarrow \mathbb{R}$ *be a differentiable mapping on* $(a_1, a_1 + {} + \eta(a_2, a_1))$ *with* $\eta(a_2, a_1) > 0$. *If* $|f'|^q$ *is preinvex on* P *for* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *then the following inequality for generalized fractional integrals holds*:

$$\left|H_{f,\Lambda^*}(a_1, a_2)\right| \leq \frac{\eta(a_2, a_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \sqrt[q]{\frac{|f'(a_1)|^q + |f'(a_2)|^q}{2}},$$

where

$$K_{\Lambda^*}(p) = \int\limits_0^1 \left|\Lambda^*(1-t) - \Lambda^*(t)\right|^p dt.$$

Proof. From Lemma 3.1, preinvexity of $|f'|^q$, Hölder's inequality and properties of the modulus, we have

$$\left|H_{f,\Lambda^*}(a_1, a_2)\right| \leq \frac{\eta(a_2, a_1)}{2\Lambda^*(1)} \int\limits_0^1 \left|\Lambda^*(1-t) - \Lambda^*(t)\right| \left|f'(a_1 + (1-t)\eta(a_2, a_1))\right| dt \leq$$

$$\leq \frac{\eta(a_2, a_1)}{2\Lambda^*(1)} \left(\int\limits_0^1 \left|\Lambda^*(1-t) - \Lambda^*(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int\limits_0^1 \left|f'(a_1 + (1-t)\eta(a_2, a_1))\right|^q dt\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{\eta(a_2, a_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \left(\int\limits_0^1 \left((1-t)\left|f'(a_1)\right|^q + t\left|f'(a_2)\right|^q\right) dt\right)^{\frac{1}{q}} =$$

$$= \frac{\eta(a_2, a_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \sqrt[q]{\frac{|f'(a_1)|^q + |f'(a_2)|^q}{2}}.$$

Theorem 3.2 is proved.

We point out some special cases of Theorem 3.2.

***Corollary* 3.7.** *Taking* $\eta(a_2, a_1) = a_2 - a_1$ *in Theorem* 3.2, *we get Theorem* 7 *of* [30].

***Corollary* 3.8.** *Taking* $p = q = 2$ *in Theorem* 3.2, *we get*

$$\left|H_{f,\Lambda^*}(a_1, a_2)\right| \leq \frac{\eta(a_2, a_1)}{2\Lambda^*(1)} \sqrt{K_{\Lambda^*}(2)} \sqrt{\frac{|f'(a_1)|^2 + |f'(a_2)|^2}{2}}.$$

***Corollary* 3.9.** *Taking* $\eta(a_2, a_1) = a_2 - a_1$ *and* $\varphi(t) = t$ *in Theorem* 3.2, *we get Theorem* 2.3 *of* [7].

***Corollary* 3.10.** *Taking* $\eta(a_2, a_1) = a_2 - a_1$ *and* $\varphi(t) = \dfrac{t^\alpha}{\Gamma(\alpha)}$ *in Theorem* 3.2, *we get Theorem* 8 *of* [27].

***Corollary* 3.11.** *Taking* $\eta(a_2, a_1) = a_2 - a_1$ *and* $\varphi(t) = \dfrac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem* 3.2, *we get Theorem* 8 *of* [12].

***Corollary* 3.12.** *Taking* $\eta(a_2,a_1)=a_2-a_1$, *where* $\varphi(t)=t(a_1+\eta(a_2,a_1)-t)^{\alpha-1}$ *and* $f(x)$ *is symmetric to* $x=a_1+\dfrac{\eta(a_2,a_1)}{2}$ *in Theorem* 3.2, *we get*

$$\left|H_{f,\Lambda_4^*}(a_1,a_2)\right|\le\frac{\sqrt[q]{\dfrac{\eta(a_2,a_1)}{2}}}{\sqrt[p]{p\alpha+1}\left[\left(a_1+\eta(a_2,a_1)\right)^{\alpha}-a_1^{\alpha}\right]}\times$$

$$\times\sqrt[p]{a_1^{p\alpha+1}+(a_1+\eta(a_2,a_1))^{p\alpha+1}-\frac{(2a_1+\eta(a_2,a_1))^{p\alpha+1}}{2^{p\alpha}}}\sqrt[q]{\frac{|f'(a_1)|^q+|f'(a_2)|^q}{2}}.$$

Theorem 3.3. *Let* $f\colon P=[a_1,a_1+\eta(a_2,a_1)]\longrightarrow\mathbb{R}$ *be a differentiable mapping on* $(a_1,a_1++\eta(a_2,a_1))$ *with* $\eta(a_2,a_1)>0$. *If* $|f'|^q$ *is preinvex on* P *for* $q\ge1$, *then the following inequality for generalized fractional integrals holds*:

$$\left|H_{f,\Lambda^*}(a_1,a_2)\right|\le\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\Big[K_{\Lambda^*}(1)\Big]^{1-\frac{1}{q}}\sqrt[q]{K_{\Lambda^*}}\sqrt[q]{|f'(a_1)|^q+|f'(a_2)|^q},$$

where

$$K_{\Lambda^*}=\int\limits_0^1 t\left|\Lambda^*(1-t)-\Lambda^*(t)\right|dt$$

and $K_{\Lambda^*}(1)$ *is defined as in Theorem* 3.2.

Proof. From Lemma 3.1, preinvexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left|H_{f,\Lambda^*}(a_1,a_2)\right|\le\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\int\limits_0^1\left|\Lambda^*(1-t)-\Lambda^*(t)\right|\left|f'(a_1+(1-t)\eta(a_2,a_1))\right|dt\le$$

$$\le\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\left(\int\limits_0^1\left|\Lambda^*(1-t)-\Lambda^*(t)\right|dt\right)^{1-\frac{1}{q}}\times$$

$$\times\left(\int\limits_0^1\left|\Lambda^*(1-t)-\Lambda^*(t)\right|\left|f'(a_1+(1-t)\eta(a_2,a_1))\right|^q dt\right)^{\frac{1}{q}}\le$$

$$\le\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\Big[K_{\Lambda^*}(1)\Big]^{1-\frac{1}{q}}\left(\int\limits_0^1\left|\Lambda^*(1-t)-\Lambda^*(t)\right|\left((1-t)\left|f'(a_1)\right|^q+t\left|f'(a_2)\right|^q\right)dt\right)^{\frac{1}{q}}=$$

$$=\frac{\eta(a_2,a_1)}{2\Lambda^*(1)}\Big[K_{\Lambda^*}(1)\Big]^{1-\frac{1}{q}}\sqrt[q]{K_{\Lambda^*}}\sqrt[q]{|f'(a_1)|^q+|f'(a_2)|^q}.$$

Theorem 3.3 is proved.

We point out some special cases of Theorem 3.3.

***Corollary* 3.13.** *Taking $q = 1$ in Theorem* 3.3, *we get*

$$\left|H_{f,\Lambda^*}(a_1,a_2)\right| \leq \frac{\eta(a_2,a_1)}{2\Lambda^*(1)} K_{\Lambda^*}\left[|f'(a_1)| + |f'(a_2)|\right].$$

***Corollary* 3.14.** *Under the assumption of Theorem* 3.3 *with $\varphi(t) = t$, we have*

$$\left|H_{f,\Lambda_1^*}(a_1,a_2)\right| \leq \frac{\eta(a_2,a_1)}{2^{2+\frac{1}{q}}} \sqrt[q]{|f'(a_1)|^q + |f'(a_2)|^q}.$$

***Corollary* 3.15.** *Under the assumption of Theorem* 3.3 *with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we obtain*

$$\left|H_{f,\Lambda_2^*}(a_1,a_2)\right| \leq \left(\frac{2^\alpha - 1}{2^{\alpha+1}}\right) \sqrt[q]{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}} \eta(a_2,a_1) \sqrt[q]{|f'(a_1)|^q + |f'(a_2)|^q}.$$

***Corollary* 3.16.** *Under the assumption of Theorem* 3.3 *with $\varphi(t) = \dfrac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$, we get*

$$\left|H_{f,\Lambda_3^*}(a_1,a_2)\right| \leq \left(\frac{2^{\frac{\alpha}{k_1}} - 1}{2^{\frac{\alpha}{k_1}+1}}\right) \sqrt[q]{\frac{\Gamma_{k_1}(\alpha+k_1)}{\Gamma_{k_1}(\alpha+k_1+1)}} \eta(a_2,a_1) \sqrt[q]{|f'(a_1)|^q + |f'(a_2)|^q}.$$

***Corollary* 3.17.** *Under the assumption of Theorem* 3.3 *with $\varphi(t) = t(a_1 + \eta(a_2,a_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = a_1 + \dfrac{\eta(a_2,a_1)}{2}$, we have*

$$\left|H_{f,\Lambda_4^*}(a_1,a_2)\right| \leq \frac{\eta(a_2,a_1)}{2\overline{\Lambda^*(1)}} \left[\overline{K_{\Lambda^*}(1)}\right]^{1-\frac{1}{q}} \sqrt[q]{\overline{K_{\Lambda^*}}} \sqrt[q]{|f'(a_1)|^q + |f'(a_2)|^q},$$

where

$$\overline{\Lambda^*(1)} = \frac{(a_1 + \eta(a_2,a_1))^\alpha}{\alpha},$$

$$\overline{K_{\Lambda^*}(1)} = \frac{2}{\alpha}\left[(a_1 + \eta(a_2,a_1))^{\alpha+1} - 2\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right)^{\alpha+1} + a_1^{\alpha+1}\right],$$

$$\overline{K_{\Lambda^*}} = \frac{1}{\alpha}\left[F_{11} - F_{12} + F_{21} - F_{22}\right],$$

and

$$F_{11} = \frac{1}{\eta^2(a_2,a_1)}\left\{\frac{(a_1 + \eta(a_2,a_1))}{\alpha+1}\left[(a_1 + \eta(a_2,a_1))^{\alpha+1} - \left(a_1 + \frac{\eta(a_2,a_1)}{2}\right)^{\alpha+1}\right] - \right.$$

$$\left. - \frac{1}{\alpha+2}\left[(a_1 + \eta(a_2,a_1))^{\alpha+2} - \left(a_1 + \frac{\eta(a_2,a_1)}{2}\right)^{\alpha+2}\right]\right\},$$

$$F_{12} = \frac{1}{\eta^2(a_2,a_1)}\left\{\frac{1}{\alpha+2}\left[\left(a_1 + \frac{\eta(a_2,a_1)}{2}\right)^{\alpha+2} - a_1^{\alpha+2}\right] - \right.$$

$$- \frac{a_1}{\alpha+1}\left[\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^{\alpha+1} - a_1^{\alpha+1}\right]\Bigg\},$$

$$F_{21} = \frac{1}{\eta^2(a_2,a_1)}\Bigg\{\frac{1}{\alpha+2}\left[(a_1+\eta(a_2,a_1))^{\alpha+2} - \left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^{\alpha+2}\right] -$$

$$- \frac{a_1}{\alpha+1}\left[(a_1+\eta(a_2,a_1))^{\alpha+1} - \left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^{\alpha+1}\right]\Bigg\},$$

$$F_{22} = \frac{1}{\eta^2(a_2,a_1)}\Bigg\{\frac{(a_1+\eta(a_2,a_1))}{\alpha+1}\left[\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^{\alpha+1} - a_1^{\alpha+1}\right] -$$

$$- \frac{1}{\alpha+2}\left[(a_1+\eta(a_2,a_1))^{\alpha+2} - a_1^{\alpha+2}\right]\Bigg\}.$$

4. Applications. Consider the following special means for different real numbers α, β and $\alpha\beta \neq 0$ as follows:

(1) the arithmetic mean

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2},$$

(2) the harmonic mean

$$H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},$$

(3) the logarithmic mean

$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|},$$

(4) the generalized log-mean

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \qquad n \in \mathbb{Z}\setminus\{-1,0\}.$$

Now, by using the theory results in Section 3, we give some applications to special means for different real numbers.

Proposition 4.1. *Let* $a_1, a_2 \in \mathbb{R}\setminus\{0\}$, *where* $a_1 < a_2$ *and* $\eta(a_2,a_1) > 0$. *Then, for* $n \in \mathbb{Z}\setminus\{-1,0\}$, *where* $q > 1$ *and* $p^{-1}+q^{-1} = 1$, *the following inequality holds*:

$$\Big|A\left(a_1^n, (a_1+\eta(a_2,a_1))^n\right) - L_n\left(a_1, a_1+\eta(a_2,a_1)\right)\Big| \leq \frac{|n|}{2}\frac{\eta(a_2,a_1)}{\sqrt[p]{p+1}} \times$$

$$\times \sqrt[q]{A\left(|a_1|^{q(n-1)}, |a_2|^{q(n-1)}\right)}.$$

Proof. Applying Theorem 3.2 for $f(x) = x^n$ and $\varphi(t) = t$, one can obtain the result immediately.

Proposition 4.2. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$ and $\eta(a_2, a_1) > 0$. Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds*:

$$\left| \frac{1}{H\left(a_1, a_1 + \eta(a_2, a_1)\right)} - \frac{1}{L\left(a_1, a_1 + \eta(a_2, a_1)\right)} \right| \leq \frac{\eta(a_2, a_1)}{2\sqrt[p]{p+1}} \frac{1}{\sqrt[q]{H\left(a_1^{2q}, a_2^{2q}\right)}}.$$

Proof. Applying Theorem 3.2 for $f(x) = \dfrac{1}{x}$ and $\varphi(t) = t$, one can obtain the result immediately.

Proposition 4.3. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$ and $\eta(a_2, a_1) > 0$. Then, for $n \in \in \mathbb{Z} \setminus \{-1, 0\}$ and $q \geq 1$, the following inequality holds*:

$$\Big| A\left(a_1^n, (a_1 + \eta(a_2, a_1))^n\right) - L_n\left(a_1, a_1 + \eta(a_2, a_1)\right) \Big| \leq \frac{|n|}{2^{2+\frac{1}{q}}} \eta(a_2, a_1) \times$$

$$\times \sqrt[q]{A\left(|a_1|^{q(n-1)}, |a_2|^{q(n-1)}\right)}.$$

Proof. Applying Theorem 3.3 for $f(x) = x^n$ and $\varphi(t) = t$, one can obtain the result immediately.

Proposition 4.4. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$ and $\eta(a_2, a_1) > 0$. Then, for $q \geq 1$, the following inequality holds*:

$$\left| \frac{1}{H\left(a_1, a_1 + \eta(a_2, a_1)\right)} - \frac{1}{L\left(a_1, a_1 + \eta(a_2, a_1)\right)} \right| \leq \frac{\eta(a_2, a_1)}{2^{2+\frac{1}{q}}} \frac{1}{\sqrt[q]{H\left(a_1^{2q}, a_2^{2q}\right)}}.$$

Proof. Applying Theorem 3.3 for $f(x) = \dfrac{1}{x}$ and $\varphi(t) = t$, one can obtain the result immediately.

***Remark* 4.1.** Applying our Theorems 3.2 and 3.3 for appropriate choices of function $\varphi(t) = = \dfrac{t^\alpha}{\Gamma(\alpha)}$, $\dfrac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$; $\varphi(t) = t(a_1 + \eta(a_2, a_1) - t)^{\alpha - 1}$, where $f(x)$ is symmetric to $x = a_1 + + \dfrac{\eta(a_2, a_1)}{2}$ and $\varphi(t) = \dfrac{t}{\alpha} \exp\left[\left(-\dfrac{1-\alpha}{\alpha}\right) t\right]$ for $\alpha \in (0, 1)$, such that $|f'|^q$ to be preinvex, we can deduce some new general fractional integral inequalities using special means. We omit their proofs and the details are left to the interested reader.

***Remark* 4.2.** Also, in Remark 4.1, if we choose $\eta(a_2, a_1) = a_2 - a_1$, we can deduce some new fascinating general fractional integral inequalities for convex functions using special means. The details are left to the interested reader.

Next, we provide some new error estimates for the trapezoidal formula.

Let Q be the partition of the points $a_1 = x_0 < x_1 < \ldots < x_n = a_2$ of the interval $[a_1, a_2]$. Let consider the quadrature formula

$$\int\limits_{a_1}^{a_2} f(x) dx = T(f, Q) + E(f, Q),$$

where

$$T(f,Q)=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$$

is the trapezoidal version and $E(f,Q)$ is denote their associated approximation error.

Proposition 4.5. *Let* $f:[a_1,a_2]\longrightarrow\mathbb{R}$ *be a differentiable function on* (a_1,a_2), *where* $a_1<a_2$. *If* $|f'|^q$ *is convex on* $[a_1,a_2]$ *for* $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds*:

$$\big|E(f,Q)\big|\leq\frac{1}{2^{\frac{q+1}{q}}\sqrt[p]{p+1}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\sqrt[q]{|f'(x_i)|^q+|f'(x_{i+1})|^q}.$$

Proof. Applying Theorem 3.2 for $\eta(a_2,a_1)=a_2-a_1$ and $\varphi(t)=t$ on the subintervals $[x_i,x_{i+1}],\ i=0,\ldots,n-1$, of the partition Q, we have

$$\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{x_{i+1}-x_i}\int\limits_{x_i}^{x_{i+1}}f(x)dx\right|\leq$$

$$\leq\frac{(x_{i+1}-x_i)}{2\sqrt[p]{p+1}}\left[\frac{|f'(x_i)|^q+|f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}}. \tag{4.1}$$

Hence from (4.1), we get

$$\big|E(f,Q)\big|=\left|\int\limits_{a_1}^{a_2}f(x)dx-T(f,Q)\right|\leq$$

$$\leq\left|\sum_{i=0}^{n-1}\left\{\int\limits_{x_i}^{x_{i+1}}f(x)dx-\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|\leq$$

$$\leq\sum_{i=0}^{n-1}\left|\left\{\int\limits_{x_i}^{x_{i+1}}f(x)dx-\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|\leq$$

$$\leq\frac{1}{2^{\frac{q+1}{q}}\sqrt[p]{p+1}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\sqrt[q]{|f'(x_i)|^q+|f'(x_{i+1})|^q}.$$

Proposition 4.5 is proved.

Proposition 4.6. *Let* $f:[a_1,a_2]\longrightarrow\mathbb{R}$ *be a differentiable function on* (a_1,a_2), *where* $a_1<a_2$. *If* $|f'|^q$ *is convex on* $[a_1,a_2]$ *for* $q\geq 1$, *then the following inequality holds*:

$$\big|E(f,Q)\big|\leq\frac{1}{2^{2+\frac{1}{q}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\sqrt[q]{|f'(x_i)|^q+|f'(x_{i+1})|^q}.$$

Proof is analogous as to that of Proposition 4.5 but use Theorem 3.3 for $\eta(a_2,a_1)=a_2-a_1$ and $\varphi(t)=t$.

***Remark* 4.3.** Applying Theorems 3.2 and 3.3 for appropriate choices of function $\varphi(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha)}$, $\dfrac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$; $\varphi(t)=t(a_2-t)^{\alpha-1}$, where $f(x)$ is symmetric to $x=\dfrac{a_1+a_2}{2}$ and

$$\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$$

for $\alpha\in(0,1)$, such that $|f'|^q$ to be convex, we can deduce some new general fractional integral inequalities using above ideas and techniques. We omit their proofs and the details are left to the interested reader.

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Received 20. 12.18