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# EXISTENCE OF SOLUTIONS FOR A FRACTIONAL-ORDER BOUNDARY-VALUE PROBLEM

# ІСНУВАННЯ РОЗВ'ЯЗКІВ КРАЙОВОЇ ЗАДАЧІ ДРОБОВОГО ПОРЯДКУ

We investigate the existence of solutions for a fractional-order boundary-value problem by using some fixed point theorems. As applications, examples are given to illustrate the main results.

За допомогою теорем про нерухому точку вивчено проблему існування розв'язків крайової задачі дробового порядку. Як застосування наведено приклади, що ілюструють отримані результати.

**1. Introduction.** The history of the theory of fractional calculus goes back to seventeenth century, when in 1695 the derivative of order $\alpha = \dfrac{1}{2}$ was defined by Leibnitz in his "Letter to L'Hospital". From that time, the theory be attractive to mathematics as well as physics, biology, engineering and economy. The first application of fractional calculus was due to Abel in his solution to the Tautochrone problem [8]. We refer to the books by Agarwal et al. [9], Kilbas et al. [1] and Podlubny [2].

The existence of positive solutions for fractional-order nonlinear boundary-value problems has been studied by many authors using the fixed point theorem in cones. To identify a few, we refer the reader to [3–7, 10–12] and references therein.

Some studies in the literature are as follows:

X. Su [11] studied the multipoint boundary-value problem

$$D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), \quad 0 < t < 1,$$

$$D^{\beta}v(t) = g(t, u(t), \quad D^{\nu}u(t)), \quad 0 < t < 1,$$

$$u(0) = u(1) = v(0) = v(1) = 0,$$

where $1 < \alpha,\ \beta < 2,\ \mu,\nu > 0,\ \alpha - \nu \geq 1,\ \beta - \mu \geq 1,\ f, g\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and $D$ is the standard *Riemann–Liouville* differentiation. X. Su obtained the existence of solutions a boundary-value problem for a coupled differential system of fractional order by using Schauder fixed point theorem.

Rehman and Khan [7] studied the multipoint boundary-value problem

$$D_t^{\alpha}y(t) = f(t, y(t), \quad D_t^{\beta}y(t)), \quad t \in (0,1),$$

$$y(0) = 0, \quad D_t^{\beta}y(1) - \sum_{i=1}^{m-2} \eta_i D_t^{\beta}y(\xi_i) = y_0,$$

where $1 < \alpha \leq 2,\ 0 < \beta < 1,\ 0 < \xi_i < 1,\ i = 1, 2, \ldots, m-2,\ \eta_i \leq 0$ with $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$ and $D_t^{\alpha}$ represents the standard *Riemann–Liouville* fractional derivative. They obtained the uniqueness existence of solutions by means of the Banach fixed point theorem.

J. Graef, L. Kong, Q. Kong and M. Wang [4] study of the nonlinear fractional boundary-value problem

$$-D_{0+}^{\alpha}u = F(t,u), \quad 0 < t < 1,$$
$$u(0) = 0, \quad u(1) - aI_{0+}^{\alpha}u(1) = b,$$

where $1 < \alpha \le 2$, $0 \le a < \Gamma(\alpha+1)$, $b \in \mathbb{R}_+$ with $\mathbb{R}_+ = [0,\infty)$, $b = 0$ if $a = 0$, and $\Gamma$ is the Gamma function. The authors study a type of nonlinear fractional boundary-value problem with nonhomogeneous integral boundary conditions and the existence and uniqueness of positive solutions are discussed.

Keyu Zhang and Jiafa Xu [5] consider the unique positive solution for the fractional boundary-value problem

$$D_{0+}^{\alpha}u(t) = -f(t,u(t)), \quad t \in [0,1],$$
$$u(0) = u'(0) = u'(1) = 0,$$

where $\alpha \in (2,3]$ is a real number, $D_{0+}^{\alpha}$ is the standard Riemann–Liouville fractional derivative of order $\alpha$. By using the method of upper and lower solutions and monotone iterative technique, they also obtain that there exists a sequence of iterations uniformly converges to the unique solution.

Motivated by the above works, we study the fractional-order nonlinear boundary-value problem

$$D^{\alpha}u(t) = f\left(t, u(t), D^{\beta}u(t)\right), \quad t \in [0,1],$$
$$u(0) = u'(0) = 0, \tag{1.1}$$
$$D^{\alpha-2}u'(1) - \sum_{i=1}^{m-2} \eta_i I^{\delta}u'(\xi_i) = A.$$

Here, $2 < \alpha \le 3$, $\beta \le \alpha - 1$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $i = 1,\ldots,m-2$, $\eta_i > 0$ and $\delta > 0$ with $\Gamma(\alpha+\delta-1) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\delta-2} \ne 0$, $A \in \mathbb{R}$, $D^{\alpha}$ standard Riemann–Liouville fractional derivative and $I^{\alpha}$ standard Riemann–Liouville fractional integral.

Throughout this paper we assume that following conditions hold:

$(H_1)$ $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function;

$(H_2)$ there exists a constant $k > 0$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le k\left(|u-\bar{u}| + |v-\bar{v}|\right)$$

for each $t \in I$ and all $u, \bar{u}, v, \bar{v} \in \mathbb{R}$.

By using Schauder fixed point theorem and the Banach contraction principle, we get the existence of at least one solution.

**2. Preliminaries.** For the convenience of the reader, we present here some necessary definitions from fractional calculus theory.

**Definition 2.1.** *The Riemann–Liouville fractional integral of order* $\alpha > 0$ *of a function* $f\colon (0,\infty) \to \mathbb{R}$ *is given by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s)ds$$

*provided that the right-hand side is pointwise defined on* $(0,\infty)$.

**Definition 2.2.** *The Riemann–Liouville fractional derivative of order* $\alpha > 0$ *of a continuous function* $f\colon (0,\infty) \to \mathbb{R}$ *is given by*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right) \int\limits_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

*where* $n-1 \leq \alpha < n$, *provided that the right-hand side is pointwise defined on* $(0,\infty)$.

**Lemma 2.1.** *Let* $\alpha > 0$, *then* $I_t^\alpha D_t^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \ldots + c_n t^{\alpha-n}$, *where* $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n,\ n = [\alpha]+1$. *Here,* $I_t^\alpha$ *stands for the standard Riemann–Liouville fractional integral of order* $\alpha > 0$ *and* $D_t^\alpha$ *denotes the Riemann–Liouville fractional derivative.*

**Lemma 2.2.** *Assume that the conditions* $(H_1)$ *and* $(H_2)$ *are satisfied. Let*

$$\Delta = \Gamma(\alpha)\left[\Gamma(\alpha+\delta-1) - \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha+\delta-2}\right].$$

*If* $h \in \mathcal{C}[0,1]$, *the fractional boundary-value problem*

$$\begin{gathered} D^\alpha u(t) = h(t), \quad t \in [0,1], \quad 2 < \alpha \leq 3, \quad \beta \leq \alpha - 1, \quad \delta > 0, \\ u(0) = u'(0) = 0, \\ D^{\alpha-2} u'(1) - \sum_{i=1}^{m-2} \eta_i I^\delta u'(\xi_i) = A, \end{gathered} \tag{2.1}$$

*has a unique solution*

$$u(t) = \int\limits_0^1 G(t,s)h(s)ds + \frac{\Gamma(\alpha+\delta-1)A}{\Delta}\, t^{\alpha-1}, \quad t \in [0,1], \tag{2.2}$$

*where* $G(t,s)$ *is the Green function and is given by*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{t^{\alpha-1}}{\Delta}\left(\sum_{j=i}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1)\right), \\ 0 \leq s \leq t, \quad \xi_{i-1} \leq s \leq \xi_i, \quad i = 1, 2, \ldots, m-1, \\[2mm] \dfrac{t^{\alpha-1}}{\Delta}\left(\sum_{j=i}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1)\right), \\ 0 \leq t \leq s, \quad \xi_{i-1} \leq s \leq \xi_i, \quad i = 1, 2, \ldots, m-1. \end{cases}$$

***Proof.*** In view of Lemma 2.1 and (2.1), we have

$$u(t) = I^{\alpha}h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} \quad \text{for} \quad c_1, c_2, c_3 \in \mathbb{R}. \tag{2.3}$$

The boundary conditions $u(0) = 0$ and $u'(0) = 0$ satisfied that $c_2 = c_3 = 0$. We can be written

$$u(t) = I^{\alpha}h(t) + c_1 t^{\alpha-1} \quad \text{for} \quad c_1 \in \mathbb{R}.$$

From here

$$u'(t) = I^{\alpha-1}h(t) + (\alpha-1)c_1 t^{\alpha-2} \quad \text{for} \quad c_1 \in \mathbb{R}.$$

Using the relations $D^{\alpha}t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, t^{\beta-\alpha}$, $I^{\alpha}t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, t^{\alpha+\beta}$, (2.3) reduces to

$$D^{\alpha-2}u'(t) = \int\limits_0^t h(s)ds + c_1\Gamma(\alpha). \tag{2.4}$$

By using the boundary condition $D^{\alpha-2}u'(1) - \sum_{i=1}^{m-2} \eta_i I^{\delta}u'(\xi_i) = A$ and (2.4), we obtain

$$c_1 = \frac{\sum_{i=1}^{m-2} \eta_i \int_0^{\xi_i} (\xi_i - s)^{\alpha+\delta-2}h(s)ds - \Gamma(\alpha+\delta-1)\int_0^1 h(s)ds + \Gamma(\alpha+\delta-1)A}{\Delta}.$$

Therefore the unique solution of problem (2.1) is given by

$$u(t) = \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}h(s)ds + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m-2}\eta_i \int\limits_0^{\xi_i} (\xi_i - s)^{\alpha+\delta-2}h(s)ds -$$

$$-\frac{t^{\alpha-1}}{\Delta}\Gamma(\alpha+\delta-1)\int\limits_0^1 h(s)ds + \frac{t^{\alpha-1}}{\Delta}\Gamma(\alpha+\delta-1)A. \tag{2.5}$$

For $0 \le t \le \xi_1$, (2.5) can be expressed as follows:

$$u(t) = \int\limits_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta}\left(\sum_{j=1}^{m-2}\eta_j(\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1)\right)\right] h(s)ds +$$

$$+\int\limits_t^{\xi_1} \frac{t^{\alpha-1}}{\Delta}\left(\sum_{j=1}^{m-2}\eta_j(\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1)\right) h(s)ds +$$

$$+\sum_{i=2}^{m-2}\int\limits_{\xi_{i-1}}^{\xi_i} \frac{t^{\alpha-1}}{\Delta}\left(\sum_{j=i}^{m-2}\eta_j(\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1)\right) h(s)ds -$$

$$-\int\limits_{\xi_{m-2}}^1 \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} h(s)ds + \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} A.$$

For $\xi_{k-1} \le t \le \xi_k,\ 2 \le k \le m-2$, (2.5) can be expressed as follows:

$$u(t) = \int_0^{\xi_1} \left[ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta} \left( \sum_{j=k}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1) \right) \right] h(s)ds +$$

$$+ \sum_{i=2}^{k-2} \int_{\xi_{i-1}}^{\xi_i} \left[ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta} \left( \sum_{j=i}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1) \right) \right] h(s)ds +$$

$$+ \int_{\xi_{k-1}}^{t} \left[ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta} \left( \sum_{j=k}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1) \right) \right] h(s)ds +$$

$$+ \int_t^{\xi_k} \frac{t^{\alpha-1}}{\Delta} \left( \sum_{j=k}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1) \right) h(s)ds +$$

$$+ \sum_{i=k+1}^{m-2} \int_{\xi_{i-1}}^{\xi_i} \frac{t^{\alpha-1}}{\Delta} \left( \sum_{j=i}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1) \right) h(s)ds -$$

$$- \int_{\xi_{m-2}}^{1} \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} h(s)ds + \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} A.$$

For $\xi_{m-2} \le t \le 1$, (2.5) can be expressed as follows:

$$u(t) = \sum_{i=1}^{m-2} \int_{\xi_{i-1}}^{\xi_i} \left[ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta} \left( \sum_{j=i}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} - \Gamma(\alpha+\delta-1) \right) \right] h(s)ds +$$

$$+ \int_{\xi_{m-2}}^{t} \left( \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} \right) h(s)ds -$$

$$- \int_t^1 \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} h(s)ds + \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)}{\Delta} A.$$

Hence, the unique solution of boundary-value problem (1.1) is given by

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\Gamma(\alpha+\delta-1)A}{\Delta} t^{\alpha-1}.$$

Lemma 2.2 is proved.

**3. Main results.** Let $\mathcal{C}[0,1]$ be space of continuous functions defined on [0,1]. The space

$$E = \left\{ u : u \in \mathcal{C}[0,1], D^{\beta} u \in \mathcal{C}[0,1] \right\}$$

equipped with the norm

$$\|u\|_E = \max_{t\in[0,1]} \left\{ \max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} \left| D^{\beta} u(t) \right| \right\}$$

is a Banach space.

For convenience, we define the following constants: $N = \max_{t\in[0,1]} \left| f\left(t, u(t), D^{\beta}u(t)\right)\right| + 1$ and

$$M = N\left[\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(\alpha)\left[\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha+\delta-1} + \Gamma(\alpha+\delta)\right]}{|\Delta|\Gamma(\alpha-\beta)(\alpha+\delta-1)}\right] + \frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)|A|}{|\Delta|\Gamma(\alpha-\beta)}.$$

**Lemma 3.1.** *Assume that* $(H_1)$ *and* $(H_2)$ *hold. Then the operator* $T\colon \Omega \to \Omega$ *is completely continuous.*

***Proof.*** Define an operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)h(s)ds + \frac{\Gamma(\alpha+\delta-1)A}{\Delta} t^{\alpha-1}.$$

Fixed points of the operator $T$ are solutions of the boundary-value problem (1.1). In view of the continuity of $f$ and $G$, the operator $T$ is continuous.

Firstly, we define

$$\Omega = \{u \in E\colon \|u\|_E \le M, t \in [0,1], M > 0\}$$

and prove that $T\colon \Omega \to \Omega$. Let $N = \max_{t\in[0,1]} |f(t, u(t), D^{\beta}u(t)| + 1$. For every $u \in \Omega$ and for every $u \in \Omega$, we have

$$|(Tu)(t)| = \left| \int_0^1 G(t,s) f\left(s, u(s), D^{\beta}u(s)\right) ds + \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)A}{\Delta}\right| \le$$

$$\le \left| \int_0^1 G(t,s) f\left(s, u(s), D^{\beta}u(s)\right) ds \right| + \left| \frac{t^{\alpha-1}\Gamma(\alpha+\delta-1)A}{\Delta}\right| \le$$

$$\le \int_0^1 |G(t,s)| \left| f(s, u(s), D^{\beta}u(s)) \right| ds + \left| \frac{\Gamma(\alpha+\delta-1)A}{\Delta}\right| \le$$

$$\le N \int_0^1 |G(t,s)| ds + \frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|} \le$$

$$\leq N\left[\frac{1}{\Gamma(\alpha+1)}+\frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\delta-1}+\Gamma(\alpha+\delta)}{|\Delta|(\alpha+\delta-1)}\right]+\frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|}\leq M<\infty.$$

In view of relation $D^\beta I^\alpha = I^{\beta-\alpha}$ and (2.5), we have the following estimates:

$$\left|D^\beta(Tu)(t)\right| = \Bigg|I^{\alpha-\beta}f\left(t,u(t),D^\beta u(t)\right)+$$

$$+\left[\frac{1}{\Delta}\sum_{i=1}^{m-2}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\delta-2}f(s,u(s),D^\beta u(s))ds-\right.$$

$$\left.-\frac{\Gamma(\alpha+\delta-1)}{\Delta}\int_0^1 f\left(s,u(s),D^\beta u(s)\right)ds+\frac{\Gamma(\alpha+\delta-1)A}{\Delta}\right]D^\beta t^{\alpha-1}\Bigg|\leq$$

$$\leq\int_0^1\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|f\left(s,u(s),D^\beta(s)\right)\right|ds+$$

$$+\left[\frac{1}{|\Delta|}\sum_{i=1}^{m-2}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\delta-2}\left|f\left(s,u(s),D^\beta(s)\right)\right|ds+\right.$$

$$\left.+\frac{\Gamma(\alpha+\delta-1)}{|\Delta|}\int_0^1\left|f\left(s,u(s),D^\beta(s)\right)\right|ds+\frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|}\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}\leq$$

$$\leq N\left[\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds+\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\beta)}\sum_{i=1}^{m-2}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\delta-2}ds+\right.$$

$$\left.+\frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\beta)}\right]+\frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\beta)}=$$

$$=N\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\Gamma(\alpha)\left[\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\delta-1}+\Gamma(\alpha+\delta)\right]}{|\Delta|\Gamma(\alpha-\beta)(\alpha+\delta-1)}\right]+$$

$$+\frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)|A|}{|\Delta|\Gamma(\alpha-\beta)}=M<\infty.$$

Therefore, $\|(Tu)(t)\|_E\leq M$. Thus we have $T\colon\Omega\to\Omega$. Now we show that $T$ is completely continuous. For this, let

$$N=\max_{t\in[0,1]}\left|f\left(t,u(t),D^\beta u(t)\right)\right|+1$$

for $u\in\Omega$ and $t_1,t_2\in[0,1]$ be such that $t_1<t_2$. Then we have

$$\left|(Tu)(t_2)-(Tu)(t_1)\right|=$$

$$=\left|\int\limits_0^1 (G(t_2,s)-G(t_1,s))f(s,u(s),D^\beta u(s))ds+\frac{\Gamma(\alpha+\delta-1)A}{\Delta}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\right|\leq$$

$$\leq\int\limits_0^1 |G(t_2,s)-G(t_1,s)|\left|f\left(s,u(s),D^\beta u(s)\right)\right|ds+\frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|}\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|\leq$$

$$\leq N\int\limits_0^1\left|\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Delta}\times\right.$$

$$\left.\times\left(\sum_{i=1}^{m-2}\eta_i(\xi_i-s)^{\alpha+\delta-2}-\Gamma(\alpha+\delta-1)\right)\right|ds+\frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\leq$$

$$\leq\frac{N\left(t_2^\alpha-t_1^\alpha\right)}{\Gamma(\alpha+1)}+\frac{N\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{|\Delta|}\left|\frac{\sum_{i=1}^{m-2}\eta_i\left(\xi_i^{\alpha+\delta-1}\right)-\Gamma(\alpha+\delta)}{\alpha+\delta-1}\right|+$$

$$+\frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right),$$

$$\left|D^\beta(Tu)(t_2)-D^\beta(Tu)(t_1)\right|=\left|I^{\alpha-\beta}\left(f\left(t_2,u(t_2),D^\beta(t_2)\right)-f\left(t_1,u(t_1),D^\beta(t_1)\right)\right)+\right.$$

$$+\left(\frac{1}{\Delta}\sum_{i=1}^{m-2}\eta_i\int\limits_0^{\xi_i}(\xi_i-s)^{\alpha+\delta-2}f\left(s,u(s),D^\beta u(s)\right)ds-\right.$$

$$\left.\left.-\frac{\Gamma(\alpha+\delta-1)}{\Delta}\int\limits_0^1 f\left(s,u(s),D^\beta u(s)\right)ds+\frac{\Gamma(\alpha+\delta-1)A}{\Delta}\right)D^\beta\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\right|\leq$$

$$\leq\left|N\left(\int\limits_0^{t_2}\frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds-\int\limits_0^{t_1}\frac{(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds\right)+\frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}\left(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}\right)\times\right.$$

$$\left.\times\left(N\sum_{i=1}^{m-2}\eta_i\int\limits_0^{\xi_i}(\xi_i-s)^{\alpha+\delta-2}ds-N\Gamma(\alpha+\delta-1)+\Gamma(\alpha+\delta-1)A\right)\right|=$$

$$=\frac{N}{\Gamma(\alpha-\beta+1)}\left|t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right|+\frac{N\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\beta)}\left|t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}\right|\sum_{i=1}^{m-2}\eta_i\frac{\xi_i^{\alpha+\delta-1}}{\alpha+\delta-1}+$$

$$+\frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\beta)}\left|t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}\right||A-N|.$$

Now using the fact that the functions $t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}$, $t_2^{\alpha-1}-t_1^{\alpha-1}$ and $t_2^{\alpha}-t_1^{\alpha}$ are uniformly continuous on [0,1], we conclude that $T\Omega$ is equicontinuous. Also $T\Omega$ is a uniformly bounded set. We have $T\Omega\subset\Omega$. By the Arzela – Ascoli theorem, $T\colon\Omega\to\Omega$ is completely continuous.

Lemma 3.1 is proved.

**Theorem 3.1** [9]. (Schauder fixed point theorem). *Let C be a convex subset of a Banach space, U be an open subset of C with $0\in U$. Then every completely continuous map $N\colon\overline{U}\to C$ has at least one of the following two properties:*

$(A_1)$ *N has a fixed point in $\overline{U}$*

*or*

$(A_2)$ *there is an $x\in\partial U$ and $\lambda\in(0,1)$ with $x=\lambda Nx$.*

**Theorem 3.2** [10]. *Let (X, d) be a complete metric space. $T\colon X\to X$ is a contraction map if there exist a constant $0\le k<1$ such that $d(Tx,Ty)\le kd(x,y)$. The set of fixed points of T is given by $F(T)=\{x\in X: Tx=x\}$. Then each contraction map $T\colon X\to X$ has an unique fixed point.*

**Theorem 3.3.** *Assume that $(H_1)$ and $(H_2)$ hold. If*

$$\frac{\rho}{M}\ge 1\quad \text{for}\quad \rho>0,$$

*where M is as previously defined, then the boundary-value problem* (1.1) *has a solution $u=u(t)$ such that*

$$0\le u(t)\le\rho,\quad t\in[0,1].$$

***Proof.*** Let $U=\{u\in E|\|u\|<\rho\}$. Our aim is to show that $u\ne\lambda Tu$ with $\lambda\in(0,1)$ and $u\in\partial U$. For this, let $u=\lambda Tu$ for $\lambda\in(0,1)$. Then, for $t\in[0,1]$, we obtain

$$|\lambda(Tu)(t)|<|(Tu)(t)|\le N\left[\frac{1}{\Gamma(\alpha+1)}+\frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\delta-1}+\Gamma(\alpha+\delta)}{|\Delta|(\alpha+\delta-1)}\right]+$$

$$+\frac{\Gamma(\alpha+\delta-1)|A|}{|\Delta|}\le M,$$

$$|\lambda D^{\beta}(Tu)(t)|<|D^{\beta}(Tu)(t)|\le$$

$$\le N\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\Gamma(\alpha)\left[\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha+\delta-1}+\Gamma(\alpha+\delta)\right]}{|\Delta|\Gamma(\alpha-\beta)(\alpha+\delta-1)}\right]+$$

$$+\frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)|A|}{|\Delta|\Gamma(\alpha-\beta)}=M.$$

Hence, we have $\lambda(Tu)(t)<M$ and $\lambda D^{\beta}(Tu)(t)<M$. Since $u=\lambda Tu$, we get $\rho=\|u\|=$ $=\|Tu\|<M$. From here $\dfrac{\rho}{M}<1$. This is a contradiction with our hypothesis. Then the boundary-value problem (1.1) has a solution $u=u(t)$ such that

$$0 \le u(t) \le \rho, \quad t \in [0,1].$$

Theorem 3.3 is proved.

**Theorem 3.4.** *Assume that* $(H_1)$ *and* $(H_2)$ *hold. If*

$$k < \left[ \frac{2}{\Gamma(\alpha-\beta+1)} + \frac{2\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)(\alpha+\delta-1)} \left( \sum_{i=1}^{m-2} \eta_i \left( \xi_i^{\alpha+\delta-1} + 1 \right) + \Gamma(\alpha+\delta) \right) \right]^{-1},$$

*then the boundary-value problem* (1.1) *has an unique solution.*

***Proof.*** By assumption $(H_2)$, we have following estimates:

$$\Big|(Tu)(t) - (Tv)(t)\Big| = \Bigg| \int_0^1 G(t,s) f\left(s, u(s), D^\beta u(s)\right) ds + \frac{\Gamma(\alpha+\delta-1)A}{\Delta} t^{\alpha-1} -$$

$$- \int_0^1 G(t,s) f\left(s, v(s), D^\beta v(s)\right) ds - \frac{\Gamma(\alpha+\delta-1)A}{\Delta} t^{\alpha-1} \Bigg| =$$

$$= \left| \int_0^1 G(t,s) \Big( f(s,u(s),D^\beta u(s)) - f(s,v(s),D^\beta v(s)) \Big) ds \right| =$$

$$= \int_0^1 |G(t,s)| \left| f\left(s,u(s),D^\beta u(s)\right) - f\left(s,v(s),D^\beta v(s)\right) \right| \le$$

$$\le k \int_0^1 \left( |u(s)-v(s)| + \left| D^\beta u(s) - D^\beta v(s) \right| \right) |G(t,s)| ds \le$$

$$\le k \int_0^1 \left( \max |u(s)-v(s)| + \max \left| D^\beta u(s) - D^\beta v(s) \right| \right) |G(t,s)| ds \le$$

$$\le 2k \|u(s)-v(s)\| \int_0^1 |G(t,s)| ds \le$$

$$\le 2k \|u(s) - v(s) \left[ \int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^1 \frac{t^{\alpha-1}}{|\Delta|} \sum_{j=1}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2} ds \right] \le$$

$$\le 2k \|u(s)-v(s)\| \left[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^1 \frac{\sum_{j=1}^{m-2} \eta_j (\xi_j - s)^{\alpha+\delta-2}}{|\Delta|} ds \right] <$$

$$< 2k\|u(s)-v(s)\| \left[\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(\alpha)\sum_{j=1}^{m-2}\eta_j\left(\xi_j^{\alpha+\delta-1}+1\right)+\Gamma(\alpha+\delta)}{|\Delta|(\alpha+\delta-1)\Gamma(\alpha-\beta)}\right] =$$

$$= \eta\|u(s)-v(s)\|,$$

$$\left|D^{\beta}(Tu)(t)-D^{\beta}(Tv)(t)\right| \leq \frac{k\left(|u(s)-v(s)|+\left|D^{\beta}u(s)-D^{\beta}v(s)\right|\right)}{\Gamma(\alpha-\beta)}\times$$

$$\times\left[\int_0^1(1-s)^{\alpha-\beta-1}ds+\frac{\Gamma(\alpha)}{|\Delta|}\sum_{i=1}^{m-2}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\delta-2}ds-\frac{\Gamma(\alpha+\delta-1)\Gamma(\alpha)}{|\Delta|}\int_0^1 ds\right]<$$

$$< 2k\|u(s)-v(s)\| \left[\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(\alpha)\sum_{i=1}^{m-2}\eta_i\left(\xi_i^{\alpha+\delta-1}+1\right)+\Gamma(\alpha+\delta)}{|\Delta|(\alpha+\delta-1)\Gamma(\alpha-\beta)}\right] =$$

$$= \eta\|u(s)-v(s)\|.$$

Hence, it follows that $\|(Tu)(t)-(Tv)(t)\| < \eta\|u(t)-v(t)\|$, where

$$\eta = 2k\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\beta)(\alpha+\delta-1)}\left\{\sum_{i=1}^{m-2}\eta_i\left(\xi_i^{\alpha+\delta-1}+1\right)+\Gamma(\alpha+\delta)\right\}\right]<1.$$

Therefore, by the contraction mapping principle the boundary-value problem (1.1) has an unique solution.

Theorem 3.4 is proved.

**4. Examples.**

***Example* 4.1.** Let $m=4,\ \eta_1=\dfrac{1}{4},\ \eta_2=\dfrac{1}{8},\ \alpha=\dfrac{5}{2},\ \delta=\dfrac{1}{2},\ \beta=\dfrac{1}{2},\ \xi_1=\dfrac{1}{2},\ \xi_2=1,\ A=\dfrac{1}{32},$ $f(t,u,v)=\dfrac{u+v}{8\left(1+t^2\right)}$. We consider the fractional boundary-value problem

$$D^{\frac{5}{2}}u(t)=f\left(t,u(t),D^{\frac{1}{2}}u(t)\right),\quad t\in[0,1],$$

$$u(0)=u'(0)=0, \tag{4.1}$$

$$D^{\frac{1}{2}}u(1)-\sum_{i=1}^{2}\eta_i I^{\frac{1}{2}}u'(\xi_i)=A.$$

It is clearly that $(H_1)$ and $(H_2)$ are provided. Also,

$$\frac{\rho}{M}\geq 1 \quad \text{for} \quad \rho\geq 3,9183673469.$$

Then all conditions of Theorem 3.3 hold. Hence with Theorem 3.3, the boundary-value problem (4.1) has a solution $u=u(t)$ such that

$$0\leq u(t)\leq\rho,\quad t\in[0,1].$$

***Example* 4.2.** Let $m = 4$, $\eta_1 = \dfrac{25}{4}$, $\eta_2 = \dfrac{9}{4}$, $\alpha = \dfrac{5}{2}$, $\beta = \dfrac{1}{2}$, $\delta = \dfrac{3}{2}$, $\xi_1 = \dfrac{1}{5}$, $\xi_2 = \dfrac{1}{3}$, $A \in \mathbb{R}$, $f(t, u, v) = \dfrac{e^t}{1013}(u + v)$. We consider the fractional boundary-value problem

$$D^{\frac{5}{2}} u(t) = f\left(t, u(t), D^{\frac{1}{2}} u(t)\right), \quad t \in [0,1],$$
$$u(0) = u'(0) = 0, \tag{4.2}$$
$$D^{\frac{1}{2}} u(1) - \sum_{i=1}^{2} \eta_i I^{\frac{3}{2}} u'(\xi_i) = A.$$

It is clearly that $(H_1)$ and $(H_2)$ are provided. For $u,\ \bar{u},\ v,\ \bar{v}$, we have

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq k\left(|u - \bar{u}| + |v - \bar{v}|\right) \quad \text{for} \quad k < \frac{135}{1013}.$$

Then all conditions of Theorem 3.4 hold. Hence with Theorem 3.4, the boundary-value problem (4.2) has an unique solution.

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