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A NOTE ON ITERATIVE SOLUTIONS OF AN ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATION*

ЗАУВАЖЕННЯ ЩОДО ІТЕРАЦІЙНИХ РОЗВ'ЯЗКІВ ІТЕРАТИВНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We propose an iterative method for solving the iterative functional differential equation

$$x''(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \ldots + \lambda_n x^{[n]}(t) + f(t).$$

Запропоновано ітераційний метод знаходження розв'язків ітеративного функціонально-диференціального рівняння

$$x''(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \ldots + \lambda_n x^{[n]}(t) + f(t).$$

1. Introduction. Second-order functional differential equation

$$x''(t) = H\Big(t, x(t-\tau_0(t)), x\big(t-\tau_1(t)\big), x^{[2]}(t), \ldots, x\big(t-\tau_n(t)\big)\Big)$$

has been studied in [1] and [5]. If take $\tau_i(t) = t - x^{[i-1]}(t)$, we obtain iterative functional differential equations of the form

$$x''(t) = H\Big(t, x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\Big),$$

where $x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)), \ldots,\ x^{[n]}(t) = x(x^{[n-1]}(t))$. Petahov [9] considers the iterative functional differential equation

$$x''(t) = cx(x(t))$$

and obtains an existence theorem for solutions. Later, Si and Wang [11] study

$$x''(x^{[r]}(t)) = c_0 t + c_1 x(t) + c_2 x^{[2]}(t) + \ldots + x^{[n]}(t),$$

and show the existence theorem of analytic solutions. Some various properties of solutions for several second-order iterative functional differential equations, we refer the interested reader to [12 – 16].

In this paper, we intend to determine explicit approximate solutions, with given initial values, of equations of the form

$$x''(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \ldots + \lambda_n x^{[n]}(t) + f(t). \tag{1.1}$$

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To the best of our knowledge, there are little results about approximate solutions for iterative functional differential equations. There exists several perturbative methods to determine explicit approximate solutions [6, 8, 10], most of them require a small perturbative parameter. In this paper, our iteration schemes inspired by [2–4, 7]. For convenience, we will make use $C^1(I,I)$ to denote the set of all continuous differential functions from closed interval I to I with the norm $\|x\| = \sup_{t\in I}|x(t)|$. For $M > 0$, define

$$C^1_M(I) = \Big\{\varphi \in C^1(I,I)\Big| |\varphi(t_2) - \varphi(t_1)| \le M|t_2 - t_1| \text{ for all } t,\ t_1,\ t_2 \in \mathbb{R}\Big\}.$$

It is easy to see $C^1_M(I)$ is closed convex and bounded subsets of $C^1(I,I)$.

2. Convergence of the sequence of approximate solutions. Now we will use iteration method to solve Eq. (1.1), where f is a continuous function on a domain $I = [\alpha - \delta, \alpha + \delta]$.

Lemma 2.1. *For any* $x, y \in C^1_M(I)$, $t_1, t_2 \in \mathbb{R}$, *the following inequality holds*:

$$\big\|x^{[k]} - y^{[k]}\big\| \le \sum_{j=0}^{k-1} M^j \|x - y\|, \quad k = 1, 2, \ldots . \tag{2.1}$$

Proof. It can be obtained by direct calculation by the definition of $C^1_M(I)$.

Noting the kth step equation for (1.1) is

$$x''_{k+1}(t) = \lambda_1 x_{k+1}(t) + \lambda_2 x_k^{[2]}(t) + \ldots + \lambda_n x_k^{[n]}(t) + f(t), \quad t \in [\alpha - \delta, \alpha + \delta], \tag{2.2}$$

with $x_{k+1}(\alpha) = \alpha, x'_{k+1}(\alpha) = \beta$ and $\lambda_1 < 0$, where $x_0(t)$ is an initial function, α and β are given real numbers. Integrating (2.2), we obtain

$$x_{k+1}(t) = \alpha \cos\big(\sqrt{-\lambda_1}(t-\alpha)\big) + \frac{\beta}{\sqrt{-\lambda_1}} \sin\big(\sqrt{-\lambda_1}(t-\alpha)\big) -$$

$$-\frac{1}{\sqrt{-\lambda_1}} \sum_{i=2}^{n} \lambda_i \cos\sqrt{-\lambda_1}t \int_\alpha^t x_k^{[i]}(s) \sin\sqrt{-\lambda_1}s ds -$$

$$-\frac{1}{\sqrt{-\lambda_1}} \cos\sqrt{-\lambda_1}t \int_\alpha^t f(s) \sin\sqrt{-\lambda_1}s ds +$$

$$+\frac{1}{\sqrt{-\lambda_1}} \sum_{i=2}^{n} \lambda_i \sin\sqrt{-\lambda_1}t \int_\alpha^t x_k^{[i]}(s) \cos\sqrt{-\lambda_1}s ds +$$

$$+\frac{1}{\sqrt{-\lambda_1}} \sin\sqrt{-\lambda_1}t \int_\alpha^t f(s) \cos\sqrt{-\lambda_1}s ds =$$

$$= \sqrt{\alpha^2 - \frac{\beta^2}{\lambda_1}} \sin\big(\sqrt{-\lambda_1}(\nu + t - \alpha)\big) +$$

$$+\frac{1}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}x_k^{[i]}(s)\sin\sqrt{-\lambda_1}(t-s)ds+$$

$$+\frac{1}{\sqrt{-\lambda_1}}\int\limits_{\alpha}^{t}f(s)\sin\sqrt{-\lambda_1}(t-s)ds, \tag{2.3}$$

where

$$\sin\sqrt{-\lambda_1}\nu=\frac{\alpha}{\sqrt{\alpha^2-\dfrac{\beta^2}{\lambda_1}}}.$$

Next we will show sequence $\{x_k(t)\}|_{k=1}^{\infty}$ convergent to $x(t)$ which is the solution of (1.1) if we take any x_0 satisfies $x_0(t)\in I$ for any $t\in I$.

Theorem 2.1. *Let* $I=[\alpha-\delta,\alpha+\delta]$, $\lambda_1<0$, *and the following conditions hold:*

(i)

$$\sqrt{\beta^2-\alpha^2\lambda_1}+L\delta\sum_{i=2}^{n}|\lambda_i|+L'\delta\le\min\{M,1\}, \tag{2.4}$$

where $L=\max\big\{|\alpha-\delta|,|\alpha+\delta|\big\}$;

(ii)

$$\frac{\delta}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}\sum_{j=0}^{i-1}M^j|\lambda_i|<1. \tag{2.5}$$

Then, for any $\|f\|\le L'$, (1.1) *has a solution in* $C_M^1(I)$.

Proof. First, we need $x_k\in C_M^1(I),\ k=1,2,\ldots,$ for any $t\in I$. We will prove it by induction. It is easy to find $x_1\in C_M^1(I)$ if we take any x_0 such that $x_0(t)\in I$ for any $t\in I$. Assume $x_k\in C_M^1(I),\ k\ge 2$. By (2.3), it is obvious that $x_{k+1}(\alpha)=\alpha, x'_{k+1}(\alpha)=\beta$. From (2.4), we have

$$\big|x_{k+1}(t)-\alpha\big|=\big|x_{k+1}(t)-x_{k+1}(\alpha)\big|\le$$

$$\le\left(\sqrt{\beta^2-\alpha^2\lambda_1}+L\delta\sum_{i=2}^{n}|\lambda_i|+L'\delta\right)|t-\alpha|\le\delta$$

and

$$\big|x_{k+1}(t_2)-x_{k+1}(t_1)\big|\le$$

$$\le\left(\sqrt{\beta^2-\alpha^2\lambda_1}+L\delta\sum_{i=2}^{n}|\lambda_i|+L'\delta\right)|t_2-t_1|\le M|t_2-t_1|.$$

This proves that x_{k+1} belongs to $C_M^1(I)$.

By (2.1), it is obviously that

$$\sup_{t\in[\alpha-\delta,\alpha+\delta]}\left|x_{k+1}(t)-x_k(t)\right|\leq\frac{\delta}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}|\lambda_i|\sup_{t\in[\alpha-\delta,\alpha+\delta]}\left|x_k^{[i]}(t)-x_{k-1}^{[i]}(t)\right|\leq$$

$$\leq\frac{\delta}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}\sum_{j=0}^{i-1}M^j|\lambda_i|\,\|x_k-x_{k-1}\|,$$

i.e.,

$$\|x_{k+1}-x_k\|\leq\Gamma\|x_k-x_{k-1}\|,$$

where

$$\Gamma=\frac{\delta}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}\sum_{j=0}^{i-1}M^j|\lambda_i|.$$

Therefore,

$$\|x_{k+1}-x_k\|\leq\Gamma^{k-1}\|x_2-x_1\|. \tag{2.6}$$

Now, let us go back to Eq. (1.1) and its solution $x_k(t)$ as given in (2.3). Let

$$x_m(t)=x_1(t)+\sum_{k=1}^{m-1}(x_{k+1}-x_k).$$

We shall show that $\sum_{k=1}^{\infty}\big(x_{k+1}(t)-x_k(t)\big)$ converges on the interval $[\alpha-\delta,\alpha+\delta]$. This would imply that $x_m(t)$ has a limit on this interval as $m\to\infty$. Clearly to show the convergence of $\sum_{k=1}^{\infty}\big(x_{k+1}(t)-x_k(t)\big)$. From (2.5), series

$$\sum_{k=1}^{\infty}\|x_{k+1}-x_k\|\leq\sum_{k=1}^{\infty}\Gamma^{k-1}\|x_2-x_1\|=\frac{1}{1-\Gamma}\|x_2-x_1\|$$

converges.

This shows that $\{x_m(t)\}$ is a Cauchy sequence under the supreme norm and, therefore, converges uniformly to a continuous function $x(t)$ on $[\alpha-\delta,\alpha+\delta]$. Thus,

$$x(t)=\lim_{k\to\infty}x_{k+1}(t)=\sqrt{\alpha^2-\frac{\beta^2}{\lambda_1}}\sin\Big(\sqrt{-\lambda_1}(\nu+t-\alpha)\Big)+$$

$$+\frac{1}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}\lim_{k\to\infty}x_k^{[i]}(s)\sin\sqrt{-\lambda_1}(t-s)ds+$$

$$+\frac{1}{\sqrt{-\lambda_1}}\int\limits_{\alpha}^{t}f(s)\sin\sqrt{-\lambda_1}(t-s)ds=$$

$$=\sqrt{\alpha^2-\frac{\beta^2}{\lambda_1}}\sin\Big(\sqrt{-\lambda_1}(\nu+t-\alpha)\Big)+$$

$$+\frac{1}{\sqrt{-\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}x_k^{[i]}(s)\sin\sqrt{-\lambda_1}(t-s)ds+$$

$$+\frac{1}{\sqrt{-\lambda_1}}\int\limits_{\alpha}^{t}f(s)\sin\sqrt{-\lambda_1}(t-s)ds. \qquad (2.7)$$

By direct substitution of (2.7) in (1.1), we show that $x(t)$ satisfies this equation. In addition (2.7) also shows that $x(\alpha)=\alpha$ and $x'(\alpha)=\beta$. Then $x(t)$ satisfies (1.1) along the required initial conditions. In consequence, the sequence of functions given by $S=\big\{x_0(t),x_1(t),\ldots,x_m(t)\ldots\big\}$ can be considered as approximate solutions of Eq. (1.1).

Theorem 2.1 is proved.

Now, we shall give the result for $\lambda_1>0$. Integrating (2.2), we obtain

$$x_{k+1}(t)=\frac{1}{2}\left(\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(t-\alpha)}+\left(\alpha-\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(\alpha-t)}+\right.$$

$$+\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i e^{\sqrt{\lambda_1}t}\int\limits_{\alpha}^{t}x_k^{[i]}(s)e^{-\sqrt{\lambda_1}s}ds+\frac{1}{\sqrt{\lambda_1}}e^{\sqrt{\lambda_1}t}\int\limits_{\alpha}^{t}f(s)e^{-\sqrt{\lambda_1}s}ds-$$

$$\left.-\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i e^{-\sqrt{\lambda_1}t}\int\limits_{\alpha}^{t}x_k^{[i]}(s)e^{\sqrt{\lambda_1}s}ds-\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}t}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}s}ds\right)=$$

$$=\frac{1}{2}\left(\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(t-\alpha)}+\left(\alpha-\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(\alpha-t)}+\right.$$

$$+\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}x_k^{[i]}(s)e^{\sqrt{\lambda_1}(t-s)}ds+\frac{1}{\sqrt{\lambda_1}}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}(t-s)}ds-$$

$$\left.-\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}x_k^{[i]}(s)e^{\sqrt{\lambda_1}(s-t)}ds-\frac{1}{\sqrt{\lambda_1}}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}(s-t)}ds\right). \qquad (2.8)$$

Theorem 2.2. *Let* $I=[\alpha-\delta,\alpha+\delta]$, $\lambda_1>0$, *and the following conditions hold*:

(i)

$$\sqrt{\lambda_1}\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}\delta}+\frac{1}{2\sqrt{\lambda_1}}\left(e^{\sqrt{\lambda_1}\delta}-e^{-\sqrt{\lambda_1}\delta}\right)\left(L'+L\sum_{i=2}^{n}|\lambda_i|\right)\le\min\{M,1\}, \qquad (2.9)$$

where $L=\max\big\{|\alpha-\delta|,|\alpha+\delta|\big\}$;

(ii)

$$\frac{1}{2\lambda_1}\left(e^{\sqrt{\lambda_1}(\alpha+\delta)}-e^{-\sqrt{\lambda_1}(\alpha+\delta)}\right)\sum_{i=2}^{n}\sum_{j=0}^{i-1}M^j|\lambda_i|<1. \qquad (2.10)$$

If we take $x_0\in C^1(I,I)$, *then for any* $\|f\|\le L'$, *Eq.* (1.1) *has a solution in* $C^1_M(I)$.

Proof. Similar as Theorem 2.1, if we take any $x_0 \in C^1(I,I)$, easy to see that x_1 belongs to $C^1_M(I)$. Assume $x_k(\alpha)=\alpha$ and $x'_k(\alpha)=\beta$, by (2.8), it is obviously $x_{k+1}(\alpha)=\alpha$ and $x'_{k+1}(\alpha)= =\beta$. Furthermore, by (2.9),

$$|x_{k+1}(t)-\alpha| = \left|x_{k+1}(t)-x_{k+1}(\alpha)\right| \leq$$

$$\leq \left(\sqrt{\lambda_1}\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}\delta}+\frac{1}{2\sqrt{\lambda_1}}\left(e^{\sqrt{\lambda_1}\delta}-e^{-\sqrt{\lambda_1}\delta}\right)\left(L'+L\sum_{i=2}^{n}|\lambda_i|\right)\right)|t-\alpha|\leq\delta$$

and

$$|x_{k+1}(t_2)-x_{k+1}(t_1)|\leq$$

$$\leq \left(\sqrt{\lambda_1}\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}\delta}+\frac{1}{2\sqrt{\lambda_1}}\left(e^{\sqrt{\lambda_1}\delta}-e^{-\sqrt{\lambda_1}\delta}\right)\left(L'+L\sum_{i=2}^{n}|\lambda_i|\right)\right)|t_2-t_1|\leq$$

$$\leq M|t_2-t_1|.$$

This proves that x_{k+1} belongs to $C^1_M(I)$.

Using (2.1), we see that

$$\sup_{t\in[\alpha-\delta,\alpha+\delta]}\left|x_{k+1}(t)-x_k(t)\right|\leq$$

$$\leq\frac{1}{2\lambda_1}\left(e^{\sqrt{\lambda_1}t}-e^{-\sqrt{\lambda_1}t}\right)\sum_{i=2}^{n}|\lambda_i|\sup_{t\in[\alpha-\delta,\alpha+\delta]}\left|x_k^{[i]}(t)-x_{k-1}^{[i]}(t)\right|\leq$$

$$\leq\frac{1}{2\lambda_1}\left(e^{\sqrt{\lambda_1}(\alpha+\delta)}-e^{-\sqrt{\lambda_1}(\alpha+\delta)}\right)\sum_{i=2}^{n}\sum_{j=0}^{i-1}M^j|\lambda_i|\|x_k-x_{k-1}\|,$$

i.e.,

$$\|x_{k+1}-x_k\|\leq\Gamma_1\|x_k-x_{k-1}\|,$$

where

$$\Gamma_1=\frac{1}{2\lambda_1}\left(e^{\sqrt{\lambda_1}(\alpha+\delta)}-e^{-\sqrt{\lambda_1}(\alpha+\delta)}\right)\sum_{i=2}^{n}\sum_{j=0}^{i-1}M^j|\lambda_i|.$$

Therefore,

$$\|x_{k+1}-x_k\|\leq\Gamma_1^{k-1}\|x_2-x_1\|. \tag{2.11}$$

Now, let us go back to Eq. (1.1) and its solution $x_k(t)$ as given in (2.8). Let

$$x_m(t)=x_1(t)+\sum_{k=1}^{m-1}(x_{k+1}-x_k).$$

We shall show that $\sum_{k=1}^{\infty}\big(x_{k+1}(t)-x_k(t)\big)$ converges on the interval $[\alpha-\delta,\alpha+\delta]$. This would imply that $x_m(t)$ has a limit on this interval as $m\to\infty$. Clearly to show the convergence of

$\sum_{k=1}^{\infty}\big(x_{k+1}(t)-x_k(t)\big)$. From (2.10), series

$$\sum_{k=1}^{\infty}\|x_{k+1}-x_k\|\leq\sum_{k=1}^{\infty}\Gamma_1^{k-1}\|x_2-x_1\|=\frac{1}{1-\Gamma_1}\|x_2-x_1\|$$

converges.

This shows that $\{x_m(t)\}$ is a Cauchy sequence under the supreme norm and, therefore, converges uniformly to a continuous function $x(t)$ on $[\alpha-\delta,\alpha+\delta]$. Thus,

$$x(t)=\lim_{k\to\infty}x_{k+1}(t)=$$

$$=\frac{1}{2}\Bigg(\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(t-\alpha)}+\left(\alpha-\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(\alpha-t)}+$$

$$+\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}\lim_{k\to\infty}x_k^{[i]}(s)e^{\sqrt{\lambda_1}(t-s)}ds+\frac{1}{\sqrt{\lambda_1}}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}(t-s)}ds-$$

$$-\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}\lim_{k\to\infty}x_k^{[i]}(s)e^{\sqrt{\lambda_1}(s-t)}ds-\frac{1}{\sqrt{\lambda_1}}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}(s-t)}ds\Bigg)=$$

$$=\frac{1}{2}\Bigg(\left(\alpha+\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(t-\alpha)}+\left(\alpha-\frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}(\alpha-t)}+$$

$$+\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}x^{[i]}(s)e^{\sqrt{\lambda_1}(t-s)}ds+\frac{1}{\sqrt{\lambda_1}}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}(t-s)}ds-$$

$$-\frac{1}{\sqrt{\lambda_1}}\sum_{i=2}^{n}\lambda_i\int\limits_{\alpha}^{t}x^{[i]}(s)e^{\sqrt{\lambda_1}(s-t)}ds-\frac{1}{\sqrt{\lambda_1}}\int\limits_{\alpha}^{t}f(s)e^{\sqrt{\lambda_1}(s-t)}ds\Bigg). \qquad (2.12)$$

By direct substitution of (2.12) in (1.1), we show that $x(t)$ satisfies this equation. In addition (2.12) also shows that $x(\alpha)=\alpha$ and $x'(\alpha)=\beta$. Then $x(t)$ satisfies (1.1) along the required initial conditions. In consequence, the sequence of functions given by $S=\big\{x_0(t),x_1(t),\ldots,x_m(t),\ldots\big\}$ can be considered as approximate solutions of Eq. (1.1).

Theorem 2.2 is proved.

3. Examples. In this section, some examples will be showed.

***Example* 3.1.** Now, we will show that the conditions in Theorem 2.1 do not self-contradict. Consider the equation

$$x''(t)=-25x(t)+x(x(t))+\sin t, \qquad (3.1)$$

where $\lambda_1=-25,\ \lambda_2=1,\ f(t)=\sin t$. Here, $\alpha=0,\ \beta=\frac{1}{3}$. Take $L=\delta=\frac{1}{10},\ M=1,$ $L'=1, x_0=\frac{1}{12}$, a simple calculation yields

$$\sqrt{\beta^2 - \alpha^2 \lambda_1} + L\delta|\lambda_2| + L'\delta = \frac{133}{300} \leq 1 = \min\{M, 1\},$$

and

$$\frac{\delta}{\sqrt{-\lambda_1}}|\lambda_2| \sum_{j=0}^{1} M^j = \frac{1}{25} < 1.$$

Then (2.4) and (2.5) are satisfied. By Theorem 2.1, equation (3.1) has sequence of approximate solutions $\{x_k\}, k \geq 0$, such that $\left|x_k(t_2) - x_k(t_1) = \right| \leq |t_2 - t_1| \ \forall t_1, t_2 \in \left[-\frac{1}{10}, \frac{1}{10}\right]$. Here,

$$x_{k+1}(t) = \frac{1}{15} \sin 5t + \frac{1}{5} \int\limits_0^t x_k^{[2]}(s) \sin 5(t-s) ds + \frac{1}{5} \int\limits_0^t \sin s \sin 5(t-s) ds =$$

$$= \frac{7}{120} \sin 5t + \frac{1}{24} \sin t + \frac{1}{5} \int\limits_0^t x_k^{[2]}(s) \sin 5(t-s) ds.$$

Moreover, we can find $x_k(0) = 0$ and $x_k'(0) = \frac{1}{3}$, then Eq. (3.1) has a solution in $C_M^1\left(\left[-\frac{1}{10}, \frac{1}{10}\right]\right)$.

***Example* 3.2.** Now, we will show that the conditions in Theorem 2.2 do not self-contradict. Consider the equation

$$x''(t) = 25x(t) + x(x(t)) + \sin t, \tag{3.2}$$

where $\lambda_1 = 25, \ \lambda_2 = 1, \ f(t) = \sin t$. Here, $\alpha = 0, \ \beta = \frac{1}{3}$. Take $L = \delta = \frac{1}{10}, \ M = 1, \ L' = 1$, $x_0 = \frac{1}{12}$, a simple calculation yields

$$\sqrt{\lambda_1}\left(\alpha + \frac{\beta}{\sqrt{\lambda_1}}\right) e^{\sqrt{\lambda_1}\delta} + \frac{1}{2\sqrt{\lambda_1}}\left(e^{\sqrt{\lambda_1}\delta} - e^{-\sqrt{\lambda_1}\delta}\right)\left(L' + L|\lambda_2|\right) < 0.665 \leq 1 = \min\{M, 1\}$$

and

$$\frac{1}{2\lambda_1}\left(e^{\sqrt{\lambda_1}(\alpha+\delta)} - e^{-\sqrt{\lambda_1}(\alpha+\delta)}\right)|\lambda_2|(1+M) < 0.209 < 1.$$

Then (2.9) and (2.10) are satisfied. By Theorem 2.2, equation (3.2) has sequence of approximate solutions $\{x_k\}, \ k \geq 0$ such that $\left|x_k(t_2) - x_k(t_1)\right| \leq |t_2 - t_1| \ \forall t_1, t_2 \in \left[-\frac{1}{10}, \frac{1}{10}\right]$. Here,

$$x_{k+1}(t) = \frac{1}{30}\left(e^{5t} - e^{-5t}\right) + \frac{1}{5} \int\limits_0^t \left(x_k^{[2]}(s) + \sin s\right)\left(e^{5(t-s)} - e^{5(s-t)}\right) ds.$$

Moreover, we can find $x_k(0) = 0$ and $x_k'(0) = \frac{1}{3}$, then Eq. (3.2) has a solution in $C_M^1\left(\left[-\frac{1}{10}, \frac{1}{10}\right]\right)$.

***Example* 3.3.** Consider the equation

$$x''(t) = \lambda x(t) + x(x(t)) + \sin t, \tag{3.3}$$

where $\lambda_1 = \lambda,\ \lambda_2 = 1,\ f(t) = \sin t$. Here, $\alpha = 0,\ \beta = \dfrac{1}{3}$, take $L = \dfrac{1}{10},\ \delta = \delta,\ M = M,\ L' = 1$. We shall study (3.3) with $\lambda < 0$ or $\lambda > 0$.

If $\lambda < 0$, then

$$\sqrt{\beta^2 - \alpha^2\lambda_1} + L\delta|\lambda_2| + L'\delta = \frac{1}{3} + \frac{11}{10}\delta \le \min\{M, 1\}$$

and

$$\frac{\delta}{\sqrt{-\lambda_1}}|\lambda_2|(1+M) = \frac{\delta(1+M)}{\sqrt{-\lambda}} < 1$$

or

$$0 < \delta \le \frac{10}{11}M - \frac{10}{33}, \qquad \delta < \frac{\sqrt{-\lambda}}{1+M}, \quad 0 < M < 1,$$

$$0 < \delta \le \frac{20}{33}, \qquad \delta < \frac{\sqrt{-\lambda}}{1+M}, \quad M \ge 1.$$

We see that the range of δ depend on the value of M and λ, i.e.,

$$\begin{gathered}
0 < \delta \le \frac{10}{11}M - \frac{10}{33}, \quad \text{if} \quad \lambda < -\frac{100}{1089}(1+M)^2(3M-1)^2, \quad 0 < M < 1, \\
0 < \delta < \frac{\sqrt{-\lambda}}{1+M}, \quad \text{if} \quad -\frac{100}{1089}(1+M)^2(3M-1)^2 \le \lambda < 0, \quad 0 < M < 1, \\
0 < \delta \le \frac{20}{33}, \quad \text{if} \quad \lambda < -\frac{400}{1089}(1+M)^2, \quad M \ge 1, \\
0 < \delta < \frac{\sqrt{-\lambda}}{1+M}, \quad \text{if} \quad -\frac{400}{1089}(1+M)^2 \le \lambda < 0, \quad M \ge 1.
\end{gathered} \tag{3.4}$$

Then (2.4) and (2.5) are satisfied. By Theorem 2.1, equation (3.3) has sequence of approximate solutions $\{x_k\},\ k \ge 0$, such that $\big|x_k(t_2) - x_k(t_1)\big| \le M|t_2 - t_1|\ \forall t_1, t_2 \in [-\delta, \delta]$. Here take $x_0 \in C^1\big([-\delta,\delta],[-\delta,\delta]\big)$ and

$$x_{k+1}(t) = \frac{1}{3\sqrt{-\lambda}}\sin(\sqrt{-\lambda}t) + \frac{1}{\sqrt{-\lambda}}\int\limits_0^t x_k^{[2]}(s)\sin\sqrt{-\lambda}(t-s)ds +$$

$$+ \frac{1}{\sqrt{-\lambda}}\int\limits_0^t f(s)\sin\sqrt{-\lambda}(t-s)ds.$$

Moreover, we see that $x_k(0) = 0$ and $x_k'(0) = \dfrac{1}{3}$. Then Eq. (3.3) has a solution in $C_M^1\big([-\delta,\delta]\big)$.

If $\lambda > 0$, then

$$\sqrt{\lambda_1}\left(\alpha + \frac{\beta}{\sqrt{\lambda_1}}\right)e^{\sqrt{\lambda_1}\delta} + \frac{1}{2\sqrt{\lambda_1}}\left(e^{\sqrt{\lambda_1}\delta} - e^{-\sqrt{\lambda_1}\delta}\right)\left(L' + L|\lambda_2|\right) =$$

$$= \left(\frac{1}{3} + \frac{11}{20\sqrt{\lambda}}\right)e^{\sqrt{\lambda}\delta} - \frac{11}{20\sqrt{\lambda}}e^{-\sqrt{\lambda}\delta} \le \min\{M, 1\}$$

and

$$\frac{1}{2\lambda_1}\left(e^{\sqrt{\lambda_1}(\alpha+\delta)} - e^{-\sqrt{\lambda_1}(\alpha+\delta)}\right)|\lambda_2|(1+M) = \frac{1}{2\lambda}\left(e^{\sqrt{\lambda}\delta} - e^{-\sqrt{\lambda}\delta}\right)(1+M) < 1$$

or

$$\delta < \frac{1}{\sqrt{\lambda}}\ln\frac{\sqrt{\lambda^2 + (1+M)^2} + \lambda}{1+M} = H_1(\lambda, M)$$

and

$$0 < \delta \le \frac{1}{\sqrt{\lambda}}\ln\left(\frac{30}{20 + 33\sqrt{\lambda}}\left(\sqrt{\sqrt{\lambda}M^2 + \frac{11}{15} + \frac{121}{100\sqrt{\lambda}}} + 1\right)\right) = H_2(\lambda, M), \quad 0 < M < 1,$$

$$0 < \delta \le \frac{1}{\sqrt{\lambda}}\ln\left(\sqrt{\frac{33}{33 + 20\sqrt{\lambda}} + \frac{90\lambda}{(20\sqrt{\lambda} + 33)^2}} + \frac{30\sqrt{\lambda}}{20\sqrt{\lambda} + 33}\right) = H_3(\lambda), \quad M \ge 1.$$

We see that the range of δ depend on the value of M and λ, i.e.,

$$\begin{gathered} 0 < \delta < H_1(\lambda, M), \quad \text{if} \quad H_1(\lambda, M) < H_2(\lambda, M), \quad 0 < M < 1, \\ 0 < \delta \le H_2(\lambda, M), \quad \text{if} \quad H_2(\lambda, M) < H_1(\lambda, M), \quad 0 < M < 1, \\ 0 < \delta \le H_3(\lambda), \quad \text{if} \quad H_3(\lambda) < H_1(\lambda, M), \quad M \ge 1, \\ 0 < \delta < H_1(\lambda, M), \quad \text{if} \quad H_1(\lambda, M) < H_3(\lambda), \quad M \ge 1. \end{gathered} \tag{3.5}$$

Then (2.9) and (2.10) are satisfied. By Theorem 2.2, equation (3.3) has sequence of approximate solutions $\{x_k\}$, $k \ge 0$, such that $|x_k(t_2) - x_k(t_1)| \le M|t_2 - t_1|$ $\forall t_1, t_2 \in [-\delta, \delta]$. Here take $x_0 \in C^1\big([-\delta, \delta], [-\delta, \delta]\big)$ and

$$x_{k+1}(t) = \frac{1}{6\sqrt{\lambda}}\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) + \frac{1}{2\sqrt{\lambda}}\int_0^t\left(\lambda_2 x_k^{[2]} + f(s)\right)\left(e^{\sqrt{\lambda}(t-s)} - e^{\sqrt{\lambda}(s-t)}\right)ds.$$

Moreover, we see that $x_k(0) = 0$ and $x_k'(0) = \dfrac{1}{3}$. Then Eq. (3.3) has a solution in $C_M^1\big([-\delta, \delta]\big)$.

***Remark* 3.1.** It is easy to see that $M = 1$, $\lambda = -25$ and $0 < \delta = \dfrac{1}{10} \le \dfrac{20}{33}$ in Example 3.1, satisfies the third line of (3.4). In Example 3.2,

$$M = 1, \qquad \lambda = 25,$$

$$H_1(\lambda, M) = \frac{1}{5}\ln\left(\frac{25 + \sqrt{629}}{2}\right) > 0.644 > 0.10026 > \frac{1}{5}\ln\left(\frac{1}{133}(\sqrt{4839} + 150)\right) = H_3(\lambda)$$

and $\delta = 0.1 < H_3$, satisfies the third line of (3.5).

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