DOI: 10.37863/umzh.v72i12.6035

UDC 517.5

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NEW GENERALIZED TRAPEZOID TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE AND APPLICATIONS

НОВІ УЗАГАЛЬНЕНІ НЕРІВНОСТІ ТИПУ ТРАПЕЦІЇ ДЛЯ КОМПЛЕКСНИХ ФУНКЦІЙ НА ОДИНИЧНОМУ КОЛІ ТА ЇХ ЗАСТОСУВАННЯ

We establish new generalized trapezoid type inequalities for complex functions defined on unit circle via the function of bounded variation and the functions satisfying Hölder type condition. Using these results, quadrature rule formula is also provided.

За допомогою функцій обмеженої варіації та функцій, які задовольняють умову типу Гельдера, отримано нові узагальнені нерівності типу трапеції для комплексних функцій на одиничному колі. З цих результатів також виведено квадратурну формулу.

1. Introduction. Over the past two decades, the field of inequalities for the function of bounded variation has undergone explosive growth. The many research paper related to some type inequalities such as Ostrowski, trapezoid, Gruss for the function of bounded variation have been written. Recently, some works have focused on Ostrowski and trapezoid type inequalities for complex functions defined on unit circle. Inspired by these inequalities, we will obtain some generalized trapezoid type inequalities.

The overall structure of the study takes the form of three sections including introduction. The remainder of this work is organized as follows: first we give the definitions of the function of bounded variation and total variation and present a trapezoid type inequality for complex functions defined on unit circle proved by Dragomir. In Section 2, a new generalized versions of this trapezoid inequality are obtained. We give also some special cases of these inequalities. Utilizing the results established in Section 2, quadrature rule formula is provided in Section 3.

First of all, we start to give the definitions the function of bounded variation and total variation.

Let $P\colon a=x_0<x_1<\ldots<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions.

Let f be of bounded variation on $[a,b]$ and $\sum(P)$ denotes the sum $\sum_{i=1}^{n}|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b(f):=\sup\left\{\sum(P)\,:\,P\in P([a,b])\right\}$$

is called the total variation of f on $[a,b]$. Here, $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [7], Dragomir proved following trapezoid type inequalities for complex functions defined on unit circle $C(0,1)$:

Theorem 1.1. *Assume that* $f: C(0,1) \to \mathbb{C}$ *satisfies the Hölder's type condition*

$$|f(z) - f(w)| \le H\,|z-w|^r \tag{1.1}$$

for any $w, z \in C(0,1)$, *where* $H > 0$ *and* $r \in (0,1]$ *are given.*

If $[a,b] \subseteq [0,2\pi]$ *and the function* $u\colon [a,b] \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$\left| \frac{f\left(e^{ib}\right) + f\left(e^{ia}\right)}{2}\left[u(b) - u(a)\right] - \int\limits_a^b f\left(e^{it}\right) du(t) \right| \le$$

$$\le 2^{r-1} H \max_{t\in[a,b]} B_r(a,b;t) \bigvee_a^b (u) \le \frac{1}{2^r} H (b-a)^r \bigvee_a^b (u) \tag{1.2}$$

for any $t \in [a,b]$, *where the bound* $B_r(a,b;t)$ *is given by*

$$B_r(a,b;t) := \sin^r\left(\frac{b-t}{2}\right) + \sin^r\left(\frac{t-a}{2}\right) \le \frac{1}{2^r}\left[(b-t)^r + (t-a)^r\right].$$

Ostrowski's type inequalities for complex functions defined on unit circle $C(0,1)$ was considered by Dragomir in [10] and the author give some application for unitary operators in Hilbert spaces. Recently, Dragomir proved also trapezoid type inequalities for complex functions defined on unit circle $C(0,1)$ and give some application in [9, 11]. The purpose of this paper is to obtain new generalized trapezoid type inequalities for complex functions defined on unit circle $C(0,1)$. For other inequalities for Riemann–Stieltjes integral, see [1–8, 12–17].

2. Main results. In this section, we present some generalized trapezoid type inequalities for complex functions defined on unit circle $C(0,1)$.

Theorem 2.1. *Suppose that* $f: C(0,1) \to \mathbb{C}$ *satisfies Hölder's type condition* (1.1). *If* $[a,b] \subseteq$ $\subseteq [0,2\pi]$ *and the mapping* $u\colon [a,b] \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, *then, for all* $s \in$ $\in \left[a, \frac{a+b}{2}\right]$, *we have the inequalities*

$$|T_c(f,u;a,b;s)| \le$$

$$\le 2^r H \left\{ \max_{t\in[a,s]} \sin^r\left(\frac{t-a}{2}\right) \bigvee_a^s (u) + \right.$$

$$+ \max_{t\in[s,a+b-s]} \left[\frac{\sin^r\left(\frac{t-a}{2}\right) + \sin^r\left(\frac{b-t}{2}\right)}{2} \right] \bigvee_s^{a+b-s} (u) +$$

$$\left. + \max_{t\in[a+b-s,b]} \sin^r\left(\frac{b-t}{2}\right) \bigvee_{a+b-s}^b (u) \right\} \le$$

$$\le \frac{H}{2^r}(b-a)^r \bigvee_a^b (u), \tag{2.1}$$

where $T_c(f,u;a,b;s)$ *defined by*

$$T_c(f,u;a,b;s):=f\left(e^{ib}\right)u(b)-f\left(e^{ia}\right)u(a)-$$

$$-\frac{f\left(e^{ib}\right)-f\left(e^{ia}\right)}{2}\left[u(s)+u(a+b-s)\right]-\int\limits_a^b f\left(e^{it}\right)du(t).$$

Proof. Obviously, we have the equality

$$T_c(f,u;a,b;s)=\int\limits_a^s\left[f\left(e^{ia}\right)-f\left(e^{it}\right)\right]du(t)+$$

$$+\int\limits_s^{a+b-s}\left[\frac{f\left(e^{ia}\right)+f\left(e^{ib}\right)}{2}-f\left(e^{it}\right)\right]du(t)+$$

$$+\int\limits_{a+b-s}^{b}\left[f\left(e^{ib}\right)-f\left(e^{it}\right)\right]du(t). \tag{2.2}$$

It is known that if $P\colon\ [c,d]\to\mathbb{C}$ is a continuous function and $v\colon\ [c,d]\to\mathbb{C}$ is of bounded variation, then the Riemann – Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the inequality holds

$$\left|\int\limits_c^d p(t)dv(t)\right|\leq\max_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.3}$$

Taking modulus in (2.2) and using the inequality (2.3), we have

$$|T_c(f,u;a,b;s)|\leq\max_{t\in[a,s]}\left|f\left(e^{ia}\right)-f\left(e^{it}\right)\right|\bigvee_a^s(u)+$$

$$+\max_{t\in[s,a+b-s]}\left|\frac{f\left(e^{ia}\right)+f\left(e^{ib}\right)}{2}-f\left(e^{it}\right)\right|\bigvee_s^{a+b-s}(u)+$$

$$+\max_{t\in[a+b-s,b]}\left|f\left(e^{ib}\right)-f\left(e^{it}\right)\right|\bigvee_{a+b-s}^{b}(u)\leq$$

$$\leq\max_{t\in[a,s]}\left|f\left(e^{ia}\right)-f\left(e^{it}\right)\right|\bigvee_a^s(u)+$$

$$+\frac{1}{2}\max_{t\in[s,a+b-s]}\left[\left|f\left(e^{ia}\right)-f\left(e^{it}\right)\right|+\left|f\left(e^{ib}\right)-f\left(e^{it}\right)\right|\right]\bigvee_s^{a+b-s}(u)+$$

$$+\max_{t\in[a+b-s,b]}\left|f\left(e^{ib}\right)-f\left(e^{it}\right)\right|\bigvee_{a+b-s}^{b}(u).$$

Since f satisfies the Hölder's type condition (1.1), we obtain

$$|T_c(f,u;a,b;s)| \le \max_{t\in[a+b-s,b]} \left|e^{ia}-e^{it}\right|^r \bigvee_a^s(u)+$$

$$+\frac{1}{2}\max_{t\in[s,a+b-s]}\left[\left|e^{ia}-e^{it}\right|^r+\left|e^{ib}-e^{it}\right|^r\right]\bigvee_s^{a+b-s}(u)+$$

$$+\max_{t\in[a+b-s,b]}\left|e^{ib}-e^{it}\right|\bigvee_{a+b-s}^{b}(u).$$

By using the fact that

$$\left|e^{ix}-e^{iy}\right|^2=\left|e^{ix}\right|^2-2\mathrm{Re}\big(e^{i(x-y)}\big)+\left|e^{iy}\right|^2=$$
$$=2-2\cos(x-y)=$$
$$=4\sin^2\left(\frac{x-y}{2}\right)$$

for any $x,y\in\mathbb{R}$, we have

$$\left|e^{ix}-e^{iy}\right|^r=2^r\left|\sin\left(\frac{x-y}{2}\right)\right|^r$$

for any $x,y\in\mathbb{R}$.

Since $[a,b]\subseteq[0,2\pi]$, we get

$$\left|e^{ia}-e^{it}\right|^r=2^r\sin^r\left(\frac{s-a}{2}\right)$$

and

$$\left|e^{ib}-e^{it}\right|^r=2^r\sin^r\left(\frac{b-s}{2}\right)$$

for any $s\in\left[a,\dfrac{a+b}{2}\right]$.

This completes the proof of the first inequality in (2.1).

For the proof of the second inequality in (2.1), by using the basic inequality $\sin x\le x$ for $x\in[0,\pi]$, we obtain

$$|T_c(f,u;a,b;s)|\le H\left\{\max_{t\in[a,s]}(t-a)^r\bigvee_a^s(u)+\right.$$

$$\left.+\max_{t\in[s,a+b-s]}\left[\frac{(t-a)^r+(b-t)^r}{2}\right]\bigvee_s^{a+b-s}(u)+\max_{t\in[a+b-s,b]}(b-t)^r\bigvee_{a+b-s}^{b}(u)\right\}=$$

$$=H\left\{(s-a)^r\bigvee_a^s(f)+\left(\frac{b-a}{2}\right)^r\bigvee_s^{a+b-s}(u)+(s-a)^r\bigvee_{a+b-s}^{b}(u)\right\}\le$$

$$\leq \frac{H}{2^r}(b-a)^r \bigvee_a^b(u).$$

Theorem 2.1 is proved.

***Remark* 2.1.** Under assumption of Theorem 2.1, if we take $s = a$, then the inequalities (2.1) reduce the inequalities (1.2).

***Corollary* 2.1.** *Under assumption of Theorem* 2.1 *with* $s = \dfrac{a+b}{2}$, *we have the inequality*

$$\left| f\left(e^{ib}\right)u(b) - f\left(e^{ia}\right)u(a) - \left[f\left(e^{ib}\right) - f\left(e^{ia}\right)\right] u\left(\frac{a+b}{2}\right) - \int_a^b f\left(e^{it}\right) du(t) \right| \leq$$

$$\leq 2^r H \left\{ \max_{t\in\left[a, \frac{a+b}{2}\right]} \sin^r\left(\frac{t-a}{2}\right) \bigvee_a^{\frac{a+b}{2}}(u) + \max_{t\in\left[\frac{a+b}{2}, b\right]} \sin^r\left(\frac{b-t}{2}\right) \bigvee_{\frac{a+b}{2}}^b(u) \right\} \leq$$

$$\leq \frac{H}{2^r}(b-a)^r \bigvee_a^b(u).$$

***Corollary* 2.2.** *Assume that* $f : C(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the unit circle* $C(0,1)$. *Then we get the inequality*

$$|T_c(f,u;a,b;s)| \leq 2L\left\{ \sin\left(\frac{s-a}{2}\right)\bigvee_a^s(u) + \sin\left(\frac{b-a}{4}\right)\bigvee_s^{a+b-s}(u) + \sin\left(\frac{s-a}{2}\right)\bigvee_{a+b-s}^b(u) \right\} \leq$$

$$\leq \frac{L}{2}(b-a)\bigvee_a^b(u).$$

Proof. The proof is obvious from the choosing $r = 1$ in Theorem 2.1 and the fact that

$$\sin^r\left(\frac{t-a}{2}\right) + \sin^r\left(\frac{b-t}{2}\right) = 2\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{t - \dfrac{a+b}{2}}{2}\right).$$

***Remark* 2.2.** If we take $s = a$ in Corollary 2.2, then we obtain the inequality

$$\left| \frac{f\left(e^{ib}\right) + f\left(e^{ia}\right)}{2}\left[u(b) - u(a)\right] - \int_a^b f\left(e^{it}\right) du(t) \right| \leq$$

$$\leq 2L\sin\left(\frac{b-a}{4}\right)\bigvee_a^b(u) \leq \frac{L}{2}(b-a)\bigvee_a^b(u).$$

The constant 2 in the first inequality is the best possible in the above sense.

This result is same as that given in [11].

***Remark* 2.3.** If we choose $[a,b]=[0,2\pi]$ and $s=\pi$ in Corollary 2.2, then we have the inequality

$$\left| f(1)\left[u(2\pi)-u(0)\right]-\int\limits_0^{2\pi} f\left(e^{it}\right)du(t)\right| \leq 2L\bigvee_0^{2\pi}(u),$$

which is given by Dragomir in [11].

***Corollary* 2.3.** *If* $0<b-a\leq\pi$, *then we get*

$$|T_c(f,u;a,b;s)| \leq 2^r H\sin^r\left(\frac{b-s}{2}\right)\bigvee_a^b(u) \leq 2^r H\sin^r\left(\frac{b-a}{2}\right)\bigvee_a^b(u)$$

for $s\in\left[a,\dfrac{a+b}{2}\right]$.

Proof. For $0<b-a\leq\pi$, we obtain

$$\max_{t\in[a,s]}\sin^r\left(\frac{t-a}{2}\right)=\max_{t\in[a+b-s,b]}\sin^r\left(\frac{b-t}{2}\right)=\sin^r\left(\frac{s-a}{2}\right)$$

and

$$\max_{t\in[s,a+b-s]}\left[\frac{\sin^r\left(\frac{t-a}{2}\right)+\sin^r\left(\frac{b-t}{2}\right)}{2}\right]=\sin^r\left(\frac{b-s}{2}\right).$$

By using these inequalities in inequality (2.1), we have

$$|T_c(f,u;a,b;s)| \leq$$

$$\leq 2^r H\left\{\sin^r\left(\frac{s-a}{2}\right)\bigvee_a^s(u)+\sin^r\left(\frac{b-s}{2}\right)\bigvee_s^{a+b-s}(u)+\sin^r\left(\frac{s-a}{2}\right)\bigvee_{a+b-s}^b(u)\right\}\leq$$

$$\leq 2^r H\max\left\{\sin^r\left(\frac{s-a}{2}\right),\sin^r\left(\frac{b-s}{2}\right)\right\}\bigvee_a^b(u)=$$

$$=2^r H\sin^r\left(\frac{b-s}{2}\right)\bigvee_a^b(u)\leq 2^r H\sin^r\left(\frac{b-a}{2}\right)\bigvee_a^b(u).$$

Theorem 2.2. *Suppose that* $f\colon C(0,1)\to\mathbb{C}$ *satisfies Hölder's type condition* (1.1). *If* $[a,b]\subseteq$ $\subseteq[0,2\pi]$ *and the mapping* $u\colon[a,b]\to\mathbb{C}$ *is Lipschitzian with the constant* $K>0$ *on* $[a,b]$, *then, for all* $s\in\left[a,\dfrac{a+b}{2}\right]$, *we have the inequality*

$$|T_c(f,u;a,b;s)| \leq$$

$$\leq 2^r KH\left\{\int\limits_a^s\sin^r\left(\frac{t-a}{2}\right)dt+\frac{1}{2}\int\limits_s^{a+b-s}\left[\sin^r\left(\frac{t-a}{2}\right)dt+\sin^r\left(\frac{b-t}{2}\right)\right]dt+\right.$$

$$\left.+\int\limits_{a+b-s}^b\sin^r\left(\frac{b-t}{2}\right)dt\right\}\leq KH\frac{(s-a)^{r+1}+(b-s)^{r+1}}{r+1}. \qquad (2.4)$$

Proof. Consider the fact that if $w\colon [a,b] \to \mathbb{C}$ is a Riemann integrable function and the mapping $v\colon [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, then the Riemann–Stieltjes integral $\int_a^b w(t)dv(t)$ exists and the inequality holds

$$\left|\int_a^b w(t)dv(t)\right| \leq M\int_a^b |w(t)|\,dt.$$

Since u is Lipschitzian with the constant $K > 0$, taking modulus in (2.2) and using the inequality (2.3), we have

$$|T_c(f,u;a,b;s)| =$$

$$= K\int_a^s \left|f\left(e^{ia}\right) - f\left(e^{it}\right)\right| dt + K\int_s^{a+b-s}\left|\frac{f\left(e^{ia}\right)+f(e^{ib})}{2} - f\left(e^{it}\right)\right| dt+$$

$$+K\int_{a+b-s}^b \left|f(e^{ib}) - f\left(e^{it}\right)\right| dt \leq$$

$$\leq K\int_a^s \left|f\left(e^{it}\right) - f\left(e^{ia}\right)\right| dt + \frac{K}{2}\int_s^{a+b-s}\left[\left|f\left(e^{it}\right) - f\left(e^{ia}\right)\right| + \left|f(e^{ib}) - f\left(e^{it}\right)\right|\right] dt+$$

$$+K\int_{a+b-s}^b \left|f(e^{ib}) - f\left(e^{it}\right)\right| dt.$$

As f satisfies the Hölder's type condition (1.1), we obtain

$$|T_c(f,u;a,b;s)| \leq$$

$$\leq KH\int_a^s \left|e^{it} - e^{ia}\right|^r dt + \frac{KH}{2}\int_s^{a+b-s}\left[\left|e^{it} - e^{ia}\right|^r + \left|e^{ib} - e^{it}\right|^r\right] dt+$$

$$+KH\int_{a+b-s}^b \left|e^{ib} - e^{it}\right|^r dt =$$

$$= 2^r KH\left\{\int_a^s \sin^r\left(\frac{t-a}{2}\right) dt + \frac{1}{2}\int_s^{a+b-s}\left[\sin^r\left(\frac{t-a}{2}\right) + \sin^r\left(\frac{b-t}{2}\right)\right] dt +\right.$$

$$\left. + \int_{a+b-s}^b \sin^r\left(\frac{b-t}{2}\right) dt\right\}, \tag{2.5}$$

which completes the proof of first inequality in (2.4).

For the proof of the second inequality in (2.4), by using the basic inequality $\sin x \leq x$ for $x \in [0, \pi]$, we have

$$\int\limits_a^s \sin^r\left(\frac{t-a}{2}\right) dt \leq \frac{1}{2^r}\int\limits_a^s (t-a)\,dt = \frac{(s-a)^{r+1}}{2^r(r+1)}, \tag{2.6}$$

$$\frac{1}{2}\int\limits_s^{a+b-s}\left[\sin^r\left(\frac{t-a}{2}\right)+\sin^r\left(\frac{b-t}{2}\right)\right] dt \leq \frac{1}{2^{r+1}}\int\limits_s^{a+b-s}[(t-a)+(b-t)]\,dt =$$

$$= \frac{(b-s)^{r+1}-(s-a)^{r+1}}{2^r(r+1)} \tag{2.7}$$

and

$$\int\limits_{a+b-s}^b \sin^r\left(\frac{b-t}{2}\right) dt \leq \frac{1}{2^r}\int\limits_{a+b-s}^b (b-t)\,dt = \frac{(s-a)^{r+1}}{2^r(r+1)}. \tag{2.8}$$

Substituting the inequalities (2.6) – (2.8) in (2.5), we obtain the required result (2.4).

Theorem 2.2 is proved.

***Remark* 2.4.** If we choose $s = a$ in Theorem 2.2, then we have the inequality

$$\left|\frac{f\left(e^{ib}\right)+f\left(e^{ia}\right)}{2}[u(b)-u(a)] - \int\limits_a^b f\left(e^{it}\right) du(t)\right| \leq$$

$$\leq 2^{r-1}KH\int\limits_a^b\left[\sin^r\left(\frac{t-a}{2}\right) dt + \sin^r\left(\frac{b-t}{2}\right)\right] dt \leq$$

$$\leq KH\frac{(b-a)^{r+1}}{r+1},$$

which is proved by Dragomir in [11].

***Corollary* 2.4.** *Under assumption of Theorem* 2.2 *with* $s = \frac{a+b}{2}$, *we have the inequality*

$$\left|f\left(e^{ib}\right)u(b) - f\left(e^{ia}\right)u(a) - \left[f\left(e^{ib}\right)-f\left(e^{ia}\right)\right]u\left(\frac{a+b}{2}\right) - \int\limits_a^b f\left(e^{it}\right) du(t)\right| \leq$$

$$\leq 2^r KH\left\{\int\limits_a^{\frac{a+b}{2}} \sin^r\left(\frac{t-a}{2}\right) dt + + \int\limits_{\frac{a+b}{2}}^b \sin^r\left(\frac{b-t}{2}\right) dt\right\} \leq KH\frac{(b-a)^{r+1}}{2^r\,(r+1)}.$$

Theorem 2.3. *Suppose that* $f : C\,(0,1) \to \mathbb{C}$ *satisfies Hölder's type condition* (1.1). *If* $[a,b] \subseteq$ $\subseteq [0, 2\pi]$ *and the mapping* $u\colon [a,b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$, *then, for all* $s \in \left[a, \frac{a+b}{2}\right]$, *we have the inequality*

$$|T_c(f,u;a,b;s)| \le 2^r H \left\{ \int\limits_a^s \sin^r\left(\frac{t-a}{2}\right) du(t) + \int\limits_s^{a+b-s} \left[\frac{\sin^r\left(\frac{b-t}{2}\right) + \sin^r\left(\frac{t-a}{2}\right)}{2}\right] du(t) + \right.$$

$$\left. + \int\limits_{a+b-s}^{b} \sin^r\left(\frac{b-t}{2}\right) du(t) \right\} \le$$

$$\le H \left\{ \int\limits_a^s (t-a)^r \, du(t) + \int\limits_s^{a+b-s} \left[\frac{(b-t)^r + (t-a)^r}{2}\right] du(t) + \right.$$

$$\left. + \int\limits_{a+b-s}^{b} (b-t)^r \, du(t) \right\}. \tag{2.9}$$

Proof. Consider the fact that if $w\colon [a,b] \to \mathbb{C}$ is a continuoud function and the mapping $v\colon [a,b] \to \mathbb{C}$ is monotonic nondecreasing on $[a,b]$, then the Riemann–Stieltjes integral $\int_a^b w(t)dv(t)$ exists and the inequality holds

$$\left| \int\limits_a^b w(t)dv(t) \right| \le \int\limits_a^b |w(t)| \, dv(t). \tag{2.10}$$

By using the inequality (2.10), we have, from (2.2),

$$|T_c(f,u;a,b;s)| \le \left| \int\limits_a^s \left[f\left(e^{ia}\right) - f\left(e^{it}\right)\right] du(t) \right| +$$

$$+ \left| \int\limits_s^{a+b-s} \left[\frac{f\left(e^{ia}\right) + f\left(e^{ib}\right)}{2} - f\left(e^{it}\right)\right] du(t) \right| +$$

$$+ \left| \int\limits_{a+b-s}^{b} \left[f\left(e^{ib}\right) - f\left(e^{it}\right)\right] du(t) \right| \le$$

$$\le \int\limits_a^s \left|f\left(e^{ia}\right) - f\left(e^{it}\right)\right| du(t) + \int\limits_s^{a+b-s} \left| \frac{f\left(e^{ia}\right) + f\left(e^{ib}\right)}{2} - f\left(e^{it}\right) \right| du(t) +$$

$$+ \int\limits_{a+b-s}^{b} \left|f\left(e^{ib}\right) - f\left(e^{it}\right)\right| du(t).$$

As f satisfies the Hölder's type condition (1.1), we get

$$|T_c(f,u;a,b;s)| \le H\int_a^s \left|e^{ia}-e^{it}\right|^r du(t) + \frac{H}{2}\int_s^{a+b-s}\left[\left|e^{ia}-e^{it}\right|^r + \left|e^{ib}-e^{it}\right|^r\right] du(t)+$$

$$+H\int_{a+b-s}^{b} \left|e^{ib}-e^{it}\right|^r du(t) \le$$

$$\le 2^r H\int_a^s \sin^r\left(\frac{t-a}{2}\right) du(t) + 2^{r-1}H\int_s^{a+b-s}\left[\sin^r\left(\frac{b-t}{2}\right)+\sin^r\left(\frac{t-a}{2}\right)\right] du(t)+$$

$$+2^r H\int_{a+b-s}^{b} \sin^r\left(\frac{b-t}{2}\right) du(t),$$

which completes the proof of first inequality in (2.9). The proof of second inequality in (2.9) is obvious from the fact that $\sin x \le x$ for $x \in [0,\pi]$.

Theorem 2.3 is proved.

***Remark* 2.5.** If we choose $s=a$ in Theorem 2.3, then we have the inequality

$$|T_c(f,u;a,b)| \le 2^{r-1}H\int_a^b \left[\sin^r\left(\frac{b-t}{2}\right)+\sin^r\left(\frac{t-a}{2}\right)\right] du(t) \le$$

$$\le \frac{H}{2}\int_s^{a+b-s} \left[(b-t)^r + (t-a)^r\right] du(t),$$

which is proved by Dragomir in [11].

***Corollary* 2.5.** *Under assumption of Theorem* 2.3 *with* $s=\dfrac{a+b}{2}$, *we have the inequality*

$$\left| f\left(e^{ib}\right)u(b) - f\left(e^{ia}\right)u(a) - \left[f\left(e^{ib}\right)-f\left(e^{ia}\right)\right]u\left(\frac{a+b}{2}\right) - \int_a^b f\left(e^{it}\right)du(t)\right| \le$$

$$\le 2^r H\left\{\int_a^{\frac{a+b}{2}} \sin^r\left(\frac{t-a}{2}\right) du(t) + \frac{a+b}{2}\sin^r\left(\frac{b-t}{2}\right) du(t)\right\} \le$$

$$\le H\left\{\int_a^{\frac{a+b}{2}} (t-a)^r du(t) + \int_{\frac{a+b}{2}}^{b} (b-t)^r du(t)\right\}.$$

3. Application to quadrature rule. We now introduce the intermediate points

$$\xi_k \in \left[x_k, \frac{x_k + x_{k+1}}{2}\right], \qquad k = 0,1,\ldots,n-1,$$

in the partition $\Delta_n : a = x_0 < x_1 < \ldots < x_n = b$. Let $h_k : = x_{k+1} - x_k$ and $v(h) = \max\{h_k : k = 0, 1, \ldots, n-1\}$ and define the sum

$$T(f,u,\Delta_n,\xi) := \sum_{k=0}^{n-1} \left\{ f\left(e^{ix_{k+1}}\right) u(x_{k+1}) - f\left(e^{ix_k}\right) u(x_k) \right\} -$$

$$-\frac{1}{2} \sum_{k=0}^{n-1} \left\{ \left[f\left(e^{ix_{k+1}}\right) - f\left(e^{ix_k}\right) \right] \left[u(\xi_k) + u(x_k + x_{k+1} - \xi_k) \right] \right\}.$$

Then the following theorem holds.

Theorem 3.1. *Let f and u be as in Theorem* 2.1. *Then*

$$\int_a^b f\left(e^{it}\right) du(t) = T(f,u,\Delta_n,\xi) + R(f,u,\Delta_n,\xi),$$

where $T(f,u,\Delta_n,\xi)$ is defined as above and the remainder term $R(f,u,\Delta_n,\xi)$ satisfies

$$|R(f,u,\Delta_n,\xi)| \le 2^r H \sin^r\left(\frac{v(h)}{2}\right) \bigvee_a^b(u) \le H v^r(h) \bigvee_a^b(u).$$

Proof. Since $v(h) \le \pi$, then application of Corollary 2.3 to the interval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, for intermediate points ξ_k, we have

$$\left| f\left(e^{ix_{k+1}}\right) u(x_{k+1}) - f\left(e^{ix_k}\right) u(x_k) - \right.$$

$$\left. - \left[\frac{f\left(e^{ix_{k+1}}\right) - f\left(e^{ix_k}\right)}{2} \right] \left[u(\xi_k) + u(x_k + x_{k+1} - \xi_k) \right] - \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) \right| \le$$

$$\le 2^r H \sin^r\left(\frac{h_k}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \tag{3.1}$$

for all $k \in \{0, 1, \ldots, n-1\}$.

Summing the inequality (3.1) over k from 0 to $n-1$ and using the generalized triangle inequality, we get

$$|R(f,u,\Delta_n,\xi)| \le 2^r H \sum_{k=0}^{n-1} \sin^r\left(\frac{h_k}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \le$$

$$\le 2^r H \max_{k\in\{0,1,..,n-1\}} \sin^r\left(\frac{h_k}{2}\right) \sum_{k=0}^{n-1} \bigvee_{x_k}^{x_{k+1}}(u) \le$$

$$\le 2^r H \sin^r\left(\frac{v(h)}{2}\right) \bigvee_a^b(u) \le H v^r(h) \bigvee_a^b(u),$$

which completes the proof.

***Remark* 3.1.** Choosing $\xi_k = x_k$ in Theorem 3.1, we obtain

$$\int\limits_a^b f\left(e^{it}\right) du(t) = T(f,u,\Delta_n) + R(f,u,\Delta_n)$$

and

$$|R(f,u,\Delta_n,\xi)| \leq 2^r H \sin^r \left(\frac{v(h)}{2}\right) \bigvee_a^b (u) \leq H v^r(h) \bigvee_a^b (u)$$

with the sum

$$T(f,u,\Delta_n) := \frac{1}{2} \sum_{k=0}^{n-1} \left\{ \left[f\left(e^{ix_{k+1}}\right) + f\left(e^{ix_k}\right) \right] [u x_{k+1}) - u(x_k)] \right\}$$

given by Dragomir in [11].

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Received 07.01.18