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FRACTIONAL TRAPEZIUM-LIKE INEQUALITIES INVOLVING GENERALIZED RELATIVE SEMI-(m, h_1, h_2)-PREINVEX MAPPINGS ON AN m-INVEX SET

ДРОБОВІ НЕРІВНОСТІ ТИПУ ТРАПЕЦІЇ З УЗАГАЛЬНЕНИМИ ВІДНОСНО НАПІВ-(m, h_1, h_2)-ПРЕІНВЕКСНИМИ ВІДОБРАЖЕННЯМИ НА m-ІНВЕКСНІЙ МНОЖИНІ

The authors derive a fractional integral equality concerning twice differentiable mappings defined on m-invex set. By using this identity, the authors obtain new estimates on generalization of trapezium-like inequalities for mappings whose second order derivatives are generalized relative semi-(m, h_1, h_2)-preinvex via fractional integrals. We also discuss some new special cases which can be deduced from our main results.

Встановлено дробову інтегральну рівність для двічі диференційовних відображень на m-інвексній множині. За допомогою цієї рівності отримано нові оцінки для узагальнених нерівностей типу трапеції для відображень, у яких похідні другого порядку є узагальненими відносно напів-(m, h_1, h_2)-преінвексними через дробові інтеграли. Також обговорено деякі нові спеціальні випадки, що випливають з отриманих результатів.

1. Introduction. In [19], Sarikaya et al. established the following interesting Hadamard-type inequalities by using Riemann – Liouville fractional integrals.

Theorem 1.1. *Let $f : [u, v] \to \mathbb{R}$ be a positive mapping along with $0 \le u < v$ and let f belongs to $L^1[u, v]$. Suppose that f is a convex function on $[u, v]$, then the following double inequalities for fractional integrals hold:*

$$f\left(\frac{u+v}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[J^\alpha_{u^+} f(v) + J^\alpha_{v^-} f(u)\right] \le \frac{f(u)+f(v)}{2}, \tag{1.1}$$

where the symbols $J^\alpha_{u^+} f$ and $J^\alpha_{v^-} f$ denote, respectively, the left- and right-sided Riemann – Liouville fractional integrals of order $\alpha \in \mathbb{R}^+$ defined by

$$J^\alpha_{u^+} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_u^x (x-t)^{\alpha-1} f(t)dt, \qquad u < x,$$

and

$$J^\alpha_{v^-} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^v (t-x)^{\alpha-1} f(t)dt, \qquad x < v.$$

Here, $\Gamma(\alpha)$ is the gamma function and its definition is $\Gamma(\alpha) = \displaystyle\int_0^\infty e^{-\mu}\mu^{\alpha-1} d\mu$.

For $\alpha = 1$, the inequality (1.1) reduces to the successive Hadamard-type inequality

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int\limits_u^v f(x)dx \leq \frac{f(u)+f(v)}{2}, \tag{1.2}$$

where $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping on the interval I of real numbers and $u, v \in I$ with $u < v$. This inequality (1.2) is also named as trapezium inequality.

In recent years, many researchers have studied error bounds with respect to the inequality (1.2); for refinements, counterparts, generalization please refer to [1, 4, 6, 11, 12, 16, 21, 22, 24] and references cited therein.

More integral inequalities via fractional integrals may be seen in [2, 7 – 10, 13, 14, 18, 20].

Our goal is to establish, employing the Riemann – Liouville fractional calculus, some new left-sided Hadamard-type integral inequalities. We deal with mappings which have absolute values of the second derivatives which are generalized relative semi-(m, h_1, h_2)-preinvex. These inequalities obtained in this paper can be viewed as generalization of the results of [14] and [15].

To end this section, we evoke some special functions and definitions as follows:

(1) the beta function

$$\beta(x,y) = \int\limits_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x, y > 0,$$

(2) the hypergeometric function

$${}_2F_1(x,y;c;z) = \frac{1}{\beta(y,c-y)}\int\limits_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}dt$$

for $|z| < 1,\ c > y > 0$.

Definition 1.1 [5]. *A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta\colon K \times \times K \times (0,1] \to \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Definition 1.2 [17]. *Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta\colon K \times K \times (0,1] \to \mathbb{R}$ and let $h_1, h_2\colon [0,1] \to \mathbb{R}_0$. A function $f\colon K \to \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex if the inequality*

$$f\left(mx + t\eta(y,x,m)\right) \leq mh_1(t)f(x) + h_2(t)f(y)$$

is valid for all $x, y \in K$ and $t \in [0,1]$.

Definition 1.3 [23]. *A set $M_\varphi \subseteq \mathbb{R}^n$ is named a relative convex (φ-convex) set if and only if there exists a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi \quad \forall x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, \quad t \in [0,1].$$

Definition 1.4 [23]. *A function f is named a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ if and only if there exists a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y))$$

$$\forall x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, \quad t \in [0,1].$$

Definition 1.5 [3]. *A function f is said to be a relative semiconvex on a relative convex set M_φ if and only if there exists a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(x) + (1-t)f(y) \qquad \forall x, y \in M_\varphi, \quad t \in [0,1].$$

2. Auxiliary results. We begin with the following new definition.

Definition 2.1. *Let $K \subseteq \mathbb{R}$ be an open nonempty m-invex set with respect to the mapping $\eta\colon K \times K \times (0,1] \to \mathbb{R}$ and $\varphi\colon I \to K$ is a continuous function. A function $f: K \to \mathbb{R}$, $h_1, h_2 \colon [0,1] \to \mathbb{R}_0$ is said to be generalized relative semi-(m, h_1, h_2)-preinvex functions if*

$$f\left(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\right) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$ with some fixed $m \in (0,1]$.

***Remark* 2.1.** In Definition 2.1, let us discuss some special cases as follows:

(I) if we take $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Breckner-preinvex functions,

(II) if we take $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m,P)-preinvex functions,

(III) if we take $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova – Levin – Dragomir-preinvex functions,

(IV) if we take $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we get generalized relative semi-(m,h)-preinvex functions,

(V) if we take $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex functions,

(VI) if we take $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex functions.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Throughout of this paper, let $\varphi\colon I \subseteq \mathbb{R} \to K$ be a continuous function with $a,\ b \in I,\ \varphi(a) < \varphi(b)$ and let $K \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta\colon K \times K \times (0,1] \to \mathbb{R}$ for some fixed $m \in (0,1]$ with $0 < \eta(\varphi(b), \varphi(a), m)$. Suppose that $f: K \to \mathbb{R}$ be a twice differentiable function on the interior K° of K and $f'' \in L^1\left[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)\right]$. Before stating the results we use following notations:

$$G(\alpha; n, m, \varphi(a), \varphi(b))(f) = \frac{(n+1)^\alpha \Gamma(\alpha+2)}{\eta^\alpha(\varphi(b), \varphi(a), m)} \left[J^\alpha_{(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b),\varphi(a),m))^-} f(m\varphi(a)) + \right.$$

$$\left. + J^\alpha_{(m\varphi(a) + \frac{n}{n+1}\eta(\varphi(b),\varphi(a),m))^+} f\left(m\varphi(a) + \eta(\varphi(b), \varphi(a), m)\right) \right] -$$

$$- \frac{\eta(\varphi(b), \varphi(a), m)}{n+1} \left[f'\left(m\varphi(a) + \frac{n}{n+1}\eta(\varphi(b), \varphi(a), m) \right) - \right.$$

$$\left. - f'\left(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b), \varphi(a), m) \right) \right] -$$

$$-(\alpha+1) \left[f\left(m\varphi(a) + \frac{n}{n+1}\eta(\varphi(b), \varphi(a), m) \right) + f\left(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b), \varphi(a), m) \right) \right].$$

Lemma 2.1. *We have the following identity for fractional integrals along with* $x \in [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $\alpha > 0$ *and* $n \in \mathbb{N}^+$:

$$G(\alpha; n, m, \varphi(a), \varphi(b))(f) =$$

$$= \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2} \int_0^1 (1-t)^{\alpha+1} \left[f''\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b),\varphi(a),m)\right) + \right.$$

$$\left. + f''\left(m\varphi(a) + \frac{n+t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right] dt. \tag{2.2}$$

Proof. Let

$$I^* = \int_0^1 (1-t)^{\alpha+1} \left[f''\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b),\varphi(a),m)\right) + \right.$$

$$\left. + f''\left(m\varphi(a) + \frac{n+t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right] dt =$$

$$= \int_0^1 (1-t)^{\alpha+1} f''\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b),\varphi(a),m)\right) dt +$$

$$+ \int_0^1 (1-t)^{\alpha+1} f''\left(m\varphi(a) + \frac{n+t}{n+1}\eta(\varphi(b),\varphi(a),m)\right) dt := I_1 + I_2. \tag{2.3}$$

Integrating I_1 on $[0,1]$ yields

$$I_1 = \frac{n+1}{\eta(\varphi(b),\varphi(a),m)} f'\left(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b),\varphi(a),m)\right) -$$

$$- \frac{(n+1)^2(\alpha+1)}{\eta^2(\varphi(b),\varphi(a),m)} f\left(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b),\varphi(a),m)\right) +$$

$$+ \frac{(n+1)^2\alpha(\alpha+1)}{\eta^2(\varphi(b),\varphi(a),m)} \int_0^1 (1-t)^{\alpha-1} f\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b),\varphi(a),m)\right) dt =$$

$$= \frac{n+1}{\eta(\varphi(b),\varphi(a),m)} f'\left(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b),\varphi(a),m)\right) -$$

$$- \frac{(n+1)^2(\alpha+1)}{\eta^2(\varphi(b),\varphi(a),m)} f\left(m\varphi(a) + \frac{1}{n+1}\eta(\varphi(b),\varphi(a),m)\right) +$$

$$+ \frac{(n+1)^{\alpha+2}\Gamma(\alpha+2)}{\eta(\varphi(b),\varphi(a),m)^{\alpha+2}} J^{\alpha}_{(m\varphi(a)+\frac{1}{n+1}\eta(\varphi(b),\varphi(a),m))^-} f(m\varphi(a)). \tag{2.4}$$

Analogously, integrating I_2 on $[0,1]$, we have

$$I_2 = -\frac{n+1}{\eta(\varphi(b), \varphi(a), m)} f'\left(m\varphi(a) + \frac{n}{n+1}\eta(\varphi(b), \varphi(a), m)\right) -$$

$$-\frac{(n+1)^2(\alpha+1)}{\eta^2(\varphi(b), \varphi(a), m)} f\left(m\varphi(a) + \frac{n}{n+1}\eta(\varphi(b), \varphi(a), m)\right) +$$

$$+\frac{(n+1)^{\alpha+2}\Gamma(\alpha+2)}{\eta(\varphi(b), \varphi(a), m)^{\alpha+2}} J^{\alpha}_{(m\varphi(a)+\frac{n}{n+1}\eta(\varphi(b),\varphi(a),m))+} f(m\varphi(a) + \eta(\varphi(b), \varphi(a), m)). \qquad (2.5)$$

Applying (2.4) and (2.5) to (2.3), then multiplying both sides by $\dfrac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}$ ends the proof.

***Remark* 2.2.** If $\eta(\varphi(b), \varphi(a), m) = \varphi(b) - m\varphi(a)$ with $m = 1$ and φ is an identity mapping, then Lemma 2.1 reduces to Lemma 1.3 in [14]. Further, if we put $n = 1$, then we obtain Lemma 1 in [15].

3. Main results. Using Lemma 2.1, we now state the following theorem.

Theorem 3.1. *If $|f''|^q$ for $q \geq 1$ is generalized relative semi-(m, h_1, h_2)-preinvex, then the following inequality for Riemann–Liouville fractional integrals along with $h_1,\ h_2 \colon [0,1] \to \mathbb{R}_0$, $\alpha > 0$ and $n \in \mathbb{N}^+$ exists:*

$$\Big|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\Big| \leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times$$

$$\times\left\{\left[\int_0^1 (1-t)^{\alpha+1}\left(mh_1\left(\frac{1-t}{n+1}\right)\Big|f''(a)\Big|^q + h_2\left(\frac{1-t}{n+1}\right)\Big|f''(b)\Big|^q\right)dt\right]^{\frac{1}{q}} +\right.$$

$$\left.+\left[\int_0^1 (1-t)^{\alpha+1}\left(mh_1\left(\frac{n+t}{n+1}\right)\Big|f''(a)\Big|^q + h_2\left(\frac{n+t}{n+1}\right)\Big|f''(b)\Big|^q\right)dt\right]^{\frac{1}{q}}\right\}. \qquad (3.1)$$

Proof. Using given hypothesis, Lemma 2.1 and the power mean inequality, we have

$$\Big|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\Big| \leq$$

$$\leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}\int_0^1 (1-t)^{\alpha+1}\left|f''\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b), \varphi(a), m)\right)\right|dt +$$

$$+\frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}\int_0^1 (1-t)^{\alpha+1}\left|f''\left(m\varphi(a) + \frac{n+t}{n+1}\eta(\varphi(b), \varphi(a), m)\right)\right|dt \leq$$

$$\leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}\left(\int_0^1 (1-t)^{\alpha+1}dt\right)^{1-\frac{1}{q}} \times$$

$$\times\left[\int_0^1 (1-t)^{\alpha+1}\left|f''\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b), \varphi(a), m)\right)\right|^q dt\right]^{\frac{1}{q}} +$$

$$+\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\int_0^1(1-t)^{\alpha+1}dt\right)^{1-\frac{1}{q}}\times$$

$$\times\left[\int_0^1(1-t)^{\alpha+1}\left|f''\left(m\varphi(a)+\frac{n+t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right|^q dt\right]^{\frac{1}{q}}\leq$$

$$\leq\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\times$$

$$\times\left\{\left[\int_0^1(1-t)^{\alpha+1}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f''(a)\right|^q+h_2\left(\frac{1-t}{n+1}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[\int_0^1(1-t)^{\alpha+1}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f''(a)\right|^q+h_2\left(\frac{n+t}{n+1}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\},$$

which completes the proof.

***Remark* 3.1.** In Theorem 3.1, if we take $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ and φ is an identity mapping along with $\eta(\varphi(b),\varphi(a),m)=\varphi(b)-m\varphi(a)$, $m=1$, then we have:

(a) for $q=1$, we get Theorem 2.1 in [14], specially, for $n=1$, we obtain Theorem 2 in [15],

(b) for $n=1$, we have Theorem 4 in [15],

(c) for $n>1$, we get Theorem 2.4 in [14].

***Corollary* 3.1.** *In Theorem* 3.1, *if we put* $h_1(t)=(1-t)^{-s}$ *and* $h_2(t)=t^{-s}$, *then we have:*

(1) *for* $n>1$, *the following inequality for generalized relative semi-*(m,s)*-Godunova – Levin – Dragomir-preinvex functions holds:*

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right|\leq\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^{2-\frac{s}{q}}}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\times$$

$$\times\left\{\left[\frac{mn^{-s}{}_2F_1\left[s,1;\alpha+3;-\frac{1}{n}\right]}{\alpha+2}\left|f''(a)\right|^q+\frac{1}{\alpha-s+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[\frac{m}{\alpha-s+2}\left|f''(a)\right|^q+\frac{n^{-s}{}_2F_1\left[s,1;\alpha+3;-\frac{1}{n}\right]}{\alpha+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},$$

(2) *for* $n=1$, *the following inequality for generalized relative semi-*(m,s)*-Godunova – Levin – Dragomir-preinvex functions exists:*

$$\frac{1}{2(\alpha+1)}\left|G(\alpha;1,m,\varphi(a),\varphi(b))(f)\right|=$$

$$= \left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} \left[J^{\alpha}_{(m\varphi(a)+\frac{1}{2}\eta(\varphi(b),\varphi(a),m))^-} f(m\varphi(a)) + \right.\right.$$

$$\left.\left. + J^{\alpha}_{(m\varphi(a)+\frac{1}{2}\eta(\varphi(b),\varphi(a),m))^+} f\left(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)\right)\right] - \left[f\left(m\varphi(a) + \frac{1}{2}\eta(\varphi(b),\varphi(a),m)\right)\right] \right| \le$$

$$\le \frac{\eta^2(\varphi(b),\varphi(a),m)}{2^{3-\frac{s}{q}}(\alpha+1)} \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times$$

$$\times \left\{ \left[m\mathcal{C}_0(s,\alpha,t)\left|f''(a)\right|^q + \frac{1}{\alpha-s+2}\left|f''(b)\right|^q \right]^{\frac{1}{q}} + \left[\frac{m}{\alpha-s+2}\left|f''(a)\right|^q + \mathcal{C}_0(s,\alpha,t)\left|f''(b)\right|^q \right]^{\frac{1}{q}} \right\},$$

where

$$\mathcal{C}_0(s,\alpha,t) = \int\limits_0^1 (1-t)^{\alpha+1}(1+t)^{-s}dt.$$

***Corollary* 3.2.** *In Theorem* 3.1, *if we take* $h_1(t) = h_2(t) = t(1-t)$, *then we have the following inequality for generalized relative semi-*(m, tgs)*-preinvex functions:*

$$\left|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\right| \le \frac{2\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^{2+\frac{2}{q}}} \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times$$

$$\times \left[\frac{n}{\alpha+3} + \frac{1}{(\alpha+3)(\alpha+4)}\right]^{\frac{1}{q}} \left(m\left|f''(a)\right|^q + \left|f''(b)\right|^q \right)^{\frac{1}{q}}.$$

***Corollary* 3.3.** *In Theorem* 3.1, *if we take* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *then we have:*

(1) *for* $n > 1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\left|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\right| \le \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2} \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times$$

$$\times \left\{ \left[\frac{mn^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;\alpha+\frac{5}{2};-\frac{1}{n}\right]}{2\alpha+3}\left|f''(a)\right|^q + \frac{n^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;\alpha+\frac{7}{2};-\frac{1}{n}\right]}{2\alpha+5}\left|f''(b)\right|^q \right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\frac{mn^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;\alpha+\frac{7}{2};-\frac{1}{n}\right]}{2\alpha+5}\left|f''(a)\right|^q + \frac{n^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;\alpha+\frac{5}{2};-\frac{1}{n}\right]}{2\alpha+3}\left|f''(b)\right|^q \right]^{\frac{1}{q}} \right\},$$

(2) *for* $n = 1$, *the following inequality for generalized relative semi-*m-MT*-preinvex functions holds:*

$$\frac{1}{2(\alpha+1)}\Big|G(\alpha;1,m,\varphi(a),\varphi(b))(f)\Big| \leq \frac{\eta^2(\varphi(b),\varphi(a),m)}{2^{3+\frac{1}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times$$

$$\times\left\{\Big[m\mathcal{C}_1(\alpha,t)\Big|f''(a)\Big|^q + \mathcal{C}_2(\alpha,t)\Big|f''(b)\Big|^q\Big]^{\frac{1}{q}} + \Big[m\mathcal{C}_2(\alpha,t)\Big|f''(a)\Big|^q + \mathcal{C}_1(\alpha,t)\Big|f''(b)\Big|^q\Big]^{\frac{1}{q}}\right\},$$

where $\mathcal{C}_1(\alpha,t) = \int_0^1 (1-t)^{\alpha+\frac{1}{2}}(1+t)^{\frac{1}{2}}dt$ *and* $\mathcal{C}_2(\alpha,t) = \int_0^1 (1-t)^{\alpha+\frac{3}{2}}(1+t)^{-\frac{1}{2}}dt.$

Now, we are ready to state the second theorem in this section.

Theorem 3.2. *If* $|f''|^q$ *for* $q > 1$ *is generalized relative semi-*(m, h_1, h_2)*-preinvex with* $\frac{1}{q} + \frac{1}{p} = 1$, *then the following inequality for Riemann–Liouville fractional integrals along with* $h_1, h_2 : [0,1] \to \mathbb{R}_0$, $\alpha > 0$ *and* $n \in \mathbb{N}^+$ *exists:*

$$\Big|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\Big| \leq \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \times$$

$$\times\left\{\left[\int_0^1 mh_1\left(\frac{1-t}{n+1}\right)dt\Big|f''(a)\Big|^q + \int_0^1 h_2\left(\frac{1-t}{n+1}\right)dt\Big|f''(b)\Big|^q\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\int_0^1 mh_1\left(\frac{n+t}{n+1}\right)dt\Big|f''(a)\Big|^q + \int_0^1 h_2\left(\frac{n+t}{n+1}\right)dt\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}\right\}. \tag{3.2}$$

Proof. Using given hypothesis, Lemma 2.1 and Hölder's inequality, we have

$$\Big|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\Big| \leq$$

$$\leq \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\int_0^1 (1-t)^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\left(m\varphi(a) + \frac{1-t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} +$$

$$+\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\int_0^1 (1-t)^{(\alpha+1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\left(m\varphi(a) + \frac{n+t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \times$$

$$\times\left\{\left[\int_0^1 mh_1\left(\frac{1-t}{n+1}\right)dt\Big|f''(a)\Big|^q + \int_0^1 h_2\left(\frac{1-t}{n+1}\right)dt\Big|f''(b)\Big|^q\right]^{\frac{1}{q}} + \right.$$

$$+\left[\int_0^1 mh_1\left(\frac{n+t}{n+1}\right)dt\Big|f''(a)\Big|^q+\int_0^1 h_2\left(\frac{n+t}{n+1}\right)dt\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}\Bigg\},$$

which completes the proof.

***Remark* 3.2.** In Theorem 3.2, if we take $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ and φ is an identity mapping along with $\eta(\varphi(b), \varphi(a), m) = \varphi(b) - m\varphi(a)$, $m = 1$, then we obtain Theorem 2.2 in [14], specially, for $n = 1$, we get Theorem 3 in [15].

***Corollary* 3.4.** *In Theorem* 3.2, *if we put* $h_1(t) = h_2(t) = t(1-t)$, *then we have the following inequality for generalized relative semi-*(m, tgs)*-preinvex functions:*

$$\Big|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\Big| \leq \frac{2\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^{2+\frac{2}{q}}}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{3n+1}{6}\right)^{\frac{1}{q}} \times$$

$$\times\left(m\Big|f''(a)\Big|^q+\Big|f''(b)\Big|^q\right)^{\frac{1}{q}}.$$

***Corollary* 3.5.** *In Theorem* 3.2, *if we put* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *then we have:*

(1) *for* $n > 1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\Big|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\Big| \leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \times$$

$$\times\left\{\left(mn^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2}, 1; \frac{3}{2}; -\frac{1}{n}\right]\Big|f''(a)\Big|^q+\frac{1}{3}n^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2}, 1; \frac{5}{2}; -\frac{1}{n}\right]\Big|f''(b)\Big|^q\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\frac{1}{3}mn^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2}, 1; \frac{5}{2}; -\frac{1}{n}\right]\Big|f''(a)\Big|^q+n^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2}, 1; \frac{3}{2}; -\frac{1}{n}\right]\Big|f''(b)\Big|^q\right)^{\frac{1}{q}}\right\},$$

(2) *for* $n = 1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\frac{1}{2(\alpha+1)}\Big|G(\alpha; 1, m, \varphi(a), \varphi(b))(f)\Big| \leq$$

$$\leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{2^{3+\frac{1}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left\{\left[m\left(\frac{\pi}{2}+1\right)\Big|f''(a)\Big|^q+\left(\frac{\pi}{2}-1\right)\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[m\left(\frac{\pi}{2}-1\right)\Big|f''(a)\Big|^q+\left(\frac{\pi}{2}+1\right)\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}\right\}.$$

Now, we are ready to state the third theorem in this section.

Theorem 3.3. *Under the assumptions of Thereom* 3.2, *then the following inequality for Riemann–Liouville fractional integrals with* $\alpha > 0$ *and* $n \in \mathbb{N}^+$ *holds:*

$$\Big|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\Big| \leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}} \times$$

$$\times\left\{\left[\int_0^1 (1-t)^{p(\alpha+1)}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f''(a)\right|^q + h_2\left(\frac{1-t}{n+1}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}} + \right.$$

$$\left.+\left[\int_0^1 (1-t)^{p(\alpha+1)}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f''(a)\right|^q + h_2\left(\frac{n+t}{n+1}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\}. \quad (3.3)$$

Proof. Using given hypothesis, Lemma 2.1 and Hölder's inequality, we have

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right| \le \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\int_0^1 \left[(1-t)^{(\alpha+1)}\right]^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}} \times$$

$$\times\left(\int_0^1 \left[(1-t)^{\alpha+1}\right]^p\left|f''\left(m\varphi(a)+\frac{1-t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} +$$

$$+\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\int_0^1 \left[(1-t)^{(\alpha+1)}\right]^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}} \times$$

$$\times\left(\int_0^1 \left[(1-t)^{\alpha+1}\right]^p\left|f''\left(m\varphi(a)+\frac{n+t}{n+1}\eta(\varphi(b),\varphi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}} \le$$

$$\le \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}} \times$$

$$\times\left\{\left[\int_0^1 (1-t)^{p(\alpha+1)}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f''(a)\right|^q + h_2\left(\frac{1-t}{n+1}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}} + \right.$$

$$\left.+\left[\int_0^1 (1-t)^{p(\alpha+1)}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f''(a)\right|^q + h_2\left(\frac{n+t}{n+1}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\},$$

which completes the proof.

***Corollary* 3.6.** *In Theorem* 3.3, *if we put* $h_1(t) = (1-t)^s$ *and* $h_2(t) = t^s$, *then we have:*

(1) *for* $n > 1$, *the following inequality for generalized relative semi-*(m, s)*-Breckner-preinvex functions holds:*

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right| \le \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^{2+\frac{s}{q}}}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}} \times$$

$$\times\left\{\left[\frac{mn^s{}_2F_1\left[-s,1;p(\alpha+1)+2;-\frac{1}{n}\right]}{p(\alpha+1)+1}\left|f''(a)\right|^q+\frac{1}{p(\alpha+1)+s+1}\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[\frac{m}{p(\alpha+1)+s+1}\left|f''(a)\right|^q+\frac{n^s{}_2F_1\left[-s,1;p(\alpha+1)+2;-\frac{1}{n}\right]}{p(\alpha+1)+1}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},$$

(2) *for* $n=1$, *the following inequality for generalized relative semi-*(m,s)*-Breckner-preinvex functions holds:*

$$\frac{1}{2(\alpha+1)}\left|G(\alpha;1,m,\varphi(a),\varphi(b))(f)\right|\leq\frac{\eta^2(\varphi(b),\varphi(a),m)}{2^{3+\frac{s}{q}}(\alpha+1)}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}}\times$$

$$\times\left\{\left[m\mathcal{C}_3(p,s,\alpha,t)\left|f''(a)\right|^q+\frac{1}{p(\alpha+1)+s+1}\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[\frac{m}{p(\alpha+1)+s+1}\left|f''(a)\right|^q+\mathcal{C}_3(p,s,\alpha,t)\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{C}_3(p,s,\alpha,t)=\int_0^1(1-t)^{p(\alpha+1)}(1+t)^s dt.$$

***Corollary* 3.7.** *In Theorem* 3.3, *if we put* $h_1(t)=h_2(t)=t(1-t)$, *then we have the following inequality for generalized relative semi-*(m,tgs)*-preinvex functions:*

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right|\leq$$

$$\leq\frac{2\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^{2+\frac{2}{q}}}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}}\times$$

$$\times\left[\frac{n}{p(\alpha+1)+2}+\frac{1}{[p(\alpha+1)+2]\,[p(\alpha+1)+3]}\right]^{\frac{1}{q}}\left(m\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}.$$

***Corollary* 3.8.** *In Theorem* 3.3, *if we put* $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, *then we have:*

(1) *for* $n>1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right|\leq\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left[\frac{q-1}{(\alpha+1)(q-p)+q-1}\right]^{\frac{q-1}{q}}\times$$

$$\times\left\{\left[\frac{mn^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;p(\alpha+1)+\frac{3}{2};-\frac{1}{n}\right]}{2p(\alpha+1)+1}\Big|f''(a)\Big|^q+\right.\right.$$

$$\left.+\frac{n^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;p(\alpha+1)+\frac{5}{2};-\frac{1}{n}\right]}{2p(\alpha+1)+3}\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}+$$

$$+\left[\frac{mn^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;p(\alpha+1)+\frac{5}{2};-\frac{1}{n}\right]}{2p(\alpha+1)+3}\Big|f''(a)\Big|^q+\right.$$

$$\left.\left.+\frac{n^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;p(\alpha+1)+\frac{3}{2};-\frac{1}{n}\right]}{2p(\alpha+1)+1}\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}\right\},$$

(2) *for* $n=1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\frac{1}{2(\alpha+1)}\Big|G(\alpha;1,m,\varphi(a),\varphi(b))(f)\Big|\leq\frac{\eta^2(\varphi(b),\varphi(a),m)}{2^{3+\frac{1}{q}}(\alpha+1)}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}}\times$$

$$\times\left\{\left[m\mathcal{C}_4(p,\alpha,t)\Big|f''(a)\Big|^q+\mathcal{C}_5(p,\alpha,t)\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}+\left[m\mathcal{C}_5(p,\alpha,t)\Big|f''(a)\Big|^q+\mathcal{C}_4(p,\alpha,t)\Big|f''(b)\Big|^q\right]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{C}_4(p,\alpha,t)=\int_0^1(1-t)^{p(\alpha+1)-\frac{1}{2}}(1+t)^{\frac{1}{2}}dt$$

and

$$\mathcal{C}_5(p,\alpha,t)=\int_0^1(1-t)^{p(\alpha+1)+\frac{1}{2}}(1+t)^{-\frac{1}{2}}dt.$$

Finally, we shall prove the following result.

Theorem 3.4. *If* $|f''|^q$ *for* $q>1$ *is generalized relative semi-*(m,h_1,h_2)*-preinvex, then the following inequality for Riemann – Liouville fractional integrals along with* $h_1,\ h_2\colon[0,1]\to\mathbb{R}_0$, $\alpha>0,\ n\in\mathbb{N}^+$ *and* $0<\mu,\ \lambda<\alpha+2$ *exists:*

$$\Big|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\Big|\leq\frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left[\frac{q-1}{q(\alpha+2-\mu)-1}\right]^{1-\frac{1}{q}}\times$$

$$\times \left[\int_0^1 (1-t)^{\mu q} \left(m h_1 \left(\frac{1-t}{n+1} \right) \left| f''(a) \right|^q + h_2 \left(\frac{1-t}{n+1} \right) \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}} +$$

$$+ \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2} \left[\frac{q-1}{q(\alpha+2-\lambda)-1} \right]^{1-\frac{1}{q}} \times$$

$$\times \left[\int_0^1 (1-t)^{\lambda q} \left(m h_1 \left(\frac{n+t}{n+1} \right) \left| f''(a) \right|^q + h_2 \left(\frac{n+t}{n+1} \right) \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}}. \tag{3.4}$$

Proof. Using given hypothesis, Lemma 2.1 and Hölder's inequality, we have

$$\left| G(\alpha; n, m, \varphi(a), \varphi(b))(f) \right| \leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2} \left[\int_0^1 (1-t)^{(\alpha+1-\mu)\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}} \times$$

$$\times \left[\int_0^1 (1-t)^{\mu q} \left| f'' \left(m\varphi(a) + \frac{1-t}{n+1} \eta(\varphi(b), \varphi(a), m) \right) \right|^q dt \right]^{\frac{1}{q}} +$$

$$+ \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2} \left[\int_0^1 (1-t)^{(\alpha+1-\lambda)\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}} \times$$

$$\times \left[\int_0^1 (1-t)^{\lambda q} \left| f'' \left(m\varphi(a) + \frac{n+t}{n+1} \eta(\varphi(b), \varphi(a), m) \right) \right|^q dt \right]^{\frac{1}{q}} \leq$$

$$\leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2} \left[\frac{q-1}{q(\alpha+2-\mu)-1} \right]^{1-\frac{1}{q}} \times$$

$$\times \left[\int_0^1 (1-t)^{\mu q} \left(m h_1 \left(\frac{1-t}{n+1} \right) \left| f''(a) \right|^q + h_2 \left(\frac{1-t}{n+1} \right) \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}} +$$

$$+ \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2} \left[\frac{q-1}{q(\alpha+2-\lambda)-1} \right]^{1-\frac{1}{q}} \times$$

$$\times \left[\int_0^1 (1-t)^{\lambda q} \left(m h_1 \left(\frac{n+t}{n+1} \right) \left| f''(a) \right|^q + h_2 \left(\frac{n+t}{n+1} \right) \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}},$$

which completes the proof.

***Corollary* 3.9.** *In Theorem* 3.4, *if we take* $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ *for* $s \in (0, 1]$ *and* $\mu = \lambda$, *then we have:*

(1) *for* $n > 1$, *the following inequality for generalized relative semi-*(m, s)*-Breckner-preinvex functions exists:*

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right| \leq \frac{\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^2}\left[\frac{q-1}{q(\alpha+2-\mu)-1}\right]^{1-\frac{1}{q}} \times$$

$$\times\left\{\left[\frac{mn^s{}_2F_1\left[-s,1;\mu q+2;-\dfrac{1}{n}\right]}{\mu q+1}\left|f''(a)\right|^q+\frac{1}{\mu q+s+1}\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[\frac{m}{\mu q+s+1}\left|f''(a)\right|^q+\frac{n^s{}_2F_1\left[-s,1;\mu q+2;-\dfrac{1}{n}\right]}{\mu q+1}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},$$

(2) *for* $n=1$, *the following inequality for generalized relative semi-*(m,s)*-Breckner-preinvex functions exists:*

$$\frac{1}{2(\alpha+1)}\left|G(\alpha;1,m,\varphi(a),\varphi(b))(f)\right| \leq \frac{\eta^2(\varphi(b),\varphi(a),m)}{2^{3+\frac{s}{q}}(\alpha+1)}\left(\frac{q-1}{q(\alpha+2-\mu)-1}\right)^{1-\frac{1}{q}} \times$$

$$\times\left\{\left[m\mathcal{C}_6(\mu,q,s,t)\left|f''(a)\right|^q+\frac{1}{\mu q+s+1}\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[\frac{m}{\mu q+s+1}\left|f''(a)\right|^q+\mathcal{C}_6(\mu,q,s,t)\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{C}_6(\mu,q,s,t)=\int_0^1(1-t)^{\mu q}(1+t)^s dt.$$

***Corollary* 3.10.** *In Theorem* 3.4, *if we take* $h_1(t)=h_2(t)=t(1-t)$ *with* $\mu=\lambda$, *then we have the following inequality for generalized relative semi-*(m,tgs)*-preinvex functions:*

$$\left|G(\alpha;n,m,\varphi(a),\varphi(b))(f)\right| \leq$$

$$\leq \frac{2\eta^2(\varphi(b),\varphi(a),m)}{(n+1)^{2+\frac{2}{q}}}\left[\frac{q-1}{q(\alpha+2-\mu)-1}\right]^{1-\frac{1}{q}}\left[\frac{n(\mu q+3)+1}{(\mu q+2)(\mu q+3)}\right]^{\frac{1}{q}} \times$$

$$\times\left(m\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}.$$

***Corollary* 3.11.** *In Theorem* 3.4, *if we take* $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *with* $\mu=\lambda$, *then we have:*

(1) *for* $n > 1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\left|G(\alpha; n, m, \varphi(a), \varphi(b))(f)\right| \leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{(n+1)^2} \left(\frac{q-1}{(\alpha+2-\mu)q-1}\right)^{1-\frac{1}{q}} \times$$

$$\times \left\{ \left[\frac{mn^{\frac{1}{2}} {}_2F_1\left[-\frac{1}{2}, 1; \mu q + \frac{3}{2}; -\frac{1}{n}\right]}{2\mu q + 1} \left|f''(a)\right|^q + \frac{n^{-\frac{1}{2}} {}_2F_1\left[\frac{1}{2}, 1; \mu q + \frac{5}{2}; -\frac{1}{n}\right]}{2\mu q + 3} \left|f''(b)\right|^q \right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\frac{mn^{-\frac{1}{2}} {}_2F_1\left[\frac{1}{2}, 1; \mu q + \frac{5}{2}; -\frac{1}{n}\right]}{2\mu q + 3} \left|f''(a)\right|^q + \frac{n^{\frac{1}{2}} {}_2F_1\left[-\frac{1}{2}, 1; \mu q + \frac{3}{2}; -\frac{1}{n}\right]}{2\mu q + 1} \left|f''(b)\right|^q \right]^{\frac{1}{q}} \right\},$$

(2) *for* $n = 1$, *the following inequality for generalized relative semi-*m*-*MT*-preinvex functions holds:*

$$\frac{1}{2(\alpha+1)}\left|G(\alpha; 1, m, \varphi(a), \varphi(b))(f)\right| \leq \frac{\eta^2(\varphi(b), \varphi(a), m)}{2^{3+\frac{1}{q}}(\alpha+1)} \left(\frac{q-1}{q(\alpha+2-\mu)-1}\right)^{1-\frac{1}{q}} \times$$

$$\times \left\{ \left[m\mathcal{C}_7(\mu,q,t)\left|f''(a)\right|^q + \mathcal{C}_8(\mu,q,t)\left|f''(b)\right|^q\right]^{\frac{1}{q}} + \left[m\mathcal{C}_8(\mu,q,t)\left|f''(a)\right|^q + \mathcal{C}_7(\mu,q,t)\left|f''(b)\right|^q\right]^{\frac{1}{q}} \right\},$$

where

$$\mathcal{C}_7(\mu, q, t) = \int_0^1 (1-t)^{\mu q - \frac{1}{2}} (1+t)^{\frac{1}{2}} dt$$

and

$$\mathcal{C}_8(\mu, q, t) = \int_0^1 (1-t)^{\mu q + \frac{1}{2}} (1+t)^{-\frac{1}{2}} dt.$$

It is worth to mention in this paper that one can calculate the value of $\mathcal{C}_0(s, \alpha, t)$, $\mathcal{C}_1(\alpha, t)$, $\mathcal{C}_2(\alpha, t), \ldots$ using some mathematical software (for example, Maple).

4. Applications to special means. Let a and b be positive real numbers such that $a < b$, we recall the following means:

$$A := A(a,b) = \frac{a+b}{2}, \qquad G := G(a,b) = \sqrt{ab}, \qquad H := H(a,b) = \frac{2ab}{a+b},$$

$$P_r := P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}, \qquad r \geq 1,$$

$$I := I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\dfrac{1}{b-a}}, & a \neq b, \\ a, & a = b, \end{cases} \qquad L := L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases}$$

and

$$L_p := L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq 0,\ -1, \text{ and } a \neq b, \\ L(a,b), & p = -1 \text{ and } a \neq b, \\ I(a,b), & p = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

Consider the function $M := M(\varphi(a),\varphi(b))\colon [\varphi(a), \varphi(a) + \eta(\varphi(b),\varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b),\varphi(a))] \to \mathbb{R}^+$, which is one of the above mentioned means and $\varphi\colon I \to K$ is a continuous function.

Replace $\eta(\varphi(y),\varphi(x),m)$ with $\eta(\varphi(y),\varphi(x))$ and setting $\eta(\varphi(y),\varphi(x)) = M(\varphi(x),\varphi(y))$ for $m = 1 = n$ in (3.1), (3.2), (3.3) and (3.4). Therefore one can obtain the following interesting inequalities involving the above means as follows:

$$\Big|G(\alpha;1,1,\varphi(a),\varphi(b))(f)\Big| \leq \frac{M^2(\varphi(b),\varphi(a))}{4}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times$$
$$\times \left\{ \left[\int_0^1 (1-t)^{\alpha+1}\left(h_1\left(\frac{1-t}{2}\right)\Big|f''(a)\Big|^q + h_2\left(\frac{1-t}{2}\right)\Big|f''(b)\Big|^q\right)dt\right]^{\frac{1}{q}} + \right.$$
$$\left. + \left[\int_0^1 (1-t)^{\alpha+1}\left(h_1\left(\frac{1+t}{2}\right)\Big|f''(a)\Big|^q + h_2\left(\frac{1+t}{2}\right)\Big|f''(b)\Big|^q\right)dt\right]^{\frac{1}{q}} \right\}, \tag{4.1}$$

$$\Big|G(\alpha;1,1,\varphi(a),\varphi(b))(f)\Big| \leq \frac{M^2(\varphi(b),\varphi(a))}{4}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \times$$
$$\times \left\{ \left[\int_0^1 h_1\left(\frac{1-t}{2}\right)dt\Big|f''(a)\Big|^q + \int_0^1 h_2\left(\frac{1-t}{2}\right)dt\Big|f''(b)\Big|^q\right]^{\frac{1}{q}} + \right.$$
$$\left. + \left[\int_0^1 h_1\left(\frac{1+t}{2}\right)dt\Big|f''(a)\Big|^q + \int_0^1 h_2\left(\frac{1+t}{2}\right)dt\Big|f''(b)\Big|^q\right]^{\frac{1}{q}} \right\}, \tag{4.2}$$

$$\Big|G(\alpha;1,1,\varphi(a),\varphi(b))(f)\Big| \leq \frac{M^2(\varphi(b),\varphi(a))}{4}\left(\frac{q-1}{(\alpha+1)(q-p)+q-1}\right)^{\frac{q-1}{q}} \times$$

$$\times\left\{\left[\int_0^1 (1-t)^{p(\alpha+1)}\left(h_1\left(\frac{1-t}{2}\right)\left|f''(a)\right|^q + h_2\left(\frac{1-t}{2}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}} +\right.$$

$$\left.+\left[\int_0^1 (1-t)^{p(\alpha+1)}\left(h_1\left(\frac{1+t}{2}\right)\left|f''(a)\right|^q + h_2\left(\frac{1+t}{2}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\}, \tag{4.3}$$

and

$$\left|G(\alpha;1,1,\varphi(a),\varphi(b))(f)\right| \leq \frac{M^2(\varphi(b),\varphi(a))}{4}\left[\frac{q-1}{q(\alpha+2-\mu)-1}\right]^{1-\frac{1}{q}}\times$$

$$\times\left[\int_0^1 (1-t)^{\mu q}\left(h_1\left(\frac{1-t}{2}\right)\left|f''(a)\right|^q + h_2\left(\frac{1-t}{2}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}} +$$

$$+\frac{M^2(\varphi(b),\varphi(a))}{(n+1)^2}\left[\frac{q-1}{q(\alpha+2-\lambda)-1}\right]^{1-\frac{1}{q}}\times$$

$$\times\left[\int_0^1 (1-t)^{\lambda q}\left(h_1\left(\frac{1+t}{2}\right)\left|f''(a)\right|^q + h_2\left(\frac{1+t}{2}\right)\left|f''(b)\right|^q\right)dt\right]^{\frac{1}{q}}, \tag{4.4}$$

where

$$G(\alpha;1,1,\varphi(a),\varphi(b))(f) = \frac{2^\alpha\Gamma(\alpha+2)}{M^\alpha(\varphi(b),\varphi(a))}\left[J^\alpha_{(\varphi(a)+\frac{1}{2}M(\varphi(b),\varphi(a)))^-}f(\varphi(a))+\right.$$

$$\left.+J^\alpha_{(\varphi(a)+\frac{1}{2}M(\varphi(b),\varphi(a)))^+}f\left(\varphi(a)+M(\varphi(b),\varphi(a))\right)\right] -$$

$$-2(\alpha+1)f\left(\varphi(a)+\frac{1}{2}M(\varphi(b),\varphi(a))\right).$$

Letting $M = A,\ G,\ H,\ P_r,\ I,\ L,\ L_p$ in (4.1), (4.2), (4.3) and (4.4), we get the inequalities involving means for a particular choice of twice differentiable generalized relative semi-(h_1,h_2)-preinvex function f. Further, applying (4.1), (4.2), (4.3) and (4.4) to generalized relative semi-s-Breckner-preinvex functions, generalized relative semi-P-preinvex functions, generalized relative semi-s-Godunova – Levin – Dragomir-preinvex functions, generalized relative semi-tgs-preinvex functions, generalized relative semi-MT-preinvex functions, respectively, one can obtain various inequalities corresponding to these functions involving means.

References

1. M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, *Two point trapezoidal like inequalities involving hypergeometric functions*, Filomat, **31**, № 8, 2281 – 2292 (2017).
2. F. X. Chen, *Extensions of the Hermite – Hadamard inequality for convex functions via fractional integrals*, J. Math. Inequal., **10**, № 1, 75 – 81 (2016).
3. X. S. Chen, *Some properties of semi-E-convex functions*, J. Math. Anal. and Appl., **275**, № 1, 251 – 262 (2002).

4. S. S. Dragomir, *Integral inequalities of the Hermite – Hadamard type for K-bounded norm-convex mappings*, Ukr. Math. J., **68**, № 10, 1530 – 1551 (2017).
5. T. S. Du, J. G. Liao, Y. J. Li, *Properties and integral inequalities of Hadamard – Simpson type for the generalized (s, m)-preinvex functions*, J. Nonlinear Sci. and Appl., **9**, № 5, 3112 – 3126 (2016).
6. T. S. Du, Y. J. Li, Z. Q. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions*, Appl. Math. and Comput., **293**, 358 – 369 (2017).
7. S.-R. Hwang, K.-L. Tseng, K.-C. Hsu, *New inequalities for fractional integrals and their applications*, Turkish J. Math., **40**, 471 – 486 (2016).
8. M. Iqbal, M. I. Bhatti, K. Nazeer, *Generalization of inequalities analogous to Hermite – Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc., **52**, 707 – 716 (2015).
9. İ. İşcan, S. H. Wu, *Hermite – Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. and Comput., **238**, 237 – 244 (2014).
10. A. Kashuri, R. Liko, *Generalizations of Hermite – Hadamard and Ostrowski type inequalities for MT_m-preinvex functions*, Proyecciones, **36**, № 1, 45 – 80 (2017).
11. M. A. Khan, T. Ali, S. S. Dragomir, M. Z. Sarikaya, *Hermite – Hadamard type inequalities for conformable fractional integrals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math., **112**, 1033 – 1048 (2018).
12. M. A. Latif, *On some new inequalities of Hermite – Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value*, Ukr. Math. J., **67**, № 10, 1552 – 1571 (2016).
13. M. Matłoka, *Some inequalities of Hadamard type for mappings whose second derivatives are h-convex via fractional integrals*, J. Fract. Calc. and Appl., **6**, № 1, 110 – 119 (2015).
14. M. A. Noor, G. Cristescu, M. U. Awan, *Generalized fractional Hermite – Hadamard inequalities for twice differentiable s-convex functions*, Filomat, **29**, № 4, 807 – 815 (2015).
15. M. A. Noor, M. U. Awan, *Some integral inequalities for two kinds of convexities via fractional integrals*, Transylv. J. Math. and Mech., **5**, № 2, 129 – 136 (2013).
16. M. E. Özdemir, A. Ekinci, *Generalized integral inequalities for convex functions*, Math. Inequal. and Appl., **19**, № 4, 1429 – 1439 (2016).
17. C. Peng, C. Zhou, T. S. Du, *Riemann – Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2)-preinvexity*, Ital. J. Pure and Appl. Math., **38**, 345 – 367 (2017).
18. S. Qaisar, M. Iqbal, M. Muddassar, *New Hermite – Hadamard's inequalities for preinvex functions via fractional integrals*, J. Comput. Anal. and Appl., **20**, № 7, 1318 – 1328 (2016).
19. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, *Hermite – Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. and Comput. Modelling, **57**, 2403 – 2407 (2013).
20. J. Wang, J. Deng, M. Fečkan, *Hermite – Hadamard-type inequalities for r-convex functions based on the use of Riemann – Liouville frantional integrals*, Ukr. Math. J., **65**, № 2, 193 – 211 (2013).
21. H. Wang, T. S. Du, Y. Zhang, *k-Fractional integral trapezium-like inequalities through (h, m)-convex and (α, m)-convex mappings*, J. Inequal. and Appl., **2017**, Article ID 311 (2017), 20 p.
22. B. Y. Xi, F. Qi, *Hermite – Hadamard type inequalities for geometrically r-convex functions*, Studia Sci. Math. Hungar., **51**, № 4, 530 – 546 (2014).
23. E. A. Youness, *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory and Appl., **102**, № 2, 439 – 450 (1999).
24. Y. C. Zhang, T. S. Du, J. Pan, *On new inequalities of Fejér – Hermite – Hadamard type for differentiable (α, m)-preinvex mappings*, ScienceAsia, **43**, № 4, 258 – 266 (2017).

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