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ON ASYMPTOTICALLY STABILITY, UNIFORMLY STABILITY AND BOUNDEDNESS OF SOLUTIONS OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*

ПРО АСИМПТОТИЧНУ СТІЙКІСТЬ, РІВНОМІРНУ СТІЙКІСТЬ ТА ОБМЕЖЕНІСТЬ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ВОЛЬТЕРРА

In this paper, two new Lyapunov functionals are defined. We apply these functionals to get sufficient conditions guaranteeing the asymptotic stability, uniform stability, and boundedness of solutions of certain nonlinear Volterra integro-differential equations of the first order. The results obtained are improvements and extensions of known results that can be found in literature. We also suggest examples to show the applicability of our results and for the sake of illustrations. Using MATLAB-Simulink, in particular cases we clearly show the behavior of orbits of Volterra integro-differential equations under consideration.

Наведено означення двох нових функціоналів Ляпунова. Ці функціонали використано для отримання достатніх умов, що гарантують асимптотичну стійкість, рівномірну стійкість та обмеженість розв'язків нелінійних інтегро-диференціальних рівнянь Вольтерра першого порядку. Отримані результати удосконалюють та розширюють відомі результати, що вже були опубліковані. Наведено приклади застосування отриманих результатів. За допомогою MATLAB-Simulink в окремих випадках показано поведінку орбіт розглянутих інтегро-диференціальних рівнянь Вольтерра.

1. Introduction. Volterra integral and integro-differential equations, integral equations and integro-differential equations have many applications in sciences and engineering (see Burton [2], Rahman [7], Wazwaz [20] and the cited references therein). Due to these facts, in the last years, stability, asymptotic stability, uniform stability, boundedness, exponentially stability, etc., of linear and nonlinear Volterra integro-differential equations, Volterra integral equations, integral equations and integro-differential equations have been discussed by many researches. In particular, as a brief information, the reader can referee to the articles of Becker [1], Furumochi and Matsuoka [3], Graef et al. [4], Mahfoud [5], Raffoul [6], Rama Mohana Rao and Srinivas [8], Tunç [9 – 13], Tunç and Mohammed [14], Tunç and Tunç [15 – 17], Wang [18, 19] and the works mentioned in that sources for the former scientific results that can be found in the literature on the diverse qualitative behaviors of various of Volterra integro-differential equations, Volterra integral equations, integral equations and integro-differential equations. For some very recent interesting works on the various qualitative properties of solutions of certain nonlinear Volterra integro-differential equations without and with delay, we would also like to mention the papers of Tunç [21, 22], Tunç and Tunç [23 – 25].

As a distinguished information from this line, the following article is notable.

In 2000, Wang [19] considers the Volterra integro-differential equation

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$$\frac{dx}{dt} = A(t)x(t) + \int\limits_0^t C(t,s)x(s)ds, \tag{1.1}$$

in which t is non-negative and real variable, $x \in \Re^n$, $n \geq 1$, $A(.)$ and $C(.)$ are $(n \times n)$-matrices, which are continuous for $0 \leq t < \infty$ and $0 \leq s \leq t < \infty$, respectively.

Wang [19] proves three theorems related to the stability, uniform stability and asymptotic stability of solutions of Volterra integro-differential equation (1.1). The author gives an example verifying the established assumptions. The results obtained in Wang [19] are variants of the results that can be found throughout Burton [2].

In this article, motivated by the results of Wang [19], we first take into consideration the nonlinear Volterra integro-differential equation

$$\frac{dx}{dt} = -A(t)x + \int\limits_0^t C(t,s)g(s,x(s))ds + h(t,x), \tag{1.2}$$

where t is non-negative and real variable, $x \in \Re^n$, $A(.)$ and $C(.)$ have the same properties as in the Volterra integro-differential equation (1.1), $g\colon \Re^+ \times \Re^n \to \Re^n$ and $h\colon \Re^+ \times \Re^n \to \Re^n$ are continuous functions with $\Re^+ = [0,\infty)$, and $g(s,0) = 0$.

We will discuss the stability, uniform stability of trivial solution and boundedness of solutions of Volterra integro-differential equation (1.2) by help of appropriate Lyapunov functionals for the cases of $h(.) \equiv 0$ and $h(.) \neq 0$, respectively.

It follows that the Volterra integro-differential equation (1.1) discussed in Wang [19] is a linear equation. However, the Volterra integro-differential equation (1.2), to be discussed here, is a nonlinear Volterra integro-differential equation. This case is a clear improvement and contribution from the linear Volterra integro-differential equation (1.1) to the nonlinear Volterra integro-differential equation (1.2). This fact is the first originality of this article.

Furthermore, Volterra integro-differential equation (1.2) includes and extends the Volterra integro-differential equation (1.1) discussed in Wang [19]. Next, Wang [19] discusses the stability and uniform stability of the solution $x(t) \equiv 0$ of Volterra integro-differential equation (1.1) when $h(.) \equiv 0$. However, in addition to these two problems, we will establish the hypotheses to get the boundedness solutions of the Volterra integro-differential equation (1.2), when $h(.) \neq 0$. This fact is the second originality of this article and its contribution to the literature.

Moreover, if choose $g(.) = x(s)$ and $h(.) = 0$, then Volterra integro-differential equation (1.2) reduces to Volterra integro-differential equation (1.1) studied by Wang [19]. This information clearly shows the other contribution of this article to the subject.

In addition, we define here two new Lyapunov functionals to proceed the proofs of main results.

It is well-known that the definition or construction of suitable Lyapunov functionals remains as an unsolved problem in the related literature by this time. Succeeding this fact for the general problems considered such as in this paper is the third original property of this article.

Furthermore, two specific examples are given to make clear the correctness and applicability of our hypotheses to be given. In particular cases, using MATLAB-Simulink, it is clearly shown the behaviors of the orbits of the Volterra integro-differential equations considered. According to our observations, this fact is not appeared in the papers or books in the references of this article and the others. This fact is another originality and contribution of this article to the topic. Indeed, we would like to state by all of these information the originality and novelty properties of the results of this paper.

Throughout this article, when we need x will represent $x(t)$.

Let $x \in \Re^n$, $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, A be an $(n \times n)$-matrix, $\|A\| = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} a_{ij}^2\right)^{1/2}$ and P be an $(n \times n)$-positive definite constant matrix.

We represent the characteristic roots of the symmetric matrix

$$\frac{1}{2}(A^T(t)P + PA(t))$$

by $\lambda_j(t, P, A),\ j = 1, 2, \ldots, n$.

We suppose that

$$\lambda_m(t, P, A) \equiv \min_{1 \le j \le n} \lambda_j(t, P, A)$$

and

$$\lambda_M(t, P, A) \equiv \max_{1 \le j \le n} \lambda_j(t, P, A),$$

which are minimum and maximum eigenvalues of the symmetric matrix $\frac{1}{2}\big(A^T(t)P + PA(t)\big)$, respectively.

Lemma 1.1 [19]. *If $x \in \Re^n$, then*

$$\lambda_m(t, P, A)\|x\|^2 \le x^T \left[\frac{1}{2}\left(A^T(.)P + PA(.)\right)\right] x \le \lambda_M(t, P, A)\|x\|^2. \tag{1.3}$$

The proof of the inequality (1.3) can be easily done. We omit the details.

The following lemma is also well-known for the linear algebra.

Lemma 1.2. *Let P be a real symmetric $(n \times n)$-matrix and*

$$a' \ge \lambda_i(P) \ge a > 0, \quad i = 1, 2, \ldots, n,$$

where a' and a are constants. Then

$$a' \langle x, x \rangle \ge \langle x, Px \rangle \ge a \langle x, x \rangle$$

and

$$a'^2 \langle x, x \rangle \ge \langle Px, Px \rangle \ge a^2 \langle x, x \rangle.$$

2. Asymptotic stability. Let $h(.) \equiv 0$.

A. Hypotheses: Suppose the following hypotheses hold:

(A_1) Let P be an $(n \times n)$-matrix with constant elements, which is positive definite and symmetric such that

$$\|P\| \le K_2, \quad \text{where} \quad K_2 > 0, \quad K_2 \in \Re;$$

(A_2) $K_3\|x\| \le \|g(t, x)\| \le K_4\|x\|$, where K_3 and K_4 are some positive real constants;

(A$_3$) $\theta(t) = 2\lambda_m(t,P,A) - K_2K_4\int_0^t \|C(t,s)\|ds - K_2K_3^{-2}K_4^3\int_t^\infty \big\|C(u,t)\big\|du \geq K_1$ such that

$$\int\limits_0^\infty \big\|C(t,s)\big\|ds < \infty \quad \text{and} \quad \int\limits_t^\infty \big\|C(u,t)\big\|du < \infty,$$

where $K_1 > 0,\ K_1 \in \Re$.

Theorem 2.1. *If hypotheses* (A$_1$) – (A$_3$) *hold, then the zero solution of Volterra integro-differential equation* (1.2) *is asymptotic stable.*

Proof. We define a Lyapunov functional $V_0 = V_0(t, x(.))$ by

$$V_0 = x^T Px + \gamma_1 \int\limits_0^t \int\limits_t^\infty \big\|C(u,s)\big\|du\big\|g(s,x(s))\big\|^2 ds, \tag{2.1}$$

where $\gamma_1 > 0,\ \gamma_1 \in \Re$, and we determine this constant later in the proof.

The positive definiteness of the Lyapunov functional V_0 and the existence of the estimate

$$x^T Px \leq V_0$$

are clear.

Differentiating the Lyapunov functional V_0 with respect to t, it can be obtained from (2.1) and Volterra integro-differential equation (1.2) that

$$\frac{d}{dt}V_0 = -x^T\big\{A^T(t)P + PA(t)\big\}x + \int\limits_0^t g^T(s,x(s))C^T(t,s)ds \times Px +$$

$$+x^T P \times \int\limits_0^t C(t,s)g(s,x(s))ds + \gamma_1\|g(t,x)\|^2 \int\limits_t^\infty \big\|C(u,t)\big\|du -$$

$$-\gamma_1 \int\limits_0^t \big\|C(t,s)\big\|\big\|g(s,x(s))\big\|^2 ds. \tag{2.2}$$

Let $\gamma_1 = \dfrac{K_2K_4}{K_3^2}$. Applying hypotheses (A$_1$) – (A$_3$) and an elementary inequality, we can get from (2.2) that

$$\frac{d}{dt}V_0 \leq -\theta(t)\|x\|^2. \tag{2.3}$$

Hence

$$\frac{d}{dt}V_0 \leq -K_1\|x\|^2$$

by (2.3) and hypothesis (A$_3$). The proof of the asymptotic stability of the zero solution of equation (1.2) is completed.

Theorem 2.1 is proved.

Remark **2.1.** We can also conclude that the sufficient hypotheses (A$_1$) – (A$_3$) guarantee the stability of the zero solution of Volterra integro-differential equation (1.2).

3. Bounded solutions. Let $h(.) \neq 0$.

B. Hypothesis: Let the following hypothesis holds:

(B_1) $\|h(.)\| \le \Omega(t)$, where Ω is a non-negative and continuous function such that $\Omega \in L^1(0,\infty)$, that is,

$$\int\limits_0^\infty \Omega(s)ds < \infty.$$

Theorem 3.1. *If hypotheses* (A_1)–(A_3) *and* (B_1) *are satisfied, then all solutions of Volterra integro-differential equation* (1.2) *are bounded.*

Proof. We keep in the mind the Lyapunov functional $V_0 = V_0(t, x(.))$ given by (2.1). Hence, in view of $h(t,x) \neq 0$ and the time derivative of functional V_0, an application of hypotheses (A_1)–(A_3) and (B_1) makes enable that

$$V_0' \le h^T(t,x)Px + x^T Ph(t,x) \le K_2\Omega(.) + \Omega(.)V_0(t, x(.)).$$

By evaluating the integral of the previous inequality between 0 and t, we arrive at

$$V_0(t,x(t)) \le V_0(0,x(0)) + K_2 \int\limits_0^t \Omega(s)ds + \int\limits_0^t \Omega(s)V_0(s,x(s))ds.$$

By an application of the Gronwall's inequality, we have

$$x^T Px \le V_0(t,x(t)) \le K_5 \exp\left(\int\limits_0^\infty \Omega(s)ds\right),$$

where $K_5 = V_0(0,x(0)) + K_2 \int_0^\infty \Omega(s)ds,\ K_5 > 0$, so that

$$x^T Px \le K_5^2.$$

Hence, we can conclude that all solutions of Volterra integro-differential equation (1.2) are bounded.

Theorem 3.1 is proved.

4. Uniform stability. In this paper, finally, we consider the nonlinear Volterra integro-differential equation

$$\frac{dx}{dt} = -A(t)x + \int\limits_0^t C(t,s,x(s))ds, \tag{4.1}$$

with $C(t,s,0) = 0$, $t \ge 0$, $x \in \Re^n$, $A(.)$ is an $(n \times n)$-matrix function and $C(.)$ is an $(n \times 1)$-vector function, which are continuous for the arguments displayed clearly.

It is clear that Volterra integro-differential equation (4.1) is a nonlinear generalization of Volterra integro-differential equation (1.1) discussed by Wang [19]. Here, we take into consideration the uniform stability of the zero solution of this Volterra integro-differential equation.

C. Hypotheses: We assume that following hypotheses are true:

(C_1) $\big\|C(t,s,x)\big\| \le K(t,s)\|x\|$, where $K(t,s)$ is a continuous scalar function for $t \ge s \ge 0$;

(C_2) $\int_0^t \int_t^\infty \|K(u,s)\|\, duds < LK_2^{-1}$, $(K_2, L \in \Re, K_2, L > 0)$, where $K(u,s)$ is a continuous scalar function for the arguments shown;

(C_3) $\rho(t) = 2\lambda_m(t,P,A) - K_2 \int_0^t \big\|K(t,s)\big\| ds - K_2 \int_t^\infty \big\|K(u,t)\big\| du \geq K_6$ with

$$\int\limits_0^\infty \big\|K(t,s)\big\| ds < \infty, \qquad \int\limits_0^\infty \big\|K(u,t)\big\| du < \infty,$$

where K_6 is a given positive constant and $K(u,t)$ and $K(t,s)$ are continuous scalar functions for the arguments shown.

Theorem 4.1. *If hypotheses* (A_1) *and* (C_1)–(C_3) *are satisfied, then the zero solution of Volterra integro-differential equation* (4.1) *is uniformly stable.*

Proof. Let us define a Lyapunov functional $V = V(t, x(.))$ by

$$V(t,x(.)) = x^T P x + \beta \int\limits_0^t \int\limits_t^\infty \|K(u,s)\|\, du\, \|x(s)\|^2\, ds, \tag{4.2}$$

where $\beta > 0$, $\beta \in \Re$, and we specific this constant in the proof.

The positive definiteness of the Lyapunov functional V is clear.

Differentiating the Lyapunov functional V with respect to t, we obtain from (4.2) and Volterra integro-differential equation (4.1) that

$$\frac{d}{dt}V = -x^T\big\{A^T(t)P + PA(t)\big\}x + \int\limits_0^t C^T\big(t,s,x(s)\big)ds \times Px +$$

$$+ x^T P \times \int\limits_0^t C(t,s,x(s))ds + \beta \int\limits_t^\infty \big\|K(u,t)\big\| du \big\|x\big\|^2 - \beta \int\limits_0^t \big\|K(t,s)\big\|\, \|x(s)\|^2\, ds. \tag{4.3}$$

Let $\beta = K_2$. Take into consideration equality (4.3), make an application of hypotheses (A_1), (C_1)–(C_3) and use an elementary inequality, we can obtain

$$\frac{d}{dt}V \leq -\rho(t)\|x\|^2. \tag{4.4}$$

Hence, we have from (4.4) and hypothesis (C_3) that

$$\frac{d}{dt}V \leq -K_6\|x\|^2. \tag{4.5}$$

Let us now show that the uniform stability of zero solution of Volterra integro-differential equation (4.1).

Since the matrix P is real, symmetric and positive definite, then it is clear that

$$a'\|x\|^2 \geq x^T P x \geq a\big\|x\big\|^2, \tag{4.6}$$

where $a' > 0$ and $a > 0$, which are some constants.

Let $x \in \Re^n$ and $|x|$ be any norm. In addition, let C denote the Banach space of continuous functions $\phi\colon [t_0 - \tau, t_0] \to \Re^n$ with

$$\|\phi\|_{t_0} := \sup_{t_0-\tau\le t\le t_0} \big|\phi(t)\big|.$$

We assume that $x(t) = x(t, t_0, \phi)$ is a solution of equation (4.1) on $[t_0 - \tau, \infty)$ such that $x(t) = \phi(t)$ on $[t_0 - \tau, t_0]$, $t_0 \ge 0$, where ϕ is the initial function with $\phi \in C[t_0 - \tau, t_0]$.

Hence, we can write form (4.5), (4.6), (C$_1$) and Lemma 1.2 that

$$a\,\|x(t,t_0,\phi)\|^2 \le \langle \phi(t_0), P\varphi(t_0)\rangle \le V(t,x(.)) \le V(t_0,\phi(.)) \le$$

$$\le \phi^T(t_0)P\phi(t_0) + \beta \int\limits_0^{t_0}\int\limits_{t_0}^{\infty} \|K(u,s)\|\,du\,\|\phi(s)\|^2\,ds \le$$

$$\le a'\,\|\phi(t_0)\|^2 + \beta L\,\|\phi\|^2\,.$$

Due to the discussion made, it follows that for each $\varepsilon > 0$, we can choose a positive constant like $\delta = \left(\dfrac{a}{a' + \beta L}\right)^2 \dfrac{\varepsilon}{4}$ such that for any solution of equation (4.1), the inequality $\|\phi(t)\| < \delta$, $t \in [t_0 - \tau, t_0]$, implies that

$$a\big\|x(t,t_0,\phi)\big\|^2 \le a'\delta^2 + \beta L\delta^2 \le \frac{a\varepsilon^2}{16},$$

that is,

$$\|x(t,t_0,\phi)\| \le \frac{\varepsilon}{4} < \varepsilon, \quad t \ge t_0.$$

It is clear that the constant δ does not depend on the constant t_0. Therefore, we can say that the solution $x(t) \equiv 0$ of Volterra integro-differential equation (4.1) is uniformly sable. Thus, we can conclude the desired result.

***Remark* 4.1.** We can also conclude that the sufficient hypotheses (A$_1$) and (C$_1$)–(C$_3$) guarantee the stability of the zero solution of Volterra integro-differential equation (4.1).

***Example* 4.1.** In particular case, we consider the nonlinear Volterra integro-differential equation of the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{3}{2}\exp(t) & \dfrac{1}{2}\exp(3t) \\ -\exp(3t) & -2\exp(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} +$$

$$+ \int\limits_0^t \begin{pmatrix} \dfrac{1}{4}\exp\big(-2t + s - x_1^2(s)\big) & 0 \\ 0 & \dfrac{1}{4}\exp\big(-2t + s - x_2^2(s)\big) \end{pmatrix}\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds,$$

where $t \ge 0$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Re^2$.

If we compare the former Volterra integro-differential equation with Volterra integro-differential equation (4.1), then it follows that

$$A(t) = \begin{pmatrix} -1 + \dfrac{3}{2}\exp(t) & -\dfrac{1}{2}\exp(3t) \\ \exp(3t) & 2\exp(t) \end{pmatrix},$$

$$C\big(t, s, x(s)\big) = \begin{pmatrix} \dfrac{1}{4}\exp\big(-2t + s - x_1^2(s)\big) & 0 \\ 0 & \dfrac{1}{4}\exp\big(-2t + s - x_2^2(s)\big) \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix},$$

$$\big\|C(t, s, x(s))\big\| \le \left\| \begin{pmatrix} \dfrac{1}{4}\exp(-t + s) & 0 \\ 0 & \dfrac{1}{4}\exp(-t + s) \end{pmatrix} \right\| \left\| \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} \right\| = \big\|K(t, s)\big\| \|x\|,$$

$$\big\|K(t, s)\big\| = \frac{1}{4}\exp(-2t + s), \quad 0 \le s \le t,$$

and

$$\big\|K(u, t)\big\| = \frac{1}{4}\exp(-2u + t), \quad 0 \le t \le u.$$

Let

$$P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, it is clear that the matrix P is positive definite and symmetric, and $\|P\| = \sqrt{5} = K_2$. Moreover, we can see that

$$\frac{1}{2}\big\{A^T(t)P + PA(t)\big\} = \begin{pmatrix} -2 + 3\exp(t) & 0 \\ 0 & 2\exp(t) \end{pmatrix},$$

so that $\lambda_1(t, P, A) = -2 + 3\exp(t)$ and $\lambda_2(t, P, A) = 2\exp(t)$.

In addition, we can obtain

$$\lambda_M(t, P, -A) = \max\big\{2 - 3\exp(t), -2\exp(t)\big\} = \begin{cases} 2 - 3\exp(t), & 0 \le t \le \ln 2, \\ -2\exp(t), & t > \ln 2, \end{cases}$$

$$\lambda_m(t, P, -A) = \min\big\{2 - 3\exp(t), -2\exp(t)\big\} = \begin{cases} -2\exp(t), & 0 \le t \le \ln 2, \\ 2 - 3\exp(t), & t > \ln 2, \end{cases}$$

$$\lambda_m(t, P, -A) \le \lambda_M(t, P, -A),$$

$$\int\limits_0^t \big\|K(t, s)\big\| ds = \frac{1}{4}\int\limits_0^t \exp(-2t + s) ds = \frac{1}{4}\big(\exp(-t) - \exp(-2t)\big) < \infty,$$

$$\int\limits_t^\infty \big\|K(u, t)\big\| du = \frac{1}{4}\int\limits_t^\infty \exp(-2u + t) du \le \frac{1}{8} < \infty, \quad 0 \le t \le u,$$

and

$$\int_0^t \int_t^\infty \|K(u,s)\|\, duds = \frac{1}{4}\int_0^t \int_t^\infty \exp(-2u+s)duds =$$

$$= \frac{1}{8}\int_0^t \exp(-2t+s)ds = \frac{1}{8}\big[\exp(-t) - \exp(-2t)\big].$$

Let us define a function F by

$$F(t) = \big[\exp(-t) - \exp(-2t)\big], \quad t > 0.$$

It can be easily seen that the function F takes its maximum value at $t = \ln 2$, which is $\frac{1}{4}$. In this case, we can conclude that

$$F(t) = \big[\exp(-t) - \exp(-2t)\big] \le \frac{1}{4}, \quad t > 0,$$

so that

$$\int_0^t \int_t^\infty \|K(u,s)\|\, duds = \frac{1}{4}\int_0^t \int_t^\infty \exp(-2u+s)duds \le \frac{1}{32} < \frac{1}{16}, \quad LK_2^{-1} = \frac{1}{16}.$$

From the above discussion, we can arrive at

$$2\lambda_m(t,P,A) - K_2\int_0^t \|K(t,s)\|\, ds - K_2\int_t^\infty \big\|K(u,t)\big\|du =$$

$$= 2\exp(t) - \frac{\sqrt{5}}{4}\big(\exp(-t) - \exp(-2t)\big) - \frac{\sqrt{5}}{8}\exp(-t) \ge \frac{16 - 3\sqrt{5}}{8} = K_6, \quad t \ge 0.$$

Thus, all assumptions of (A_1), (C_1) – (C_3) hold. Hence, we can conclude that the zero solution of the given Volterra integro-differential equation is asymptotic stable, and it is also stable.

The asymptotic stability of the zero solution for the considered Volterra integro-differential equation is shown by Figs. 1 – 3.

***Example* 4.2.** Consider the nonlinear Volterra integro-differential equation in Example 4.1 for the case $h(t,x) \neq 0$ as given below:

$$h(t,x) = \begin{pmatrix} (1+t^2+x_1^2)^{-1}\sin t \\ (1+t^2+x_2^2)^{-1}\cos t \end{pmatrix}, \quad \text{where} \quad t \ge 0 \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Re^2.$$

Then

$$\|h(t,x)\| = \left\| \begin{pmatrix} (1+t^2+x_1^2)^{-1}\sin t \\ (1+t^2+x_2^2)^{-1}\cos t \end{pmatrix} \right\| \le \frac{1}{1+t^2} = \Omega(t), \qquad \int_0^\infty \frac{1}{1+t^2}dt = \frac{\pi}{2}.$$

Thus, it is clear that $\Omega \in L^1(0,\infty)$.

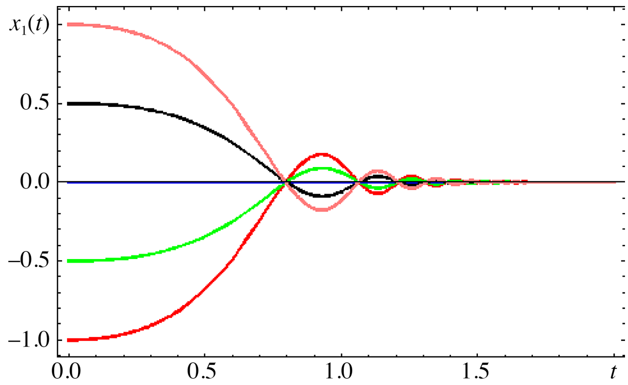

Fig. 1. Trajectory of $x_1(t)$ for Example 4.1.

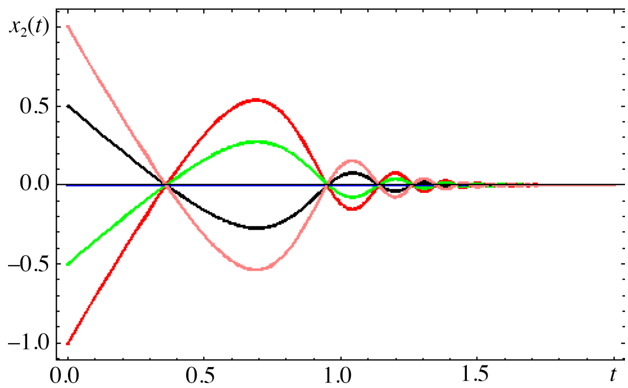

Fig. 2. Trajectory of $x_2(t)$ for Example 4.1.

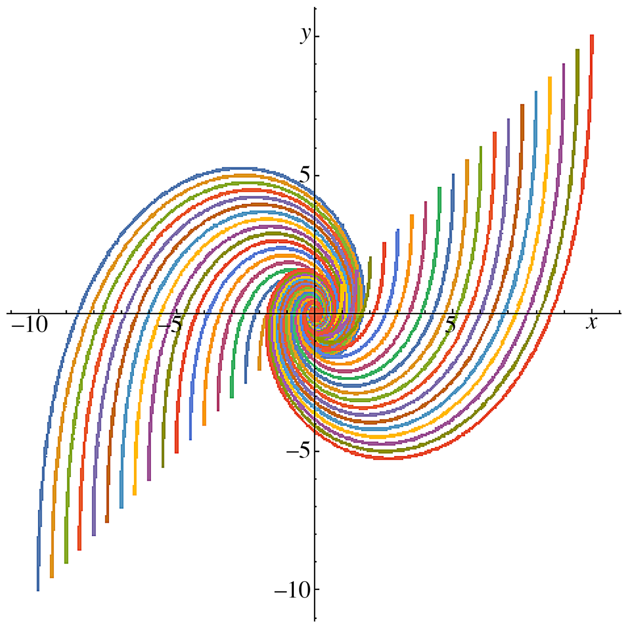

Fig. 3. Numerical plots of solutions $x_1(t)$ and $x_2(t)$ for Example 4.1.

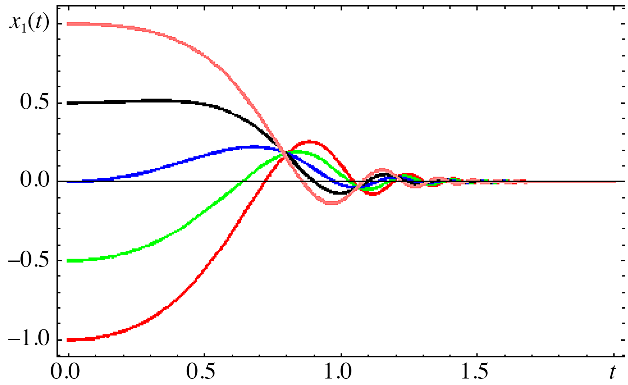

Fig. 4. Trajectory of $x_1(t)$ for Example 4.2.

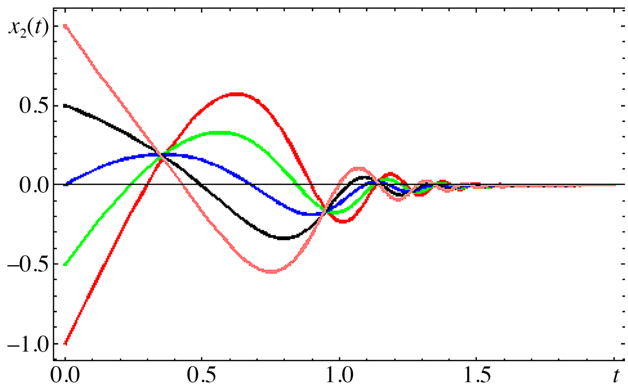

Fig. 5. Trajectory of $x_2(t)$ for Example 4.2.

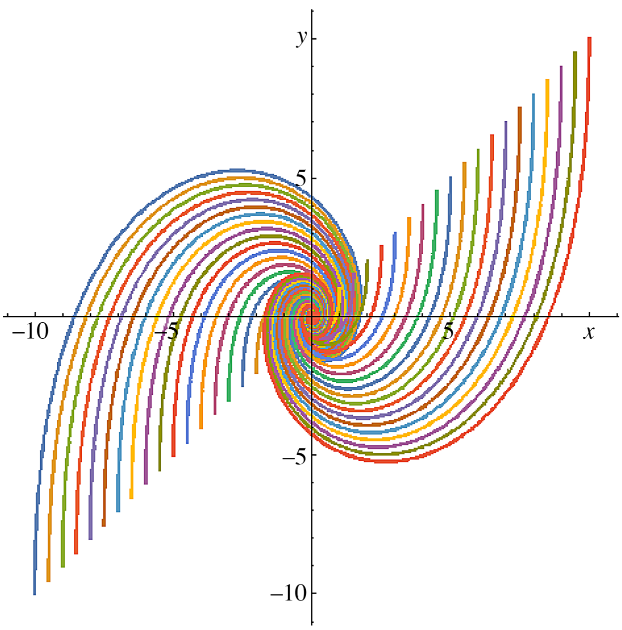

Fig. 6. Numerical plots of solutions $x_1(t)$ and $x_2(t)$ for Example 4.2.

In view of the proof of the boundedness theorem, that is, the proof of Theorem 3.1, we can reconsider the inequality

$$x^T P x \leq V_0(t, x(t)) \leq K_5 \exp\left(\int\limits_0^\infty \Omega(s) ds\right), \quad K_5 > 0.$$

For the choice of $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that

$$x^T P x = [x_1, x_2] \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + x_2^2.$$

By noting the above relations, we have

$$2x_1^2 + x_2^2 \leq V_0(t, x) \leq K_5 \exp\left(\int\limits_0^\infty \frac{1}{1+s^2} ds\right) = K_5 \left[\exp\left(\frac{\pi}{2}\right) - 1\right].$$

Hence, in the particular case, we can conclude that the boundedness of the solutions of the considered Volterra integro-differential equation.

Further, the boundedness of the solutions of the considered Volterra integro-differential equation is shown by Figs. 4 – 6.

Hence, it is true to say that the solutions of the Volterra integro-differential equation considered are bounded.

5. Conclusion. We pay our attention to a class of first order nonlinear Volterra integro-differential equation. We establish new sufficient conditions for the asymptotic stability, uniform stability and boundedness of solutions to the considered Volterra integro-differential equations by defining two new Lyapunov functionals. That is, by Theorems 2.1 and 3.1, we improve and extend the stability and uniformly stability results from linear Volterra integro-differential equations to nonlinear Volterra integro-differential equations (see [19], Theorems 1 and 2). In addition, boundedness of solutions is not discussed in Wang [19]. However, by Theorem 4.1 we communicate a new result on the boundedness of solutions of Volterra integro-differential equation (1.2). By means of the obtained results, we improved and extended the previous results that can be found in the literature from linear cases to their nonlinear cases and give an additional result on the boundedness of solutions with specific applications.

References

1. L. C. Becker, *Uniformly continuous L^1 solutions of Volterra equations and global asymptotic stability*, Cubo, **11**, № 3, 1 – 24 (2009).
2. T. A. Burton, *Volterra integral and differential equations*, Second ed., Math. Sci. and Eng., Vol. 202, Elsevier B. V., Amsterdam (2005).
3. T. Furumochi, S. Matsuoka, *Stability and boundedness in Volterra integro-differential equations*, Mem. Fac. Sci. Eng. Shimane Univ. Ser. B, Math. Sci., **32**, 25 – 40 (1999).
4. J. R. Graef, C. Tunç, S. Şevgin, *Behavior of solutions of non-linear functional Volterra integro-differential equations with multiple delays*, Dynam. Systems and Appl., **25**, № 1-2, 39 – 46 (2016).
5. W. E. Mahfoud, *Boundedness properties in Volterra integro-differential systems*, Proc. Amer. Math. Soc., **100**, № 1, 37 – 45 (1987).

6. Y. Raffoul, *Boundedness in nonlinear functional differential equations with applications to Volterra integro-differential equations*, J. Integral Equat. and Appl., **16**, № 4, 375 – 388 (2004).
7. M. Rahman, *Integral equations and their applications*, WIT Press, Southampton (2007).
8. M. Rama Mohana Rao, P. Srinivas, *Asymptotic behavior of solutions of Volterra integro-differential equations*, Proc. Amer. Math. Soc., **94**, № 1, 55 – 60 (1985).
9. C. Tunç, *A note on the qualitative behaviors of nonlinear Volterra integro-differential equation*, J. Egyptian Math. Soc., **24**, № 2, 187 – 192 (2016).
10. C. Tunç, *New stability and boundedness results to Volterra integro-differential equations with delay*, J. Egyptian Math. Soc., **24**, № 2, 210 – 213 (2016).
11. C. Tunç, *Properties of solutions to Volterra integro-differential equations with delay*, Appl. Math. Inf. Sci., **10**, № 5, 1775 – 1780 (2016).
12. C. Tunç, *Qualitative properties in nonlinear Volterra integro-differential equations with delay*, J. Taibah Univ. Sci., **11**, № 2, 309 – 314 (2017).
13. C. Tunç, *On the qualitative behaviors of a functional differential equation of second order*, Appl. Appl. Math., **12**, № 2, 813 – 842 (2017).
14. C. Tunç, S. A. Mohammed, *A remark on the stability and boundedness criteria in retarded Volterra integro-differential equations*, J. Egyptian Math. Soc., **25**, № 4, 363 – 368 (2017).
15. C. Tunç, O. Tunç, *On the exponential study of solutions of Volterra integro-differential equations with time lag*, Electron. J. Math. Anal. and Appl., **6**, № 1, 253 – 265 (2018).
16. C. Tunç, O. Tunç, *New results on the stability, integrability and boundedness in Volterra integro-differential equations*, Bull. Comput. Appl. Math., **6**, № 1, 41 – 58 (2018).
17. C. Tunç, O. Tunç, *On behaviors of functional Volterra integro-differential equations with multiple time-lags*, J. Taibah Univ. Sci., **12**, № 2, 173 – 179 (2018).
18. Ke Wang, *Uniform asymptotic stability in functional-differential equations with infinite delay*, Ann. Different. Equat., **9**, № 3, 325 – 335 (1993).
19. Q. Wang, *The stability of a class of functional differential equations with infinite delays*, Ann. Different. Equat., **16**, № 1, 89 – 97 (2000).
20. A. M. Wazwaz, *Linear and nonlinear integral equations*, Methods and applications. Higher Education Press, Beijing; Springer, Heidelberg (2011).
21. C. Tunç, *A remark on the qualitative conditions of nonlinear IDEs*, Int. J. Math. Comput. Sci., **15**, № 3, 905 – 922 (2020).
22. C. Tunç, *Asymptotic stability and boundedness criteria for nonlinear retarded Volterra integro-differential equations*, J. King Saud Univ.-Sci., **30**, № 4, 3531 – 3536 (2018).
23. O. Tunç, *On the qualitative analysis of integro-differential equations with constant time lag*, Appl. Math. Inf. Sci., **14**, № 1, 57 – 63 (2020).
24. C. Tunç, O. Tunç, *New qualitative criteria for solutions of Volterra integro-differential equations*, Arab J. Basic and Appl. Sci., **25**, № 3, 158 – 165 (2018).
25. C. Tunç, O. Tunç, *A note on the qualitative analysis of Volterra integro-differential equations*, J. Taibah Univ. Sci., **13**, № 1, 490 – 496 (2019).

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