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A REMARK ON COVERING OF COMPACT KÄHLER MANIFOLDS AND APPLICATIONS*

ЗАУВАЖЕННЯ ЩОДО ПОКРИТТЯ КОМПАКТНИХ КЕЛЕРОВИХ МНОГОВИДІВ ТА ЇХ ЗАСТОСУВАННЯ

Recently, Kolodziej proved that, on a compact Kähler manifold M, the solutions to the complex Monge – Ampère equation with the right-hand side in L^p, $p > 1$, are Hölder continuous with the exponent depending on M and $\|f\|_p$ (see [Math. Ann., **342**, 379 – 386 (2008)]). Then, by the regularization techniques in [J. Algebraic Geom., **1**, 361 – 409 (1992)], the authors in [J. Eur. Math. Soc., **16**, 619 – 647 (2014)] have found the optimal exponent of the solutions. In this paper, we construct a cover of the compact Kähler manifold M which only depends on curvature of M. Then, as an application, base on the arguments in [Math. Ann., **342**, 379 – 386 (2008)], we show that the solutions are Hölder continuous with the exponent just depending on the function f in the right-hand side and upper bound of curvature of M.

Нещодавно Колодзей довів, що на компактному келеровому многовиді M розв'язки комплексного рівняння Монжа – Ампера із правою частиною у L^p, $p > 1$, є неперервними за Гельдером з експонентою, що залежить від M та $\|f\|_p$ (див. [Math. Ann., **342**, 379 – 386 (2008)]). Після цього, за допомогою методу регуляризації з [J. Algebraic Geom., **1**, 361 – 409 (1992)], автори роботи [J. Eur. Math. Soc., **16**, 619 – 647 (2014)] знайшли оптимальну експоненту розв'язків. У цій роботі ми будуємо покриття компактного келерового многовиду M, яке залежить лише від кривини M. Далі, як застосування, використовуючи аргументацію з [Math. Ann., **342**, 379 – 386 (2008)], доводимо, що розв'язки є неперервними за Гельдером з експонентою, що залежить лише від функції f у правій частині та верхньої межі кривини M.

1. Introduction. Let M be a compact n-dimensional Kähler manifold with the fundamental form ω given in local coordinates by

$$\omega = \frac{i}{2}\,\Sigma_{k,j} g_{k\bar{j}}\, dz^k \wedge d\bar{z}^j.$$

An upper semicontinuous function u on M is called ω-plurisubharmonic if $dd^c u + \omega \geq 0$.

Consider the Monge – Ampère equation

$$(dd^c u + \omega)^n = f\omega^n, \tag{1.1}$$

where the given function $f \in L^1(M),\ f \geq 0$ and $\displaystyle\int_M f\omega^n = \int_M \omega^n$.

Now, we recall some results achieved on the equation (1.1) recently. In [20], by using the continuous method, S. T. Yau has shown that the equation (1.1) has solutions belong to $PSH \cap \cap\, C^\infty(M)$, when $f \in C^\infty(M),\ f > 0,\ \displaystyle\int_M f\omega^n = 1$, with a constant error. Then, in [10], S. Kolodziej has proven that it has solutions belong to $PSH \cap C(M)$, when $f \in L^p(M),\ f \geq 0$, $\displaystyle\int_M f\omega^n = 1,\ p > 1$. Recall that this result solves in particular the Calabi conjecture and allows to construct Ricci flat metrics on X whenever $c_1(X) = 0$. In [11], the author has proven that L^∞-norm of a difference of solutions is controlled by L^1-norm of the difference of functions on the right-hand side (see Theorem 2.1 below). Continuing research the results in this direction, in [12], the author

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has shown that the solutions to the equation (1.1) are Hölder continuous with the exponent depending on $M,\ \|f\|_p$. A similar result was also proved in [8], when M is a bounded strongly pseudoconvex domain. By demonstrating the opposite case of the main result in [7], P. H. Hiep gave a result that is stronger than the result in [12] (see [14]). More exactly, P. H. Hiep proved, in a special case of μ measure, for every $f \in L^p(\mu)$ with $p > 1$ there exists a Hölder continuous ω-plurisubharmonic function u such that $(dd^c u + \omega)^n = f\mu$. Then, by the regularization techniques in [5], the authors in [6] have found the optimal exponent and other interesting results.

In this paper, we construct a cover for the compact Kähler manifold M which depends on the curvature of M. Then, as an application, we show that the solutions are Hölder continuous with the exponent just depending on L^p-norm of the function f in the right-hand side of (1.1) and upper bound of curvature of M.

The paper is organized as follows. In Section 2, after two necessary lemmas (Lemmas 2.1 and 2.2), we present main result, that is Theorem 2.3).

In Section 3, we show that the solutions are Hölder continuous with the exponent depends only on the L^p-norm of the function on the right-hand side of (1.1) and the upper bound of the curvature of M in Theorem 3.1.

2. A covering on compact Kähler manifolds. First, recall that we use the normalization $d = \partial + \bar{\partial},\ d^c = i\left(\bar{\partial} - \partial\right)$. According to [1, 2], the Monge–Ampère operator $(dd^c.)^n$ is well defined on the class of locally bounded plurisubharmonic functions (see also [3, 9]). Moreover, if $u \in PSH \cap L^\infty_{\mathrm{loc}}(M)$ then by [1] $(dd^c u)^n$ is a non-negative Borel measure.

On a compact Kähler manifold M with fundamental form ω, the L^p-norm of function $f \in$ $\in L^p(M),\ p > 0$ is defined by

$$\|f\|_p = \left(\int\limits_M |f|^p \omega^n\right)^{1/p}.$$

Here we cite the result about stability of solutions that is set up in [11].

Theorem 2.1. *Given $p > 1,\ \varepsilon > 0,\ c_0 > 0$ and $\|f\|_p < c_0,\ \|g\|_p < c_0$ satisfying the normalizing condition in* (1.1) *there exists $c(\varepsilon, c_0)$ such that*

$$\|\varphi - \psi\|_\infty \le c(\varepsilon, c_0)\|f - g\|_1^{1/(n+3+\varepsilon)}.$$

Here $\varphi,\ \psi$ are solutions of (1.1) *corresponding to the functions $f,\ g$ on the right-hand side.*

Proof. See [11].

Let Ω be a domain in $\mathbb{C}^n$. For fixed $\delta > 0$ we consider $\Omega_\delta = \{z \in \Omega\colon \operatorname{dist}(z, \partial\Omega) > \delta\}$. With $u \in PSH(\Omega)$, we define a function $\widetilde{u}_\delta$ on Ω_δ as follows:

$$\widetilde{u}_\delta(z) = \left[\tau(n)\delta^{2n}\right]^{-1} \int\limits_{|\zeta|\le\delta} u(z+\zeta)dV(\zeta), \qquad \tau(n) = \int\limits_{|\zeta|\le 1} dV(\zeta),$$

where dV denotes the Lebesgue measure. Then $\widetilde{u}_\delta$ is a plurisubharmonic in Ω_δ. On the other hand, by [8] we have the following inequality:

$$\int\limits_{\Omega_\delta} \left(\widetilde{u}_\delta - u\right)(\zeta)dV(\zeta) \le c_1\|\Delta u\|_1\delta^2 \tag{2.1}$$

with the constant c_1 depending only on the dimension.

The following main results on compact Kähler manifolds (Theorems 2.2 and 2.3) are the basis for expanding the main results in [12]. Before presenting the theorems, we shall prove two lemmas to prove the main theorems.

For each matrix $A = (a_{ij})_{i,j=\overline{1,n}}$, $a_{ij} \in \mathbb{C}$, we set A^* is the conjugate transpose matrix of A (i.e., $A^* = (\bar{a}_{ji})_{i,j=\overline{1,n}}$). Set I is an unit matrix and $\|A\|$ is a norm of matrix A.

Lemma 2.1. *Let C be a matrix such that $A = CBC^*$ with $\|A-I\| < \epsilon$, $\|B-I\| < \epsilon$, $\epsilon < 1/3$. Then $\|CC^* - I\| < 5\epsilon$ and $\|C^*C - I\| < 5\epsilon$.*

Proof. Set $A = I + E$ and $B = I + F$ with $\|E\| < \epsilon$, $\|F\| < \epsilon$. We have

$$\|CC^* - I\| = \|CC^* - A + E\| = \|CC^* - CBC^* + E\| =$$
$$= \|E - CFC^*\| \leq \|E\| + \|C\|\|F\|\,\|C^*\|\,.$$

Hence

$$\|CC^*\| \leq \|I\| + \|E\| + \|C\|\|F\|\,\|C^*\| \leq 1 + \epsilon + \epsilon\|C\|\,\|C^*\|\,.$$

Moreover, since $\|C\|^2 = \|C^*\|^2 = \|CC^*\|$, we obtain

$$\|C\|^2 \leq 1 + \epsilon + \epsilon\|C\|^2 \Leftrightarrow \|C\|^2 \leq \frac{1+\epsilon}{1-\epsilon} < 4 \Rightarrow \|C\| < 2.$$

From this, we infer that

$$\|CC^* - I\| = \|E - CFC^*\| \leq \|E\| + \|C\|\|F\|\,\|C^*\| < \|E\| + 4\|F\| < 5\epsilon.$$

On the other hand, since $B = C^{-1}A\left(C^{-1}\right)^*$, applying the above result for C and A, B invert each other we have $\left\|C^{-1}\left(C^{-1}\right)^* - I\right\| < 5\epsilon$. Now, from this we get $\|C^*C - I\| < 5\epsilon$.

***Remark* 2.1.** i) With $z = [z_1, z_2, \ldots, z_n]^t$ and C be a matrix, we have

$$\|z\|^2 = z^*z = z_1\bar{z}_1 + \ldots + z_n\bar{z}_n,$$
$$\|Cz\|^2 = (Cz)^*(Cz) = z^*C^*Cz.$$

From these formulas, we obtain

$$\|Cz\|^2 - \|z\|^2 = z^*C^*Cz - z^*z = z^*\left(C^*C - I\right)z.$$

So, if $\|C^*C - I\| < \epsilon$, then

$$(1-\epsilon)\|z\|^2 < \|Cz\|^2 < (1+\epsilon)\|z\|^2.$$

ii) We denote by $\mathbb{B}_r$ is the open ball of radius $r > 0$ and $\mathbb{B}_r(z)$ is the open ball of radius r centered at z in $\mathbb{C}^n$.

Lemma 2.2. *Let $U \subset \mathbb{C}^n$ and $f : U \longrightarrow \mathbb{C}^n$ be a holomorphic function. Then we have the following estimate:*

$$\|f(\omega) - f(z)\| \leq \|Df(z)(\omega - z)\| + \sup_{\mathbb{B}_r(z)} \left\|D^2 f(z)\right\| \|\omega - z\|^2 \tag{2.2}$$

for all $z, \omega \in U$, $\|\omega - z\| < r$ and $\mathbb{B}_r(z) \subset U$.

Proof. This lemma as follows from the Taylor inequality for real functions.

Now we prove the main result about a covering for Kähler compact manifolds as follows.

Theorem 2.2. *Let (M,ω) be a Kähler compact manifold. Then there exist charts $\{U_j\}_{j=\overline{1,m}}$ of M and holomorphic bijective functions $f_j : U_j \longrightarrow \mathbb{B}_{3r}$ such that*

$$\bigcup_{j=1}^{m} f_j^{-1}(\mathbb{B}_r) = M,$$

$$\frac{i}{2}\sum_{l=1}^{n} dz_l \wedge d\overline{z}_l \leq \left(f_j^{-1}\right)^* \omega \leq 2i\sum_{l=1}^{n} dz_l \wedge d\overline{z}_l \quad on \quad \mathbb{B}_{3r}, \tag{2.3}$$

and

$$\|D(f_{jk})D(f_{jk})^* - I\| < \epsilon r^2 \quad on \quad f_j(U_j \cap U_k), \tag{2.4}$$

where $f_{jk} = f_k \circ f_j^{-1} : f_j(U_j \cap U_k) \longrightarrow f_k(U_j \cap U_k)$ is the local translate function on $U_j \cap U_k$ and $\epsilon > 0$ depending on the curvature of M.

Proof. First, in order to prove (2.3), we use the techniques in the proof of Theorem 4.8 in [4]. Let T_M and T_M^* are the tangent and cotangent bundles of M. Then, for each $a \in M$, since ω is Kähler form, we can choose the local coordinates $z' = (z'_1, \ldots, z'_n)$ such that $(dz'_1, \ldots, dz'_n)$ is an ω-orthonormal basis of T_a^*M. Hence,

$$\omega = i\sum_{l=1}^{n} \omega_{lm} dz'_l \wedge d\overline{z}'_m,$$

where

$$\omega_{lm} = \delta_{lm} + O\left(\|z'\|\right) = \delta_{lm} + \sum_{j=1}^{n}\left(a_{jlm}z'_j + a'_{jlm}\overline{z}'_j\right) + O(\|z'\|^2).$$

By ω is real and Kähler form, we have $a'_{jlm} = \overline{a}_{jml}$ and $a_{jlm} = a_{ljm}$. Put

$$z_m = z'_m + \frac{1}{2}\sum_{j,l=1}^{n} a_{jlm} z'_j z'_l, \quad 1 \leq m \leq n.$$

Then $(z_1, \ldots, z_n)$ is a coordinates system at a and

$$dz_m = dz'_m + \sum_{j=1}^{n} a_{jlm} z'_j dz'_l.$$

It follows that

$$i\sum_{m=1}^{n} dz_m \wedge d\overline{z}_m = i\sum_{m=1}^{n} dz'_m \wedge d\overline{z}'_m + i\sum_{j,l,m=1}^{n} a_{jlm} z'_j dz'_l \wedge d\overline{z}'_m +$$

$$+ i\sum_{j,l,m=1}^{n} \overline{a}_{jlm} \overline{z}'_j dz'_m \wedge d\overline{z}'_l + O(\|z'\|^2) =$$

$$= i \sum_{j,l,m=1}^{n} \omega_{lm} dz'_l \wedge d\overline{z}'_m + O(\|z'\|^2) = \omega + O(\|z'\|^2).$$

So we can conclude that, at every points $a \in M$, we have a holomorphic coordinate system $(z'_n, \ldots, z'_n)$ centered at a such that

$$\omega = i \sum_{l,m=1}^{n} \omega_{lm} dz'_l \wedge d\overline{z}'_m, \quad \omega_{lm} = \delta_{lm} + O(\|z'\|^2). \tag{2.5}$$

Assume that the coordinates $(z'_n, \ldots, z'_n)$ are chosen such that (2.5) is satisfied. Then by the Taylor expansion we get

$$\omega_{lm} = \delta_{lm} + O(\|z'\|^2) = \delta_{lm} + \sum_{j,k=1}^{n} \left(a_{jklm} z'_j \overline{z}'_k + a'_{jklm} z'_j z'_k + a''_{jklm} \overline{z}'_j \overline{z}'_k \right) + O\left(\|z'\|^3\right). \tag{2.6}$$

However $a'_{jklm} = a'_{kjlm}$, $a''_{jklm} = \overline{a}'_{jkml}$, $\overline{a}_{jklm} = a_{kjml}$. Moreover, by Kähler condition $\partial\omega_{lm}/\partial z'_j = = \partial\omega_{lm}/\partial z'_j$ at $z' = 0$ we have $a'_{jklm} = a'_{lkjm}$, i.e., a'_{jklm} is invariant under all permutations of j, k, l. Next, if we put

$$z_m = z'_m + \frac{1}{3} \sum_{j,k,l=1}^{n} a'_{jklm} z'_j z'_k z'_l, \quad 1 \le m \le n,$$

then by (2.6) we infer that

$$dz_m = dz'_m + \sum_{j,k,l=1}^{n} a'_{jklm} z'_j z'_k dz'_l, \quad 1 \le m \le n,$$

$$\omega = i \sum_{m=1}^{n} dz_m \wedge d\overline{z}_m + i \sum_{j,k,l,m=1}^{n} a_{jklm} z'_j \overline{z}'_k dz'_l \wedge d\overline{z}'_m + O\left(\|z'\|^3\right),$$

$$\omega = i \sum_{m=1}^{n} dz_m \wedge d\overline{z}_m + i \sum_{j,k,l,m=1}^{n} a_{jklm} z_j \overline{z}_k dz_l \wedge d\overline{z}_m + O\left(\|z\|^3\right). \tag{2.7}$$

Now we have

$$\left\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle = \delta_{lm} + \sum_{j,k=1}^{n} a_{jklm} z_j \overline{z}_k + O\left(\|z\|^3\right),$$

$$\partial \left\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle = \left\{ D' \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\} = \sum_{j,k=1}^{n} a_{jklm} \overline{z}_k dz_j + O\left(\|z\|^2\right).$$

Then the Chern curvature tensor $\Theta(T_M)_a$ can find by

$$\Theta(T_M) \frac{\partial}{\partial z_l} = D'' D' \left(\frac{\partial}{\partial z_l} \right) = - \sum_{j,k,m=1}^{n} a_{jklm} dz_j \wedge d\overline{z}_k \otimes \frac{\partial}{\partial z_m} + O(\|z\|).$$

So $-a_{jklm}$ are the coefficients of the Chern curvature tensor $\Theta(T_M)_a$. On the other hand, from (2.7), we have

$$\omega_{lm} = \left\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle = \delta + \sum_{j,k=1}^{n} a_{jklm} z_j \overline{z}_k + O\left(\|z\|^3\right).$$

This gives (2.3).

In order to obtain (2.4), we proceed as follows. From the above, we can assume that, at each $a \in M$, we can find a neighbourhood V_a of a and a holomorphic bijective function $f_a : V_a \longrightarrow \mathbb{B}_{s_a}$ such that

$$\left(f_a^{-1}\right)^* \omega(z) = i \sum_{l=1}^{n} dz_l \wedge d\overline{z}_l + O\left(\|z\|^2\right), \quad z \in \mathbb{B}_{s_a},$$

with $O\left(\|z\|^2\right)$ depending on the curvature of M. Now, by compactness of M we can assume that $s_a \geq s > 0$ $\forall a \in M$ and

$$\left(f_a^{-1}\right)^* \omega(z) = i \sum_{l=1}^{n} dz_l \wedge d\overline{z}_l + O\left(\|z\|^2\right), \quad z \in \mathbb{B}_{s},$$

uniformly for $a \in M$. Hence, with $\epsilon > 0$ depending on the curvature of the M, we can choose $r = r(\varepsilon)$ small enough such that

$$\left\| \left(f_a^{-1}\right)^* \omega - i \sum_{l=1}^{n} dz_l \wedge d\overline{z}_l \right\| < \frac{\epsilon}{5} r^2$$

on $\mathbb{B}_{3r}$. Set $U_a = f_a^{-1}(\mathbb{B}_{3r})$, then by the compactness of M, there exist $m = m(r)$ points $a_1, a_2, \ldots, a_m \in M$ such that the family $\left\{f_{a_j}^{-1}(\mathbb{B}_r)\right\}_{j=\overline{1,m}}$ is open cover of M. Set

$$U_j = f_j^{-1}(\mathbb{B}_{3r}) \quad \text{and} \quad f_j = f_{a_j}.$$

Fixed $1 \leq j, k \leq m$, we set that

$$\left(f_j^{-1}\right)^* \omega = i \sum_{1 \leq l,t \leq n}^{n} a_{lt} dz_l \wedge d\overline{z}_t,$$

$$\left(f_k^{-1}\right)^* \omega = i \sum_{1 \leq l,t \leq n}^{n} b_{lt} dz_l \wedge d\overline{z}_t.$$

Since $\left(f_j^{-1}\right)^* \omega = (f_{jk})^* \left(\left(f_k^{-1}\right)^* \omega\right)$ on $f_j(U_j \cap U_k)$, we get $A = Df_{jk} B Df_{jk}^*$ on $f_j(U_j \cap U_k)$, where $A = \left(f_j^{-1}\right)^* \omega$, $B = \left(f_k^{-1}\right)^* \omega$. Now using Lemma 2.1 we obtain

$$\|D(f_{jk}) D(f_{jk})^* - I\| < \epsilon r^2 \quad \text{on} \quad f_j(U_j \cap U_k).$$

Theorem 2.2 is proved.

From Lemma 2.2, Theorem 2.2 and Remark 2.1, we are ready to prove the following main result.

Theorem 2.3. *Let (M, ω) be a Kähler compact manifolds. Then there exist charts $\{U_j\}_{j=\overline{1,m}}$ of M and holomorphic bijective functions $f_j : U_j \longrightarrow \mathbb{B}_{3r}$ such that*

$$\bigcup_{j=1}^{m} f_j^{-1}(\mathbb{B}_r) = M,$$

$$\frac{i}{2}\sum_{l=1}^{n} dz_l\wedge d\overline{z}_l \le \left(f_j^{-1}\right)^*\omega \le 2i\sum_{l=1}^{n} dz_l\wedge d\overline{z}_l \quad \textit{on} \quad \mathbb{B}_{3r},$$

and

$$f_{jk}\left(\mathbb{B}_\delta(z)\right)\subset \mathbb{B}_{C_0\delta}(f_{jk}(z)) \quad \forall z\in f_j\big(f_j^{-1}(\mathbb{B}_{2r})\cap f_k^{-1}(\mathbb{B}_{2r})\big) \quad \forall\delta\in(0,\delta_\epsilon),$$

where $C_0 = 1+\epsilon r^2$ *and* $\epsilon>0$ *depending on the curvature of* M.

Proof. First of all, we use the same notation as in Theorem 2.2. Now we wish to apply the Lemma 2.2 for f_{jk}, with $w\in\mathbb{B}_\delta(z)$. Indeed, by (2.2) with f replaced by f_{jk}, we have

$$\|f_{jk}(w)-f_{jk}(z)\| \le \|Df_{jk}(w-z)\| + \sup_{\mathbb{B}_\delta(z)}\left\|D^2(f_{jk})\right\|\,\|w-z\|^2.$$

Therefore, in view of Theorem 2.2 and Remark 2.1, we get

$$\|Df_{jk}(w-z)\| < \sqrt{1+\epsilon r^2}\delta < \left(1+\frac{\epsilon r^2}{2}\right)\delta.$$

Set $d=\sup_{\mathbb{B}_\delta(z)}\left\|D^2(f_{jk})\right\|<\infty$ and choose $\delta_\epsilon=\delta(\varepsilon)<\dfrac{\epsilon r^2}{2d}$, we conclude that

$$\|f_{jk}(w)-f_{jk}(z)\| < \left(1+\epsilon r^2\right)\delta.$$

Therefore,

$$f_{jk}(w)\in\mathbb{B}_{C_0\delta}(f_{jk}(z)) \quad \text{with} \quad C_0=1+\epsilon r^2 \quad \text{and for all} \quad \delta\in(0,\delta_\epsilon).$$

Theorem 2.3 is proved.

3. Applications to the complex Monge – Ampère equation. In this section, we apply the main result to show that the solutions to the equation (1.1) are Hölder continuous with the exponent just depending on the upper bound of the curvature of M.

Theorem 3.1. *Assume that* $p>1$ *and* $f\in L^p(M)$ *satisfying the normalizing condition in* (1.1). *Then the solutions to the equation* (1.1) *are Hölder continuous with the Hölder exponent which depends on* $\|f\|_p$ *and upper bound of curvature of* M.

Proof. Take $\epsilon>0$ which only depends on curvature of M. By Theorem 2.3, there exist charts $\{U_j\}_{j=\overline{1,m}}$ of M and holomorphic bijective functions $f_j: U_j \longrightarrow \mathbb{B}_{3r}$ such that

$$\bigcup_{j=1}^{m} f_j^{-1}(\mathbb{B}_r) = M,$$

$$\frac{i}{2}\sum_{l=1}^{n} dz_l\wedge d\overline{z}_l \le \left(f_j^{-1}\right)^*\omega \le 2i\sum_{l=1}^{n} dz_l\wedge d\overline{z}_l \quad \text{on} \quad \mathbb{B}_{3r},$$

and

$$f_{jk}(\mathbb{B}_\delta)(z)\subset \mathbb{B}_{C_0\delta}(f_{jk}(z)) \quad \forall z\in f_j\left(f_j^{-1}(\mathbb{B}_{2r})\cap f_k^{-1}(\mathbb{B}_{2r})\right) \quad \forall\delta\in(0,\delta_\epsilon), \tag{3.1}$$

where $C_0=1+\epsilon r^2$. For each $j=1,2,\ldots,m$, we set $B_j''=f_j^{-1}(\mathbb{B}_{3r})$, $B_j'=f_j^{-1}(\mathbb{B}_r)$, $B_j = f_j^{-1}(\mathbb{B}_{2r})$. Choose $h\in C^\infty(\mathbb{C}^n)$ such that $-1\le h\le 0$ on $\mathbb{C}^n$, $h=0$ on $\mathbb{B}_1$ and $h=-1$ on

$\mathbb{C}^n \setminus \mathbb{B}_{\frac{3}{2}}$. Set $\widehat{\rho}(z) = h\left(\frac{z}{r}\right)$. We have $\widehat{\rho} \in C^\infty(\mathbb{C}^n)$ such that $-1 \leq \widehat{\rho} \leq 0$ on $\mathbb{C}^n$, $\widehat{\rho} = 0$ on $\mathbb{B}_r$, $\widehat{\rho} = -1$ on $\mathbb{C}^n \setminus \mathbb{B}_{\frac{3r}{2}}$ and

$$dd^c\widehat{\rho} \geq -\frac{c(n)}{2r^2}\, i \sum_{l=1}^{n} dz_l \wedge d\bar{z}_l,$$

where $c(n)$ is a constant depending on n. Set $\rho_j = \widehat{\rho} \circ f_j$, then we obtain $\rho_j \in C^\infty(B_j'')$, $-1 \leq$ $\leq \rho_j \leq 0$ on B_j'', $\rho_j = 0$ on B_j' and $\rho_j = -1$ on the neighbourhood of ∂B_j. Since $dd^c\widehat{\rho} \geq$ $\geq -\frac{c(n)}{r^2}\left(f_j^{-1}\right)^*\omega$ on $\mathbb{B}_{3r}$, we get $dd^c\rho_j \geq -\frac{c(n)}{r^2}\,\omega$ on B_j''. Set $C = \frac{c(n)}{r^2}$ and fixed $N > 4$ big enough. Then, by $\log C_0 = \log\left(1 + \epsilon r^2\right) \leq \epsilon r^2$ and $C \log C_0 \leq \epsilon c(n)$, we can choose $r = r(\epsilon)$ small enough such that $2C_0 < N$, $\alpha < \frac{1}{q(n+3+\epsilon)+1}$ (where p, q conjugate) and

$$2(2C\|u\|_\infty + 1)\log C_0 < N^{-\alpha}\log N.$$

From the above it follows that

$$2(2C\|u\|_\infty + 1) < N^{-\alpha}\,\frac{\log N}{\log C_0}. \tag{3.2}$$

On the local chart B_j'', we define regularizations

$$\widehat{u}_{j,\delta}(z) = \max_{|t|<\delta} u(z+t), \quad z \in B_j.$$

Set $u_{j,\delta} = \widehat{u}_{j,\delta} \circ f_j^{-1}$ (function $u_{j,\delta}$ defined locally on the neighbourhood of 0 in $\mathbb{C}^n$). We also define two auxiliary functions

$$\chi(\delta) = \delta^{-\alpha} \max_j \max_{z\in\mathbb{B}_{2r}} \left(u_{j,\delta} - u \circ f_j^{-1}\right)(z),$$

$$\eta(\delta) = \max_j \max_{z\in\mathbb{B}_{2r}} (u_{j,C_0\delta} - u_{j,\delta})(z).$$

According to (3.1), we have

$$\max_{z\in\mathbb{B}_{2r}} |(u_{j,\delta} - u_{k,\delta})(z)| \leq \eta(\delta). \tag{3.3}$$

We will approximate the function u by ω-plurisubharmonic functions u_δ which are created by gluing together the local regularization $u_{j,\delta}$ (see [5]). Then by (3.3) the function $\eta(\delta)$ plays adjustment functions $u_{j,\delta}$ in the intersection of the charts when moving from local definition to global definition. Note that, due to the continuity of the function u (see [10]) should have $\lim_{\delta\to 0}\eta(\delta) = 0$. Set

$$u_\delta(z) = (1 + C_1\eta(\delta))^{-1} \max_j (u_{j,\delta}(z) + \eta(\delta)\rho_j(z)), \quad C_1 = 2C.$$

By (3.3) and the property of ρ_j the maximum in the above definition should always be achieved on $\mathbb{B}_{2r}$, so the function u_δ is continuous on $\bigcup_{j=1}^m f_j(B_j'')$. Moreover, by $C\eta(\delta) < 1$, we get

$$dd^c\left(u_{j,\delta}(z) + \eta(\delta)\rho_j(z)\right) \geq -\left(1 + C\eta(\delta)\right)\left(f_j^{-1}\right)^*\omega.$$

From the above results and the inequality approximately $dd^c \max(u, v)$ (see [1]) we derive

$$dd^c u_\delta + \omega > 0 \quad \text{(with } \delta \text{ is sufficiently small).} \tag{3.4}$$

To finish the proof, we need to verify the following proposition.

Proposition 3.1. *The function χ is bounded on some nonempty interval $(0, \widetilde{\delta})$.*

Proof. Suppose that $\chi(\delta) > \max(9, \chi(N\delta))$ and $N\delta < r/2$. Then we have

$$E = \bigcup_j \left\{ z \in \mathbb{B}_{2r} : \left(u_{j,\delta} - u \circ f_j^{-1}\right)(z) > \left(\frac{\chi(\delta)}{3} - 2\right)\delta^\alpha \right\} \neq \varnothing.$$

Choose the function g such that $g \circ f_j^{-1} = 0$ on E and $g = C_2 f$ on $M \setminus \bigcup_j f_j^{-1}(E)$ with C_2 as a constant satisfying condition $\int_M f\omega^n = \int_M \omega^n$. Set

$$\widetilde{u}_{j,\delta}(z) = \left[\tau(n)\delta^{2n}\right]^{-1} \int\limits_{|\zeta|\le\delta} u \circ f_j^{-1}(z+\zeta)\,dV(\zeta), \quad \tau(n) := \int\limits_{|\zeta|\le 1} dV(\zeta).$$

We will compare $u_{j,\delta}$ and $\widetilde{u}_{j,\delta}$ as follows. Given $z \in \mathbb{B}_{2r}$ we find t_z with $|t_z| = \delta$ such that

$$u_{j,\delta}(z) = u \circ f_j^{-1}(z + t_z) \le \widetilde{u}_{j,\sqrt{\delta}}(z + t_z) \le \widetilde{u}_{j,\sqrt{\delta}}(z) + 2\|u\|_\infty \sqrt{\delta}.$$

Since $\alpha < \dfrac{1}{2}$, we conclude from above estimate for $\delta < \delta_0$ and δ_0 small enough that

$$E \cap \mathbb{B}_{2r} \subset \left\{u_{j,\delta} - u \circ f_j^{-1} > \delta^\alpha\right\} \subset \left\{\widetilde{u}_{j,\sqrt{\delta}} - u \circ f_j^{-1} > \delta^\alpha/2\right\}.$$

As $\|\Delta u\|_1$ is bounded on every B_j'', thus, applying formula (2.1) we have

$$\int\limits_{E\cap\mathbb{B}_{2r}} \omega^n < C_3 \delta^{1-\alpha} \quad \text{for all} \quad j.$$

Therefore,

$$\int\limits_E \omega^n < C_4 \delta^{1-\alpha}.$$

So, by Hölder inequality, we get

$$\int\limits_E f\omega^n \le \|f\|_p \left(\int\limits_E \omega^n\right)^{1/q} \le C_5 \delta^{(1-\alpha)/q},$$

where C_5 depends only on $\|f\|_p$. So, if v is a solution of $(\omega + dd^c v)^n = g\omega^n$, then by Theorem 2.1 with $\delta < \delta_1$ (δ_1 small enough) we conclude that

$$\|u - v\|_\infty \le \|f - g\|^{\frac{1}{(n+3+\epsilon)}} \le C_6 \delta^{\frac{1-\alpha}{q(n+3+\epsilon)}} \le \delta^\alpha, \tag{3.5}$$

where α is choosen such that $\alpha < \dfrac{1-\alpha}{q(n+3+\epsilon)}$.

To end the proof of Proposition 3.1, we will prove the following lemma.

Lemma 3.1. *If $z_0 \in \mathbb{B}_{2r}$ such that $\left(u_{j_0,\delta} - u \circ f_{j_0}^{-1}\right)(z_0) = \chi(\delta)\delta^\alpha$, then we have*

$$\sup_{\cup_j f_j(B_j)\setminus E} \left(u_\delta - v \circ f_j^{-1}\right) < (u_\delta - v)(z_0).$$

Proof. We can assume that $u > 1$. Take $z \in \left(\bigcup_j f_j(B_j) \setminus E\right) \cap \mathbb{B}_{2r}$, then

$$\big(u_{j,\delta} - u \circ f_j^{-1}\big)(z) \leq \left(\frac{\chi(\delta)}{3} - 2\right) \delta^{\alpha}.$$

So, by (3.5), we get

$$\big(u_{j,\delta} - v \circ f_j^{-1}\big)(z) \leq \left(\frac{\chi(\delta)}{3} - 1\right) \delta^{\alpha}.$$

Since $u > 1$, we infer that

$$\big(u_{\delta} - v \circ f_j^{-1}\big)(z) \leq \max_{j\,:\,z\in B_j} \big(u_{j,\delta} - v \circ f_j^{-1}\big)(z) \leq \left(\frac{\chi(\delta)}{3} - 1\right) \delta^{\alpha}. \tag{3.6}$$

Again, by (3.5) we conclude similarly that

$$\big(u_{j_0,\delta} - v \circ f_{j_0}^{-1}\big)(z_0) \geq (\chi(\delta) - 1)\, \delta^{\alpha}.$$

So, by definition of the functions, we have

$$\big(u_{\delta} - v \circ f_{j_0}^{-1}\big)(z_0) \geq (\chi(\delta) - 1)\, \delta^{\alpha} - \eta(\delta)(2C\|u\|_{\infty} + 1). \tag{3.7}$$

With δ small enough and by the three circles theorem we get

$$\big(u_{j,N\delta} - u_{j,\delta}\big) \geq \frac{\log N}{\log C_0}\, \left(u_{j,C_0\delta} - u_{j,\delta}\right).$$

Choose j such that

$$\eta(\delta) = \max_{z\in\mathbb{B}_{2r}} \left(u_{j,C_0\delta} - u_{j,\delta}\right)(z).$$

From this we obtain

$$(N\delta)^{\alpha}\chi(N\delta) \geq \frac{\log N}{\log C_0}\, \eta(\delta).$$

On the other hand, since $\chi(\delta) \geq \chi(N\delta)$ then, from (3.2), we get the following result:

$$\delta^{\alpha}\chi(\delta) \geq \frac{\log N}{\log C_0}\, \eta(\delta) N^{-\alpha} > 2\eta(\delta)(2C\|u\|_{\infty} + 1).$$

The above result combined with (3.7), we obtain

$$\big(u_{\delta} - v \circ f_{j_0}^{-1}\big)(z_0) \geq \left(\frac{\chi(\delta)}{2} - 1\right) \delta^{\alpha}.$$

From this and (3.6), we complete the proof of Lemma 3.1.

We proceed to finish the proof of the theorem.

Applying the Lemma 3.1, we can find C_7 such that

$$z_0 \in U = \left\{v \circ f_j^{-1} < u_{\delta} - C_7\right\} \subset E.$$

By the comparison principle (see [10]) and (3.4), we lead to a contradiction because

$$0 < \int\limits_U \left(dd^c u_\delta + \left(f_j^{-1}\right)^* \omega\right)^n \leq \int\limits_U (dd^c v + \omega) \leq \int\limits_E (dd^c v + \omega)^n = \int\limits_E g\omega^n = 0.$$

This contradiction shows that the choice of small enough δ such that

$$\chi(\delta) > \max(9, \chi(N\delta))$$

is impossible. Thus, the proof of Proposition 3.1 and, so, of Theorem 3.1 is completed.

Finally, we get the following corollary as a special case of Theorem 3.1 when M be a compact Kähler manifold of zero curvature (such as in [17 – 19]).

***Corollary* 3.1.** *Let M be a compact Kähler manifold of zero curvature. Then the solutions of* (1.1) *in Theorem* 3.1 *are Hölder continuous with the Hölder exponent which depends only on $\|f\|_p$.*

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