DOI: 10.37863/umzh.v73i1.6039

UDC 512.7

M. Masjed-Jamei, M. R. Beyki (Dep. Math., K. N. Toosi Univ. Technology, Tehran, Iran)

ON A BIVARIATE KIND OF q-EULER AND q-GENOCCHI POLYNOMIALS*

ПРО БІВАРІАНТНІ ПОЛІНОМИ ТИПУ q-ЕЙЛЕРА І q-ДЖЕНОКІ

Two bivariate kinds of q-Euler and q-Genocchi polynomials are introduced and their basic properties are stated and proved.

Визначено біваріантні поліноми типу q-Ейлера і q-Дженокі. Також сформульовано і доведено їхні основні властивості.

1. Introduction. Euler and Genocchi polynomials have found valuable applications in various branches of mathematics such as analytic number theory, numerical analysis, geometric design and mathematical physics. For instance, Euler numbers are directly related to the Brouwer fixed point theorem and vector fields [12]. These numbers are extended by Carlitz in [1] and called q-Euler numbers. In [10], the authors have presented a new q-analogue of the exponential generating function of Euler polynomials and in [5] a new q-extension of Euler numbers and polynomials are introduced. In [2], the authors have obtained some new symmetric identities for q-Genocchi polynomials arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$. Finally, in [8], a new type of Euler polynomials and numbers are introduced.

In this paper, we first give some preliminary definitions of q-calculus and the q-analogue of some elementary functions, which are required in Section 3, in order to extend both ordinary q-Euler and q-Genocchi polynomials. In this sense, we introduce a bivariate kind of q-Euler and q-Genocchi polynomials in Section 3 and present some basic properties of the extended q-Euler polynomials. Of course, because of similarity, we only give the properties of bivariate q-Genocchi polynomials without proof in Section 4.

2. Preliminaries and definitions. If $q \neq 1$ and α is a real number, the q-analogue of α is defined by [3, 4]

$$[\alpha]_q = \frac{1-q^\alpha}{1-q},$$

and

$$[n]_q! = \prod_{k=1}^{n} [k]_q = [n]_q[n-1]_q \ldots [1]_q, \quad n \in \mathbb{N},$$

is the q-analogue of $n!$ where $\lim_{q\to 1}[\alpha]_q = \alpha$ and $[0]_q! = 1$.

The q-derivative operator of an arbitrary function defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

satisfies the rules

* This work was supported by the Alexander von Humboldt Foundation (grant number Ref 3.4-IRN-1128637-GF-E).

$$D_q\big(f(x) \pm kg(x)\big) = D_q f(x) \pm k D_q g(x),$$

$$D_q\big(f(x)g(x)\big) = f(x)D_q g(x) + g(qx)D_q f(x) = g(x)D_q f(x) + f(qx)D_q g(x),$$

and

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)} = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)}.$$

Although there is not a general chain rule for q-derivatives, we have

$$D_q\big(f(\alpha x^\beta)\big) = \alpha[\beta]_q x^{\beta-1}(D_{q^\beta} f)(\alpha x^\beta)$$

and

$$D_q\big(f(\alpha x)\big) = \alpha(D_q f)(\alpha x).$$

The function

$$(x-a)_q^n = \begin{cases} (x-a)(x-aq)(x-aq^2)\ldots(x-aq^{n-1}), & n = 1, 2, \ldots, \\ 1, & n = 0, \end{cases} \tag{1}$$

is the q-analogue of $(x-a)^n$, which can be extended to

$$(x-a)_q^{-n} = \frac{1}{(x-aq^{-n})_q^n}, \quad n \in \mathbb{N}.$$

It is easy to check that $D_q(x-a)_q^n = [n]_q(x-a)_q^{n-1}$.

The q-Pochhammer symbol is indeed a particular case of (1) for $x = 1$ and is defined as

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{with} \quad (a;q)_0 = 1, \quad n \in \mathbb{N}. \tag{2}$$

When $n \to \infty$, the limit relation of (2) is denoted by $(a;q)_\infty$ (provided that $|q| < 1$) and in the sequel we have

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{N}_0, \quad |q| < 1,$$

while for any complex number α, it reads as

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \quad |q| < 1.$$

The q-binomial coefficient is defined for positive integers n and k by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

In [9], Schork studied Ward's "Calculus of sequences" and introduced a q-addition symbol as

$$(x \oplus_q y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

It is clear that the q-subtraction can be defined in the same way as

$$(x \ominus_q y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (-y)^{n-k} = \big(x \oplus_q (-y)\big)^n.$$

A q-analogue of the classical exponential function e^x is defined by [3, 6]

$$D_q e_q^x = e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \qquad 0 < |q| < 1, \quad |x| < 1,$$

where

$$e_q^x e_q^y = e_q^{x \oplus_q y} \qquad \text{and} \qquad e_q^{a(x \oplus_q y)} = e_q^{ax \oplus_q ay}.$$

Another q-type of the exponential function is defined by

$$E_q^x = \sum_{n=0}^{\infty} q^{\binom{n}{k}} \frac{x^n}{[n]_q!}, \qquad 0 < |q| < 1,$$

so that these two q-exponential functions are closely related to each other by the relation

$$e_q^x E_q^{-x} = 1. \tag{3}$$

Finally, in this section we state q-Taylor's theorem for formal power series [4].

Theorem 2.1. *For any polynomial $p(x)$ of degree n and any arbitrary point $x = a$, we have*

$$p(x) = \sum_{j=0}^{n} D_q^{(j)} f(a) \frac{(x-a)_q^j}{[j]_q!}.$$

Hence, any formal power series $f(x) = \sum_{j=0}^{\infty} c_j x^j$ can be expressed in terms of a generalized Taylor series $\sum_{j=0}^{\infty} D_q^{(j)} f(0) \dfrac{x^j}{[j]_q!}$ such that

$$c_j = \frac{D_q^{(j)} f(0)}{[j]_q!} \quad \forall j \in \mathbb{N}_0 \qquad \textit{and} \qquad D_q f(x) = \sum_{j=1}^{\infty} [j]_q c_j x^{j-1}.$$

2.1. q-Appell sets, q-Euler and q-Genocchi polynomials and some related properties. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a polynomial set in which $P_n(x)$ is of exact degree n. $\{P_n(x)\}_{n=0}^{\infty}$ is a q-Appell set if

$$D_q P_{n+1}(x) = [n+1]_q P_n(x).$$

Such sets were first introduced by Sharma and Chak [11]. The following characterization theorem holds in this regard.

Theorem 2.2 [11]. *Let $\{P_n(x)\}_{n=0}^{\infty}$ ba a polynomial set. The following assertions are equivalent:*

1. $\{P_n(x)\}$ *is a q-Appell polynomial set.*

2. *There exists a sequence* $(a_k)_{k\geq 0}$ *independent of* $n,\ a_0 = 1,$ *such that*

$$P_n(x) = \sum_{k=0}^{n} a_k \frac{[n]_q!}{[n-k]_q!} x^{n-k}.$$

3. $\{P_n(x)\}$ *is generated by*

$$A(t)e_q(xt) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{[n]_q!},$$

where

$$A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}, \quad a_0 = 1.$$

The q-Euler polynomials are defined by [5]

$$\frac{2e_q^{xt}}{e_q^t + 1} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!},$$

leading to the representation

$$E_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}(0) x^{n-k},$$

and the q-Genocchi polynomials are defined by [2]

$$\frac{2te_q^{xt}}{e_q^t + 1} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{[n]_q!},$$

leading to the representation

$$G_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}(0) x^{n-k}.$$

It is not difficult to verify for every $n \in \mathbb{N}$ that

$$D_q E_{n,q}(x) = [n]_q E_{n-1,q}(x) \qquad \text{and} \qquad D_q G_{n,q}(x) = [n]_q G_{n-1,q}(x).$$

Hence, q-Euler and q-Genocchi polynomials belong to q-Appell set.

3. A bivariate kind of q-Euler polynomials. Let $x, y \in \mathbb{R}$. Then the Taylor expansion of the two functions $e^{xt} \cos yt$ and $e^{xt} \sin yt$ are respectively as follows [7]:

$$e^{xt} \cos yt = \sum_{k=0}^{\infty} C_k(x, y) \frac{t^k}{k!}$$

and

$$e^{xt} \sin yt = \sum_{k=0}^{\infty} S_k(x, y) \frac{t^k}{k!},$$

where

$$C_k(x,y)=\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\binom{k}{2j}x^{k-2j}y^{2j} \tag{4}$$

and

$$S_k(x,y)=\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1}x^{k-2j-1}y^{2j+1}. \tag{5}$$

Now, a q-extension of the bivariate polynomials (4) and (5) can be considered.

If $x,y\in\mathbb{R}$, then

$$e_q^{xt}\cos_q yt=\sum_{k=0}^{\infty}C_{k,q}(x,y)\frac{t^k}{[k]_q!} \tag{6}$$

and

$$e_q^{xt}\sin_q yt=\sum_{k=0}^{\infty}S_{k,q}(x,y)\frac{t^k}{[k]_q!},$$

where

$$\cos_q z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{[2n]_q!}=\sum_{n=0}^{\infty}\frac{1+(-1)^n}{2}\frac{(\mathrm{i}z)^n}{[n]_q!}$$

and

$$\sin_q z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{[2n+1]_q!}=\mathrm{i}\sum_{n=0}^{\infty}\frac{(-1)^n-1}{2}\frac{(\mathrm{i}z)^n}{[n]_q!}.$$

In this sense, we have

$$\left(\sum_{k=0}^{\infty}a_k\frac{t^k}{[k]_q!}\right)\left(\sum_{k=0}^{\infty}b_k\frac{t^k}{[k]_q!}\right)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}_q a_j b_{k-j}\right)\frac{t^k}{[k]_q!}. \tag{7}$$

Proposition 3.1. *The polynomials $C_{k,q}(x,y)$ and $S_{k,q}(x,y)$ can be explicitly represented as*

$$C_{k,q}(x,y)=\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\begin{bmatrix}k\\ 2j\end{bmatrix}_q x^{k-2j}y^{2j} \tag{8}$$

and

$$S_{k,q}(x,y)=\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\begin{bmatrix}k\\ 2j+1\end{bmatrix}_q x^{k-2j-1}y^{2j+1}. \tag{9}$$

Proof. We have

$$e_q^{xt}\cos_q yt = \left(\sum_{k=0}^{\infty}\frac{(xt)^k}{[k]_q!}\right)\left(\sum_{k=0}^{\infty}\frac{1+(-1)^k}{2}\frac{(\mathrm{i}yt)^k}{[k]_q!}\right) =$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{1+(-1)^j}{2}(\mathrm{i}y)^j x^{k-j}\right)\frac{t^k}{[k]_q!} =$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\begin{bmatrix} k \\ 2j \end{bmatrix}_q x^{k-2j}y^{2j}\right)\frac{t^k}{[k]_q!}.$$

The proof of (9) is similar.

Proposition 3.2. *The following derivative rules are valid:*

$$D_{q,x}C_{k,q}(x,y) = [k]_q C_{k-1,q}(x,y), \tag{10}$$

$$D_{q,y}C_{k,q}(x,y) = -[k]_q S_{k-1,q}(x,y), \tag{11}$$

$$D_{q,x}S_{k,q}(x,y) = [k]_q S_{k-1,q}(x,y), \tag{12}$$

and

$$D_{q,y}S_{k,q}(x,y) = [k]_q C_{k-1,q}(x,y). \tag{13}$$

Proof. Relation (6) yields

$$\sum_{n=1}^{\infty} D_{q,x}C_{n,q}(x,y)\frac{t^n}{[n]_q!} = te_q^{xt}\cos_q yt = \sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!} =$$

$$= \sum_{n=1}^{\infty} C_{n-1,q}(x,y)\frac{t^n}{[n-1]_q!} = \sum_{n=0}^{\infty}[n]_q C_{n-1,q}(x,y)\frac{t^n}{[n]_q!},$$

proving (10). Other equations (11), (12) and (13) can be similarly derived.

Proposition 3.3. *The following identities hold:*

$$C_{k,q}(x,y) = \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q C_{k-j,q}(0,y)x^j \tag{14}$$

and

$$S_{k,q}(x,y) = \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q S_{k-j,q}(0,y)x^j. \tag{15}$$

Proof. By Proposition 3.2, for $j = 0, 1, \ldots, k$ we have

$$\frac{\partial_q^j}{\partial_q x^j}C_{k,q}(x,y) = [k]_q[k-1]_q\ldots[k-j+1]_q C_{k-j,q}(x,y),$$

while for $j > k$ we obtain

$$\frac{\partial_q^j}{\partial_q x^j} C_{k,q}(x,y) = 0,$$

because $C_{k,q}(x,y)$ is a polynomial of degree k in terms of x. Hence, the q-Taylor expansion of $C_{k,q}(x,y)$ at x gives

$$C_{k,q}(x+h,y) = \sum_{j=0}^{k} \frac{\partial_q^j}{\partial_q x^j} C_{k,q}(x,y) \frac{h^j}{[j]_q!} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q C_{k-j}(x,y) h^j,$$

in which $h \in \mathbb{R}$. It is now enough to take $x = 0$ and $h = x$ to reach (14). In a similar way, (15) can be derived.

Proposition 3.4. *For any $n \in \mathbb{N}_0$, the following power representations hold:*

$$\sum_{k=0}^{2n} (-1)^{n-k} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q C_{2n-k,q}(x,y) x^k = y^{2n}, \tag{16}$$

$$\sum_{k=0}^{2n+1} (-1)^{k} q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q C_{2n+1-k,q}(x,y) x^k = 0, \tag{17}$$

$$\sum_{k=0}^{2n} (-1)^{k} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q S_{2n-k,q}(x,y) x^k = 0, \tag{18}$$

and

$$\sum_{k=0}^{2n+1} (-1)^{n-k} q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q S_{2n+1-k,q}(x,y) x^k = y^{2n+1}. \tag{19}$$

Proof. Multiplying both sides of (6) by E_q^{-xt} and using (3), it follows that

$$\sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} \mathrm{i}^n y^n \frac{t^n}{[n]_q!} =$$

$$= \left(\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} (-1)^n x^n t^n \right) \left(\sum_{n=0}^{\infty} C_{n,q}(x,y) \frac{t^n}{[n]_q!} \right) =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-1)^k x^k C_{n-k,q}(x,y) \right) \frac{t^n}{[n]_q!}.$$

By setting $n \to 2n$ and $n \to 2n+1$ in the above relation, (16) and (17) are proved respectively. The proof of (18) and (19) is similar.

Based on previous comments, we are now in a good position to introduce two kinds of bivariate q-Euler polynomials as

$$\frac{2e_q^{xt}}{e_q^t + 1} \cos_q yt = \sum_{n=0}^{\infty} E_{n,q}^{(c)}(x,y) \frac{t^n}{[n]_q!}$$

and

$$\frac{2e_q^{xt}}{e_q^t+1}\sin_q yt=\sum_{n=0}^{\infty}E_{n,q}^{(s)}(x,y)\frac{t^n}{[n]_q!},$$

and give some basic properties of them in the sequel.

Proposition 3.5. *$E_{n,q}^{(c)}(x,y)$ and $E_{n,q}^{(s)}(x,y)$ can be represented in terms of q-Euler numbers as follows:*

$$E_{n,q}^{(c)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,q}(0)C_{n-k,q}(x,y) \tag{20}$$

and

$$E_{n,q}^{(s)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,q}(0)S_{n-k,q}(x,y). \tag{21}$$

Proof. By using the relation (7), we have

$$\sum_{n=0}^{\infty}E_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}=\frac{2}{e_q^t+1}e_q^{xt}\cos_q yt=$$

$$=\left(\sum_{n=0}^{\infty}E_{n,q}(0)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}C_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)=$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,q}(0)C_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!},$$

which proves (20). The proof of (21) is similar.

Proposition 3.6. *$E_{n,q}^{(c)}(x,y)$ and $E_{n,q}^{(s)}(x,y)$ can be represented in terms of $E_{n,q}(x)$ as follows:*

$$E_{n,q}^{(c)}(x,y)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\begin{bmatrix}n\\2k\end{bmatrix}_q E_{n-2k,q}(x)y^{2k} \tag{22}$$

and

$$E_{n,q}^{(s)}(x,y)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\begin{bmatrix}n\\2k+1\end{bmatrix}_q E_{n-2k-1,q}(x)y^{2k+1}. \tag{23}$$

Proof. The relation (22) follows since

$$\sum_{n=0}^{\infty}E_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}=\frac{2e_q^{xt}}{e_q^t+1}\cos_q yt=$$

$$=\left(\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{1+(-1)^n}{2}\mathrm{i}^n y^n\frac{t^n}{[n]_q!}\right)=$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1+(-1)^k}{2}\mathrm{i}^k y^k E_{n-k,q}(x)\right)\frac{t^n}{[n]_q!} =$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\begin{bmatrix} n \\ 2k \end{bmatrix}_q E_{n-2k,q}(x)y^{2k}\right)\frac{t^n}{[n]_q!}.$$

Similarly, (23) can be proved.

Proposition 3.7. *For every* $n \in \mathbb{N}_0$, *the following identities hold:*

$$E^{(c)}_{n,q}\big((1 \oplus_q x), y\big) + E^{(c)}_{n,q}(x,y) = 2C_{n,q}(x,y) \tag{24}$$

and

$$E^{(s)}_{n,q}\big((1 \oplus_q x), y\big) + E^{(s)}_{n,q}(x,y) = 2S_{n,q}(x,y). \tag{25}$$

Proof. We have

$$\sum_{n=0}^{\infty} E^{(c)}_{n,q}\big((1 \oplus_q x), y\big)\frac{t^n}{[n]_q!} = \frac{2e_q^{(1\oplus_q x)t}}{e_q^t+1}\cos_q yt =$$

$$= \frac{2e_q^{xt}(e_q^t+1-1)}{e_q(t)+1}\cos_q yt = 2e_q^{xt}\cos_q(yt) - \frac{2e_q^{xt}}{e_q^t+1}\cos_q yt =$$

$$= 2\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} E^{(c)}_{n,q}(x,y)\frac{t^n}{[n]_q!},$$

which proves (24). The relation (25) can be similarly proved.

***Corollary* 3.1.** *The following relations hold:*

$$E^{(c)}_{2n,q}(1,y) + E^{(c)}_{2n,q}(0,y) = 2(-1)^n y^{2n}$$

and

$$E^{(s)}_{2n+1,q}(1,y) + E^{(s)}_{2n+1,q}(0,y) = 2(-1)^n y^{2n+1}.$$

Proof. If n is replaced by $2n$ in (24) and x by 0, we obtain

$$E^{(c)}_{2n,q}(1,y) + E^{(c)}_{2n,q}(0,y) = 2C_{2n,q}(0,y),$$

which proves the first relation because from (8) we have $C_{2n,q}(0,y) = (-1)^n y^{2n}$. The second relation can be similarly proved.

Proposition 3.8. *For every* $n \in \mathbb{N}$, *the following identities hold:*

$$E^{(c)}_{n,q}\big((x \oplus_q z), y\big) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E^{(c)}_{k,q}(x,y)z^{n-k} \tag{26}$$

and

$$E^{(s)}_{n,q}\big((x \oplus_q z), y\big) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E^{(s)}_{k,q}(x,y)z^{n-k}. \tag{27}$$

Proof. We have

$$\sum_{n=0}^{\infty} E_{n,q}^{(c)}\big((x\oplus_q z),y\big)\frac{t^n}{[n]_q!}=\frac{2e_q^{(x\oplus_q z)t}}{e_q^t+1}\cos_q yt=$$

$$=\frac{2e_q^{xt}}{e_q^t+1}e_q^{zt}\cos_q yt=\left(\sum_{n=0}^{\infty}E_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{(zt)^n}{[n]_q!}\right)=$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,q}^{(c)}(x,y)z^{n-k}\right)\frac{t^n}{[n]_q!},$$

which proves (26). The proof of (27) is similar.

***Corollary* 3.2.** *For every $n\in\mathbb{N}$, the following partial q-differential equations hold:*

$$D_{q,x}E_{n,q}^{(c)}(x,y)=[n]_qE_{n-1,q}^{(c)}(x,y),$$

$$D_{q,y}E_{n,q}^{(c)}(x,y)=-[n]_qE_{n-1,q}^{(s)}(x,y),$$

$$D_{q,x}E_{n,q}^{(s)}(x,y)=[n]_qE_{n-1,q}^{(s)}(x,y),$$

and

$$D_{q,y}E_{n,q}^{(s)}(x,y)=[n]_qE_{n-1,q}^{(c)}(x,y).$$

4. A bivariate kind of q-Genocchi polynomials. In this section, we introduce a bivariate kind of q-Genocchi polynomials and just present some basic propositions of them as their proofs are similar to the previous section.

Based on pervious comments, we can introduce two kinds of bivariate q-Genocchi polynomials as follows:

$$\frac{2te_q^{xt}}{e_q^t+1}\cos_q yt=\sum_{n=0}^{\infty}G_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}$$

and

$$\frac{2te_q^{xt}}{e_q^t+1}\sin_q yt=\sum_{n=0}^{\infty}G_{n,q}^{(s)}(x,y)\frac{t^n}{[n]_q!},$$

where they can be represented in terms of q-Genocchi numbers as

$$G_{n,q}^{(c)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,q}(0)C_{n-k,q}(x,y)$$

and

$$G_{n,q}^{(s)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,q}(0)S_{n-k,q}(x,y).$$

They can also be represented in terms of $G_{n,q}(x)$ as follows:

$$G^{(c)}_{n,q}(x,y)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\begin{bmatrix} n\\ 2k\end{bmatrix}_q G_{n-2k,q}(x)y^{2k}$$

and

$$G^{(s)}_{n,q}(x,y)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q G_{n-2k-1,q}(x)y^{2k+1}.$$

For every $n\in\mathbb{N}$, the following identities hold:

$$G^{(c)}_{n,q}\big((1\oplus_q x),y\big)+G^{(c)}_{n,q}(x,y)=2[n]_qC_{n-1,q}(x,y)$$

and

$$G^{(s)}_{n,q}\big((1\oplus_q x),y\big)+G^{(s)}_{n,q}(x,y)=2[n]_qS_{n-1,q}(x,y).$$

Consequently, we have

$$G^{(c)}_{2n+1,q}(1,y)+G^{(c)}_{2n,q}(0,y)=2[2n+1]_q(-1)^ny^{2n}$$

and

$$G^{(s)}_{2n,q}(1,y)+G^{(s)}_{2n,q}(0,y)=2[2n]_q(-1)^{n+1}y^{2n+1}.$$

Moreover, for every $n\in\mathbb{N}$,

$$G^{(c)}_{n,q}\big((x\oplus_q z),y\big)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q G^{(c)}_{k,q}(x,y)z^{n-k}$$

and

$$G^{(s)}_{n,q}\big((x\oplus_q z),y\big)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q G^{(s)}_{k,q}(x,y)z^{n-k}.$$

Finally, for every $n\in\mathbb{N}$, the following partial q-differential equations hold:

$$D_{q,x}G^{(c)}_{n,q}(x,y)=[n]_qG^{(c)}_{n-1,q}(x,y),$$
$$D_{q,y}G^{(c)}_{n,q}(x,y)=-[n]_qG^{(s)}_{n-1,q}(x,y),$$
$$D_{q,x}G^{(s)}_{n,q}(x,y)=[n]_qG^{(s)}_{n-1,q}(x,y),$$

and

$$D_{q,y}G^{(s)}_{n,q}(x,y)=[n]_qG^{(c)}_{n-1,q}(x,y).$$

References

1. L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76**, 332 – 350 (1954).
2. U. Duran, M. Acikgoz, A. Esi, S. Araci, *Some new symmetric identities involving q-Genocchi polynomials under S_4*, J. Math. Anal., **6**, 22 – 31 (2015).
3. T. Ernst, *A comprehensive treatment of q-calculus*, Birkhäuser (2012).
4. V. Kac, P. Cheung, *Quantum calculus*, Springer, 2001.
5. D. S. Kim, T. Kim, *Some identities of q-Euler polynomials arising from q-umbral calculus*, J. Inequal. and Appl., **2014**, № 1 (2014), 12 p.
6. T. Kim, *q-Extension of the Euler formula and trigonometric functions*, Russ. J. Math. Phys., **14**, 275 – 278 (2007).
7. M. Masjed-Jamei, W. Koepf, *Symbolic computation of some power-trigonometric series*, J. Symbolic Comput., **80**, 273 – 284 (2017).
8. M. Masjed-Jamei, M. R. Beyki, W. Koepf, *A new type of Euler polynomials and numbers*, Mediterr. J. Math., **15**, Article 138 (2018).
9. M. Schork, *Wards 'calculus of sequences' q-calculus and the limit $q \to -1$*, Adv. Stud. Contemp. Math., **13**, 131 – 141 (2006).
10. J. Shareshian, M. L. Wachs, *q-Eulerian polynomials: excedance number and major index*, Electron. Res. Announc. Amer. Math. Soc., **13**, 33 – 45 (2007).
11. A. Sharma, A. Chak, *The basic analogue of a class of polynomials*, Riv. Math. Univ. Parma, **5**, 325 – 337 (1954).
12. Y. Simsek, *Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Theory and Appl., **2013**, № 87 (2013), doi:10.1186/1687-1812-2013-87.

Received 24.03.18