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# HYPERBOLICALLY LIPSCHITZ CONTINUITY, AREA DISTORTION AND COEFFICIENT ESTIMATES FOR $(K, K')$-QUASICONFORMAL HARMONIC MAPPINGS OF UNIT DISK

# ГІПЕРБОЛІЧНА НЕПЕРЕРВНІСТЬ ЗА ЛІПШИЦЕМ, СПОТВОРЕННЯ ОБЛАСТЕЙ ТА ОЦІНКИ КОЕФІЦІЄНТІВ ДЛЯ $(K, K')$-КВАЗІКОНФОРМНИХ ГАРМОНІЧНИХ ВІДОБРАЖЕНЬ ОДИНИЧНОГО ДИСКА

We study the hyperbolically Lipschitz continuity, Euclidean and hyperbolic area distortion theorem, and coefficient estimate for the classes of $(K, K')$-quasiconformal harmonic mappings from the unit disk onto itself.

Вивчаються гіперболічна неперервність за Ліпшицем, теорема про спотворення евклідових та гіперболічних областей, а також оцінки коефіцієнтів для $(K, K')$-квазіконформних гармонічних відображень одиничного диска в себе.

**1. Introduction.** Before stating some backgrounds and our main results, we firstly introduce some terminologies. Suppose that $\gamma$ is a rectifiable curve in the complex plane. Denote by $l$ the length of $\gamma$ and let $\Gamma\colon [0,1] \mapsto \gamma$ be the natural parameterization of $\gamma$, i.e., the parameterization satisfying the condition

$$|\dot{\Gamma}(s)| = 1 \quad \text{for all} \quad s \in [0,1].$$

We will say that $\gamma$ is of class $C^{n,\mu}$ for $n \in \mathbb{N}$, $0 < \mu < 1$, if $\Gamma$ is of class $C^n$ and

$$\sup_{t,s\in[0,1]} \frac{|\Gamma^{(n)}(t) - \Gamma^{(n)}(s)|}{|t-s|^{\mu}} < \infty.$$

We will call a Jordan $C^{n,\mu}$ domain in $\mathbb{C}$, if is bounded by $C^{n,\mu}$ Jordan curve.

Let $D$ and $G$ be subdomains of the complex plane $\mathbb{C}$. We say that a function $u\colon D \mapsto \mathbb{R}$ is absolutely continuous on line in the region $D$ if for every closed rectangle $R \subset D$ with sides parallel to the axes $x$ and $y$, $u$ are absolutely continuous on almost every horizontal line and almost every vertical line in $R$. Such a function has, of course, partial derivatives $u_x$ and $u_y$ everywhere in $D$. A topological mapping $f = u + iv\colon D \to G$ is said to be $(K, K')$-quasiconformal if it satisfies:

(a) $f$ is absolutely continuous on lines in $D$;

(b) there are constants $K \geq 1$ and $K' \geq 0$ such that

$$L_f^2 \leq KL_f l_f + K', \text{ a.e.} \quad \text{in} \quad D,$$

where $L_f = |f_z(z)| + |f_{\overline{z}}(z)|$, $l_f = ||f_z(z)| - |f_{\overline{z}}(z)||$, $f_z = \dfrac{1}{2}(f_x - if_y)$ and $f_{\overline{z}} = \dfrac{1}{2}(f_x + if_y)$. If $K' = 0$, then $f$ is called a $K$-quasiconformal mapping.

Let $\rho(z)|dz|^2$ be a conformal $C^1$ metric defined on $D$. A map $f \in C^2(D, G)$ is called a $\rho$-harmonic mapping if

$$f_{z\overline{z}} + (\log \rho)_\omega \circ f \cdot f_z f_{\overline{z}} = 0.$$

In particular, 1-harmonic mapping is called an Euclidean harmonic function. In what follows, we say a function is harmonic always means that it is Euclidean harmonic.

Let $\lambda_D(z)|dz|$ be the hyperbolic metric of the domain $G$ having constant Gaussian curvature $-1$. The hyperbolic distance $d_{h_D}(z_1, z_2)$ between two points $z_1$ and $z_2$ in $D$ is defined by

$$\inf_{\gamma} \left\{ \int\limits_{\gamma} \lambda_D(z)|dz| \right\},$$

where infimum is taken over all rectifiable curves $\gamma$ in $D$ connecting $z_1$ and $z_2$. It is known that if $D = \mathbb{D}$, then

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2} \quad \text{and} \quad d_{h_{\mathbb{D}}}(z_1, z_2) = \log \frac{|1 - z_1\overline{z_2}| + |z_1 - z_2|}{|1 - z_1\overline{z_2}| - |z_1 - z_2|}.$$

A mapping $h$ of $D$ onto $G$ is said to be hyperbolically Lipschitz if there exists a constant $L_1 > 0$, such that the inequality

$$d_{h_G}(h(z_1), h(z_2)) \leq L_1 \, d_{h_D}(z_1, z_2)$$

holds for every $z_1,\ z_2 \in D$.

We will say that a mapping $f : \overline{\mathbb{D}} \to \overline{\Omega}$ is normalized if $f(t_i) = \omega_i$, $i = 0, 1, 2$, where $\{t_0 t_1, t_1 t_2, t_2 t_0\}$ and $\{\omega_0\omega_1, \omega_1\omega_2, \omega_2\omega_1\}$ are arcs of $\mathbb{T} = \partial\mathbb{D}$ and of $\gamma = \partial\Omega$, respectively, having the same length $2\pi/3$ and $|\gamma|/3$, respectively. Let $\gamma \in C^{1,\mu}, 0 < \mu < 1$, be a Jordan curve, $g$ be the arc length parameterization of $\gamma$ and $l = |\gamma|$ be the length of $\gamma$. Let $d_\gamma$ be the distance between $g(s)$ and $g(t)$ along the curve $\gamma$, i.e.,

$$d_\gamma(g(s), g(t)) = \min\{|s - t|, (l - |s - t|)\}.$$

A closed rectifiable Jordan curve $\gamma$ enjoys a $b$-chord-arc condition for some constant $b > 1$ if for all $z_1, z_2 \in \gamma$ there holds the inequality

$$d_\gamma(z_1, z_2) \leq b|z_1 - z_2|. \tag{1.1}$$

It is clear that if $\gamma \in C^{1,\mu}$ then $\gamma$ enjoys a chord-arc condition for some $b_\gamma > 1$.

***1.1. Background and main results.*** Martio [12] was the first one to consider quasiconformal harmonic mappings for the unit disk, and Kalaj [4] extended the domain to the unit ball. In [14], Wan showed that every hyperbolically harmonic quasiconformal diffeomorphism from $\mathbb{D}$ onto itself is a quasiisometry of the Poincaré disk. In [11], Parlović proved that a $K$-quasiconformal harmonic mapping of $\mathbb{D}$ onto itself is bi-Lipschitz with respect to Euclidean distence. Its explicit bi-Lipschitz constants were given by Partyka and Sakan [5]. In 2007, Knežević and Matecljević [10] showed that a $K$-quasiconformal harmonic mapping of the unit disk onto itself is a $\left(\dfrac{1}{K}, K\right)$-quasiisometry with respect to Poincaré metric. Recently, Kalaj and Mateljević [2] studied the class of $(K, K')$-quasiconformal mappings with bounded image domains. They obtained the following intrigue results [2].

**Theorem 1.1** [2]. *Suppose that $\Omega$ is a Jordan domain with $C^2$ boundary and $\omega$ is $(K, K')$-quasiconformal harmonic mapping between the unit disk $\mathbb{D}$ and $\Omega$. Then:*

(a) *$\omega$ has a continuous extension to $\overline{\mathbb{D}}$, whose restriction to $\mathbb{T}$ we denote by $f$;*

(b) *furthermore, $\omega$ is Lipschitz continuous on $\mathbb{D}$;*

(c) *if $f$ is normalized, there exists a constant $L = L(K, K', \partial\Omega)$ such that*

$$|f'(t)| \leq L \quad for \quad almost \quad every \quad t \in [0, 2\pi],$$

*and*

$$|\omega(z_1) - \omega(z_2)| \leq (KL + \sqrt{K'})\, |z_1 - z_2| \quad for \quad z_1, z_2 \in \mathbb{D}.$$

*Here,*

$$L \leq \left(K\lambda k_0 b(L_\lambda(K, K'))^{1+1/\lambda}\pi^{1/\lambda} + \sqrt{K'}\right)^\lambda, \quad \alpha = \frac{1}{K(1+2b)^2}, \quad \lambda = \frac{2-\alpha}{\alpha}, \quad k_0 = \sup_s |k_s|,$$

*and $k_s$ is the curvature of $\partial\Omega$ at the point $g(s)$, $b$ is a constant such that $\partial\Omega$ satisfies $b$-chord-arc condition in* (1.1),

$$L_\lambda(K, K') = 4(1+2b)2^\alpha \sqrt{\max\left\{\frac{2\pi K|\Omega|}{\log 2}, \frac{2\pi K'}{K(1+2b)^2 + 4}\right\}}.$$

The hyperbolically Lipschitz continuity for $(K, K')$-quasiconformal harmonic mapping from upper half-plane onto itself was obtained by Min Chen and Xingdi Chen (see [8], Theorem 2.2). The first aim of this paper, we study the hyperbolically Lipschitz continuity for the class of $(K, K')$-quasiconformal harmonic mappings from unit disk onto itself. Our result reads as follows.

**Theorem 1.2.** *Suppose that $\omega$ is $(K, K')$-quasiconformal harmonic mapping from unit disk onto itself satisfying $|\omega^{-1}(0)| \leq l < 1$, where $l$ is a constant, then $\omega$ is hyperbolically Lipschitz continuity.*

In 1994, Astala [8] proved that if $f$ is a $K$-quasiconformal mapping from the unit disk $\mathbb{D}$ onto itself, normalized by $f(0) = 0$, and if $E$ is any measurable subset of the unit disk, then $A_e(f(E)) \leq$ $\leq a(K)\, A_e(E)^{1/K}$, where $A_e(\cdot)$ denotes the Euclidean area and $a(K) \to 1$ when $K \to 1^+$. In 1998, Porter and Reséndis [13] obtained some results about area distortion under quasiconformal mappings on the unit disk $\mathbb{D}$ onto itself with respect to the hyperbolic measure. They also showed the existence of explodable sets; this kind of sets has bounded hyperbolic area, but under a specific quasiconformal mapping its image has infinite hyperbolic area. In [1], Hernándezmontes and Reséndis studied the hyperbolic and Euclidean area distortion of measurable sets under some classes of $K$-quasiconformal mappings from the upper half-plane and the unit disk onto themselves, respectively. The Euclidean and hyperbolic area distortion theorems for $(K, K')$-quasiconformal harmonic mapping from upper half-plane onto itself were obtained by Min Chen and Xingdi Chen [9]. It was showed by Kalaj and Mateljević [2] (Example 1.5) that a $(K, K')$-quasiconformal harmonic mapping from unit disk onto itself is generally not a $(K, 0)$-quasiconformal harmonic mapping. So, it is interesting to study the Euclidean and hyperbolic area distortion for $(K, K')$-quasiconformal harmonic mapping from unit disk onto itself. We have the following theorem.

**Theorem 1.3.** *Let $\omega$ be a $(K, K')$-quasiconformal harmonic mapping from unit disk $\mathbb{D}$ onto itself satisfying $\omega(0) = 0$. If $\omega|_{\mathbb{T}} = f$ is normalized, then, for any measurable set $E \subset \mathbb{D}$, we have*

$$A_e(\omega(E)) \leq (K\phi(K, K') + \sqrt{K'})^2 \, A_e(E)$$

*and*

$$A_{\mathcal{H}}(\omega(E)) \leq \frac{\pi^2}{4} \, (K\phi(K, K') + \sqrt{K'})^2 \, A_{\mathcal{H}}(E),$$

*where $A_e(\cdot)$ and $A_{\mathcal{H}}(\cdot)$ denote the Euclidean and hyperbolic area, respectively. Here,*

$$\phi(K, K') = \left( \frac{\pi K[2K(1+\pi)^2 - 1]}{2} \, \varphi(K, K') \, \pi^{\frac{1}{2K(1+\pi)^2-1}} + \sqrt{K'} \right)^{2K(1+\pi)^2-1}$$

*and*

$$\varphi(K, K') = \left( 4(1+\pi) \cdot 2^{1/K(1+\pi)^2} \sqrt{\max\left\{ \frac{2K\pi^2}{\log 2}, \frac{2\pi K'}{K(1+\pi)^2+4} \right\}} \right)^{\frac{2K(1+\pi)^2}{2K(1+\pi)^2-1}}.$$

In [7], Zhu obtained the coefficient estimates for $K$-quasiconformal harmonic mappings from unit disk onto itself. Here we consider the coefficient estimates for the $(K, K')$-quasiconformal harmonic mappings of unit disk $\mathbb{D}$ onto itself. We have the following theorem.

**Theorem 1.4.** *Given $K \geq 1, K' \geq 0$. Let $\omega(z) = h(z) + \overline{g(z)}$ be a $(K, K')$-quasiconformal harmonic mapping from unit disk onto itself satisfying $\omega(0) = 0$, where*

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

*are analytic in $\mathbb{D}$. If the boundary function $f$ of $\omega$ is normalized, then*

$$|a_n| + |b_n| \leq \frac{4\phi(K, K')}{n\pi}, \qquad n = 1, 2, \ldots,$$

*where*

$$\phi(K, K') = \left( \frac{\pi K[2K(1+\pi)^2 - 1]}{2} \, \varphi(K, K') \, \pi^{\frac{1}{2K(1+\pi)^2-1}} + \sqrt{K'} \right)^{2K(1+\pi)^2-1}$$

*and*

$$\varphi(K, K') = \left( 4(1+\pi) \cdot 2^{1/K(1+\pi)^2} \sqrt{\max\left\{ \frac{2K\pi^2}{\log 2}, \frac{2\pi K'}{K(1+\pi)^2+4} \right\}} \right)^{\frac{2K(1+\pi)^2}{2K(1+\pi)^2-1}}.$$

The remainder of this paper are devoted to prove Theorems 1.2, 1.3 and 1.4, which will be presented in Sections 2, 3 and 4, respectively.

**2. Hyperbolically Lipschitz continuity.** The aim of this section is to prove Theorem 1.2. We need the following lemma which will be used in the proof of Theorem 1.2. (See [3], Remark 2.4, for the case of $a = 0$.)

**Lemma 2.1.** *Let $\omega$ be a harmonic mapping from unit disk into itself satisfying $\omega(a) = 0$, then*

$$\frac{1-|z|^2}{1-|\omega(z)|^2} \leq \frac{\pi}{2}\frac{1+|a|}{1-|a|}. \tag{2.1}$$

***Proof.*** Let $\varphi(z) = \dfrac{z+a}{1+\overline{a}z}$ and $F(z) = \omega(\varphi(z))$, then $F(z)$ is a harmonic mapping from unit disk onto itself satisfying $F(0) = 0$. Hence by harmonic Schwarz lemma [5] (Lemma) and the elementary inequality

$$\left|\frac{z-a}{1-\overline{a}z}\right| \leq \frac{|z|+|a|}{1+|a||z|}, \qquad z, a \in \mathbb{D},$$

we have

$$|\omega(z)| \leq \frac{4}{\pi}\arctan\left|\frac{z-a}{1-\overline{a}z}\right| \leq \frac{4}{\pi}\arctan\frac{|z|+|a|}{1+|a||z|}. \tag{2.2}$$

Consider the function $\varphi\colon [0,1) \longrightarrow \mathbb{R}$ and $\varphi(t) = \dfrac{4}{\pi}\arctan t - \dfrac{2}{\pi}(t-1) - 1$. As $\varphi'(t) = \dfrac{2}{\pi}\dfrac{1-t^2}{1+t^2} >$ $> 0$, we get

$$\frac{4}{\pi}\arctan t \leq \frac{2}{\pi}(t-1) + 1, t \in [0,1). \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\frac{1-|\omega(z)|^2}{1-|z|^2} \geq \frac{1-\left(\dfrac{4}{\pi}\arctan\left(\dfrac{|z|+|a|}{1+|a||z|}\right)\right)^2}{1-|z|^2} \geq \frac{1-\left(\dfrac{2}{\pi}\left(\dfrac{|z|+|a|}{1+|a||z|}-1\right)+1\right)^2}{1-|z|^2} =$$

$$= \frac{4}{\pi}(1-|a|)\left[\frac{1}{(1+|a||z|)(1+|z|)} - \frac{1}{\pi}\frac{(1-|a|)(1-|z|)}{(1+|a||z|)^2(1+|z|)}\right]. \tag{2.4}$$

Let

$$\phi(t) = \frac{1}{(1+tm)(1+t)} - \frac{1}{\pi}\frac{(1-m)(1-t)}{(1+tm)^2(1+t)}, \qquad t, m \in [0,1),$$

then

$$\phi'(t) = -\frac{2tm+m+1}{(1+tm)^2(1+t)^2} - \frac{2(1-m)}{\pi}\frac{t^2-tm-m-1}{(1+tm)^3(1+t)^2} =$$

$$= -\frac{(2tm+m+1)(1+tm) + \dfrac{2(1-m)(t^2m-tm-m-1)}{\pi}}{(1+tm)^3(1+t)^2} <$$

$$< -\frac{(2tm+m+1)(1+tm) + (1-m)(t^2m-tm-m-1)}{(1+tm)^3(1+t)^2} =$$

$$= -\frac{t^2m^2 + 2tm^2 + t^2m + tm + m^2 + m}{(1+tm)^3(1+t)^2} \le 0.$$

Therefore, $\phi(t)$ is monotonically decreasing on $[0,1)$, so we get

$$\phi(t) = \frac{1}{(1+tm)(1+t)} - \frac{1}{\pi}\frac{(1-m)(1-t)}{(1+tm)^2(1+t)} \ge \phi(1) = \frac{1}{2(1+m)}. \tag{2.5}$$

Thus, (2.1) is immediately derived from (2.4) and inequality (2.5).

Lemma 2.1 is proved.

***Proof of Theorem* 1.2.** In order to prove Theorem 1.2, we only need to prove that $\dfrac{|\nabla\omega(z)|(1-|z|^2)}{1-|\omega(z)|^2} < +\infty$ holds for every $z \in \mathbb{D}$. Since $\omega$ is harmonic from unit disk onto itself satisfying $|\omega^{-1}(0)| \le l < 1$, hence, by Lemma 2.1, we have

$$\frac{1-|z|^2}{1-|\omega(z)|^2} \le \frac{\pi}{2}\frac{1+|\omega^{-1}(0)|}{1-|\omega^{-1}(0)|} \le \frac{\pi}{2}\frac{1+l}{1-l}. \tag{2.6}$$

Moreover, by Theorem 1.1, there exists a positive constant $M$ such that the inequality

$$|\nabla\omega| \le M \tag{2.7}$$

holds, for every $z \in \mathbb{D}$. Combining (2.6) and (2.7), we get

$$\frac{|\nabla\omega(z)|\,(1-|z|^2)}{1-|\omega(z)|^2} \le \frac{\pi M}{2}\frac{1+l}{1-l} < +\infty.$$

Theorem 1.2 is proved.

**3. Area distortion.** In this section, we will prove Theorem 1.3. In order to derive an explicit Lipschitz constant in Theorem 1.2 in the setting of $(K, K')$-quasiconformal harmonic mapping from unit disk onto itself, we need the following lemma.

**Lemma 3.1.** *$\gamma = \partial\mathbb{D}$ satisfies the $\frac{\pi}{2}$-chord-arc condition. Namely, for all $g(s)$, $g(t) \in \partial\mathbb{D}$, there exists $b = \frac{\pi}{2} > 1$ such that*

$$d_\gamma(g(s), g(t)) \le \frac{\pi}{2}|g(s) - g(t)|. \tag{3.1}$$

***Proof.*** Suppose that $g(s) = e^{i\alpha}$, $g(t) = e^{i\beta} \in \gamma = \partial\mathbb{D}$. Without loss of generality, we can assume the angle, denoted by $\theta$, between $g(s)$ and $g(t)$ satisfies $0 < \theta < \pi$. Namely, $0 < \theta = |\alpha - \beta| < \pi$, then $d_\gamma(g(s), g(t)) = \theta$. Since $|g(s) - g(t)| = 2\sin\frac{\theta}{2}$, by Jordan inequality, we get

$$\frac{2}{\pi}\frac{\theta}{2} \le \sin\frac{\theta}{2} = \frac{|g(s)-g(t)|}{2},$$

which yields (3.1).

***Proof of Theorem* 1.3.** Considering the case for $\mathbb{D} = \Omega$ in Theorem 1.1. By Lemma 3.1, we have $\alpha = \dfrac{1}{K(1+\pi)^2}$ and $\lambda = 2K(1+\pi)^2 - 1$. By Theorem 1.1, together with $k_0 = 1$, we obtain

$$L \le \phi(K, K'), \tag{3.2}$$

where

$$\phi(K,K') = \left( \frac{\pi K[2K(1+\pi)^2 - 1]}{2}\, \varphi(K,K')\, \pi^{\frac{1}{2K(1+\pi)^2-1}} + \sqrt{K'} \right)^{2K(1+\pi)^2-1}$$

and

$$\varphi(K,K') = \left( 4(1+\pi)\cdot 2^{1/K(1+\pi)^2} \sqrt{\max\left\{ \frac{2K\pi^2}{\log 2}, \frac{2\pi K'}{K(1+\pi)^2+4} \right\}} \right)^{\frac{2K(1+\pi)^2}{2K(1+\pi)^2-1}}.$$

Using Theorem 1.1 again, we get

$$|\omega_z(z)| \le |\nabla\omega| \le K\,\phi(K,K') + \sqrt{K'}, \tag{3.3}$$

Combining (2.1) and (3.3), we have

$$\frac{|\omega_z(z)|}{1 - |\omega(z)|^2} \le \frac{\pi}{2}\, \frac{K\,\phi(K,K') + \sqrt{K'}}{1 - |z|^2}. \tag{3.4}$$

Furthermore, the Jacobian $J_\omega$ of $\omega$ satisfies

$$J_\omega = |\omega_z(z)|^2 - |\omega_{\overline{z}}(z)|^2 \le |\omega_z(z)|^2 \le \left( K\,\phi(K,K') + \sqrt{K'} \right)^2,$$

hence, for any measurable set $E \subset \mathbb{D}$, we obtain

$$A_e(\omega(E)) = \int\limits_{\omega(E)} dudv = \int\limits_E J_\omega(z)dxdy \le \left( K\,\phi(K,K') + \sqrt{K'} \right)^2 A_e(E).$$

In addition, by (3.4), we have

$$A_{\mathcal{H}}(\omega(E)) = \int\limits_{\omega(E)} \frac{4dudv}{(1-|\omega(z)|^2)^2} = \int\limits_E \frac{4J_\omega(z)}{(1-|\omega(z)|^2)^2} dxdy \le \int\limits_E \frac{4|\omega_z(z)|^2}{(1-|\omega(z)|^2)^2} dxdy \le$$

$$\le \int\limits_E \frac{4\left[\frac{\pi}{2}\left(K\,\phi(K,K') + \sqrt{K'}\right)\right]^2}{(1-|z|^2)^2} dxdy = \frac{\pi^2}{4} \left( K\,\phi(K,K') + \sqrt{K'} \right)^2 A_{\mathcal{H}}(E).$$

Theorem 1.3 is proved.

**4. Coefficient estimates.** In this section, we will prove Theorem 1.4. We follow the idea in [7] (Theorem 3).

***Proof of Theorem* 1.4.** For every $z = re^{i\theta} \in \mathbb{D}$, $\omega(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \overline{\sum_{n=1}^{\infty} b_n r^n e^{in\theta}}$. Hence,

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} \omega(re^{i\theta}) e^{-in\theta} d\theta, \qquad n = 1, 2, \ldots,$$

and

$$\overline{b_n} r^n = \frac{1}{2\pi} \int_0^{2\pi} \omega(re^{i\theta}) e^{in\theta} d\theta, \qquad n = 1, 2, \ldots.$$

For every $n$, setting $a_n = |a_n| e^{i\alpha_n}, b_n = |b_n| e^{i\beta_n}$, and $\theta_n = \dfrac{\alpha_n + \beta_n}{2n}$. Then

$$(|a_n| + |b_n|) r^n = \left| \frac{1}{2\pi} \int_0^{2\pi} \omega(re^{i\theta}) [e^{-i\alpha_n} e^{-in\theta} + e^{i\beta_n} e^{in\theta}] d\theta \right| =$$

$$= \left| \frac{1}{2\pi} \int_0^{2\pi} \omega(re^{i\theta}) [e^{-in(\theta+\theta_n)} + e^{in(\theta+\theta_n)}] d\theta \right| = \left| \frac{1}{\pi} \int_0^{2\pi} \omega(re^{i\theta}) \cos n(\theta + \theta_n) d\theta \right|.$$

Integrating by parts, we have

$$(|a_n| + |b_n|) r^n = \left| \frac{1}{n\pi} \int_0^{2\pi} \omega_\theta(re^{i\theta}) \sin n(\theta + \theta_n) d\theta \right|. \tag{4.1}$$

By Theorem 1.1, we can see that $f$ is absolutely continuous, hence,

$$\frac{\partial \omega}{\partial \theta}(z) = P[f'](z),$$

where $P(r, x) = \dfrac{1 - r^2}{2\pi(1 - 2r\cos x + r^2)}$ and $P[f](z) = \displaystyle\int_0^{2\pi} P(r, x - \varphi) f(e^{ix}) dx$. In addition, by [2] (Lemma 4.1), the radial limits of $\omega_\theta$ exist almost everywhere and $\lim_{r \to 1^-} \omega_\theta(re^{i\theta}) = f'(\theta)$. Hence, tending $r \to 1^-$ in (4.1) and also by Theorem 1.1 and (3.2), we obtain

$$|a_n| + |b_n| \leq \frac{1}{n\pi} \int_0^{2\pi} |f'(\theta)| |\sin n(\theta + \theta_n)| d\theta \leq \frac{4\phi(K, K')}{n\pi}.$$

Theorem 1.4 is proved.

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