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THE BIRMAN – HILDEN PROPERTY OF COVERING SPACES OF NONORIENTABLE SURFACES

ВЛАСТИВІСТЬ БІРМАНА – ХІЛЬДЕНА ДЛЯ ПРОСТОРІВ НАКРИТТЯ НЕОРІЄНТОВНИХ ПОВЕРХОНЬ

Let $p\colon \widetilde{N} \to N$ be a finite covering space of nonorientable surfaces, where $\chi(\widetilde{N}) < 0$. We search whether or not p has the Birman – Hilden property.

Нехай $p\colon \widetilde{N} \to N$ — скінченний простір накриття неорієнтовних поверхонь, де $\chi(\widetilde{N}) < 0$. Ми з'ясуємо, чи має p властивість Бірмана – Хільдена.

1. Introduction and statements of the results. Let Σ (resp., N) denote the connected, orientable (resp., nonorientable) surface of genus g with n marked points. Let $\mathrm{Mod}(\Sigma)$ (resp., $\mathrm{Mod}(N)$) denote the mapping class group of Σ (resp., N), which is the group of isotopy classes of orientation-preserving (resp., all) diffeomorphisms of Σ (resp., N), where diffeomorphisms and isotopies fix the set of marked points. In this paper, we assume that the set of marked points is the set of branched points. Hence, the mapping classes must fix the set of branch points.

Our aim is to get a better understanding of the algebraic structure of the mapping class groups. In this paper, we will take into consideration mainly nonorientable surfaces and their mapping class groups, which are far less understood compared to their orientable counter parts.

In this work, we will consider the Birman – Hilden property of branched covers of surfaces, both orientable and nonorientable. Let S denote a surface (orientable or nonorientable). Fix a covering (possibly branched) $p\colon \tilde{S} \to S$. Let $\mathrm{LMod}(S)$ be the finite index subgroup of $\mathrm{Mod}(S)$ consisting of mapping classes of S, which lift to diffeomorphisms of $\tilde{S}$. Let $\mathrm{SMod}(\tilde{S})$ be the subgroup of $\mathrm{Mod}(\tilde{S})$ consisting of fiber preserving (or symmetric) mapping classes of $\tilde{S}$. If the surjective homomorphism $\Phi\colon \mathrm{LMod}(S) \to \mathrm{SMod}(\tilde{S})/\mathrm{Deck}(p)$ is an isomorphism, then we say that the covering space $p\colon \tilde{S} \to S$ has the Birman – Hilden property. This property plays an important role in understanding the algebraic structure of mapping class groups.

It is important to note that, in Birman – Hilden theory branch points in a surface S and their preimages in $\tilde{S}$ are treated differently. Any liftable diffeomorphism should leave the set of branch points invariant and therefore, we regard the branch points in S as marked points. However, on $\tilde{S}$ the preimages of the branch points are considered as ordinary points, not as marked points. Therefore, the diffeomorphisms of $\tilde{S}$ do not need to leave invariant the preimage of the set of branch points.

It is known that all unbranched coverings and regular coverings are fully ramified so that they satisfy the Birman – Hilden property [1 – 3, 6]. However, there are irregular branched covering spaces which have the Birman – Hilden property and there are coverings that do not have the Birman – Hilden property (see [8]).

Winarski gave one necessary condition and one sufficient condition for a covering space of orientable surfaces to have the Birman – Hilden property in [8]. Later, Margalit and Winarski present a new article related to the Birman – Hilden property [7]. In this paper, our goal is to investigate whether or not similar results for nonorientable surfaces hold. The organization of the paper is as follows. In Section 2, we will give some notations and definitions, secondly, we will state some results of Winarski. In Section 3, we extend these results to nonorientable surfaces. In particular, we prove the following theorem in this section.

Theorem 1.1. *Let $p\colon \widetilde{N} \to N$ be a finite covering space of nonorientable surfaces, where $\chi(\widetilde{N}) < 0$. If p is fully ramified, then it has the Birman – Hilden property.*

In Section 4, we will introduce a blowing up process to obtain covering spaces of nonorientable surfaces from those of orientable surfaces. By using this process, we prove the following theorem.

Theorem 1.2. *If a covering space does not have the Birman – Hilden property then the covering space that is obtained by blowing up the given covering spaces does not have the Birman – Hilden property either.*

In the same section, using Theorem 1.2 repeatedly, we construct a covering space of nonorientable surfaces, which does not have the Birman – Hilden property. We also prove in Theorem 4.1 that the blowing up process preserves the weak curve lifting property.

Lastly, the final section contains some immediate consequences of these results.

2. Preliminaries. Throughout this paper, let S denote a surface (orientable or nonorientable).

2.1. A branched covering space over a surface. Let S be a surface. A space $\tilde{S}$ is said to be a branched covering space over S if there is a map $p\colon \tilde{S} \to S$ such that other than the inverse image of some finite set B in S, p is a covering map, where for all $s \in B$ there is an open neighborhood U such that $U \cap B = \{s\}$. Moreover, $p^{-1}(U)$ is the disjoint finite union some open subsets V_i so that the restriction map of p on $V_i \backslash \{\tilde{s_i}\}$, $p_| : V_i \backslash \{\tilde{s_i}\} \to U \backslash \{s\}$ is a degree r_i covering map and each V_i contains exactly one point $\tilde{s_i} \in p^{-1}(s)$, and at least one $r_i > 1$.

2.2. Ramification number. The number r_i is called a ramification number. A preimage $\tilde{s_i}$ is said to be ramified, if $r_i > 1$, otherwise it is said to be unramified.

2.3. Fully ramified. A covering space is said to be fully ramified, if all points in $p^{-1}(B)$ are ramified.

2.4. Simple cover. A degree n branched covering space is said to be a simple cover if every branch point has $n-1$ preimages.

2.5. Essential simple closed curves. Let a be an unoriented simple closed curve on any surface (marked surface or not) S. According to whether a regular neighbourhood of a is an annulus or a Möbius band, we call a two-sided or one-sided, respectively. An *essential simple closed curve* on an orientable surface Σ is a curve which is not isotopic to a single point on the surface Σ (see [8]). We note that all simple closed curves on Σ are two-sided. On the other hand, on a nonorientable surface N, an *essential simple closed curve* is a two-sided curve, which is not isotopic to a single point or which is not the boundary of a Möbius band. For example, on Fig. 1, the simple closed curve c on the surface N is essential but the preimages c_1 and c_2 on the cover $\widetilde{N}$ are inessential, because each c_i bounds a Möbius band. The other preimage c_3 on $\widetilde{N}$ is also inessential, since it bounds a disc. As it is indicated in the introduction, on the cover $\tilde{S}$ the preimages of the branch points are considered as ordinary points, not as marked points.

2.6. The weak curve lifting property. A covering space of surfaces is said to have the weak curve lifting property if the preimage in $\widetilde{\Sigma}$ (resp., $\widetilde{N}$) of every essential simple closed curve on Σ (resp., N) has at least one essential connected component (see [8]).

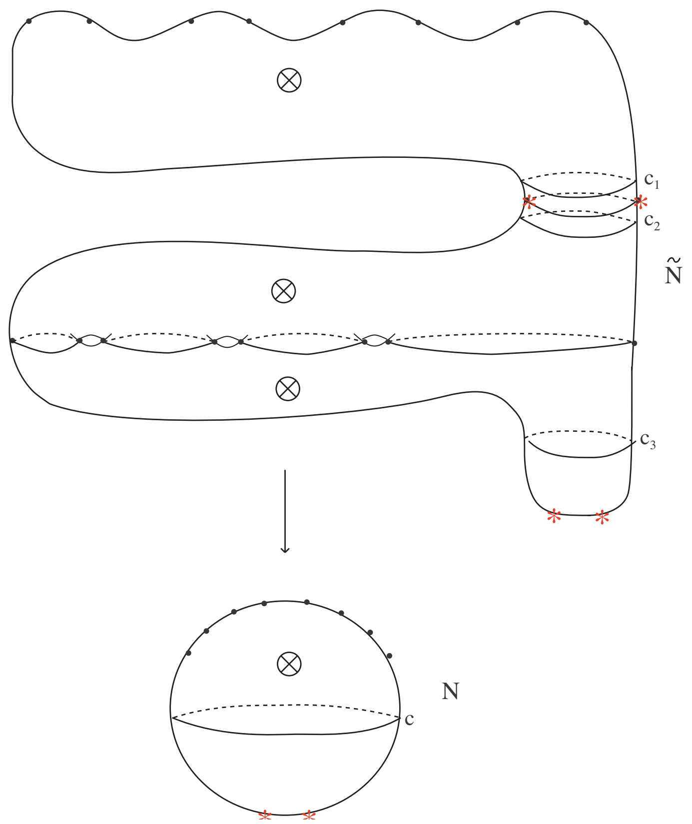

Fig. 1

2.7. The orientation double cover of N. Let $\pi : \Sigma \to N$ be the orientation double cover of the nonorientable surface N with the deck transformation $\tau : \Sigma \to \Sigma$. Then $\pi_*(\Pi_1(\Sigma, x))$ is a normal subgroup of index 2 in $\Pi_1(N, \pi(x))$ and the deck transformation group is $\langle \tau | \tau^2 = \text{identity} \rangle \cong \mathbb{Z}_2$. If F' is a diffeomorphism of Σ such that $\tau \circ F' = F' \circ \tau$, then F' induces a diffeomorphism F on N. On the other hand, any diffeomorphism F of the nonorientable surface N has a unique lift to an orientation preserving diffeomorphism $F' : \Sigma \to \Sigma$.

Indeed, any isotopy of the nonorientable surface N lifts to the surface Σ. Moreover, if the Euler characteristic of the surface is negative (hence it carries a complete hyperbolic metric), then any isotopy of orientation preserving diffeomorphisms of Σ, whose end maps commute with the deck transformation τ, can be deformed into one which commutes with τ, and hence descends to an isotopy of diffeomorphisms of N (cf. Lemma 5 in [9]).

2.8. Pull back map. Let $\pi : E \to B$ be a fiber bundle, which is a continuous surjective locally trivial map so that for each point $b \in B, \pi^{-1}(p) = F$. Obviously covering spaces are fiber bundles.

If $f : B' \to B$ is a continuous map, the set

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$$

which gives the following commutative diagram is called pull back bundle:

$$\begin{array}{ccc} f^*E & \xrightarrow{\pi^*} & B' \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\pi} & B\,, \end{array}$$

where $\pi^* : f^*E \to B'$ and $f^* : f^*E \to E$ are projection maps, first one projects onto first component and the second one projects onto second component.

2.9. One necessary condition and one sufficient condition for orientable surfaces. Winarski proved the following two results.

Theorem 2.1 [8]**.** *Let $p\colon \widetilde{\Sigma} \to \Sigma$ be a finite covering space of orientable surfaces such that $\chi(\widetilde{\Sigma}) < 0$. If p is fully ramified, then p has the Birman – Hilden property.*

Proposition 2.1 [8]**.** *Let $p\colon \widetilde{\Sigma} \to \Sigma$ be a finite covering space of orientable surfaces such that $\chi(\widetilde{\Sigma}) < 0$. If $p\colon \widetilde{\Sigma} \to \Sigma$ has the Birman – Hilden property, then it has the weak curve lifting property.*

3. One necessary condition and one sufficient condition for nonorientable surfaces. ***3.1. The necessary part.*** The following proposition and its proof are analogous to the orientable case (see Proposition 2.1 [8]) and, therefore, we only state the proposition and omit its proof.

Proposition 3.1. *Let $p\colon \widetilde{N} \to N$ be a finite covering space of nonorientable surfaces such that $\chi(\widetilde{N}) < 0$. If $p\colon \widetilde{N} \to N$ has the Birman – Hilden property, then it has the weak curve lifting property.*

3.2. The sufficient part. 3.2.1. Proof of Theorem **1.1.** The proof has two steps:

Step 1. Let $\pi : \Sigma \to N$ be the orientation double cover and let $\tilde{\pi} : \widetilde{\Sigma} \to \widetilde{N}$ be the pull back of $\pi : \Sigma \to N$ by p:

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\tilde{p}} & \Sigma \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{N} & \xrightarrow{p} & N\,, \end{array}$$

where $p^*(\Sigma) = \widetilde{\Sigma} = \big\{(x, y) \in \widetilde{N} \times \Sigma \mid p(x) = \pi(y)\big\}$. Since $\pi : \Sigma \to N$ is double covering, $\tilde{\pi} : \widetilde{\Sigma} \to \widetilde{N}$ is double covering where $\tilde{\pi}(x, y) = x$.

Now, if $\tau : \Sigma \to \Sigma$ is the deck transformation, then $\tilde{\tau} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ given by $\widetilde{\tau}(x, y) = (x, \tau(y))$ is the deck transformation of $\tilde{\pi} : \widetilde{\Sigma} \to \widetilde{N}$. Indeed,

$$\widetilde{\tau}^2(x, y) = \widetilde{\tau}(\widetilde{\tau}(x, y)) = \widetilde{\tau}(x, \tau(y)) = (x, \tau^2(y)) = (x, y).$$

Thus, $\widetilde{\tau}^2$ is the identity.

The map $\tilde{p}\colon \widetilde{\Sigma} \to \Sigma$ given by $\tilde{p}(x, y) = y$ is also fully ramified. Since Σ is orientable, $\widetilde{\Sigma}$ is orientable. Indeed, since $\tilde{\pi} : \widetilde{\Sigma} \to \widetilde{N}$ is a double covering and $\widetilde{\Sigma}$ is orientable, we deduce that $\tilde{\pi} : \widetilde{\Sigma} \to \widetilde{N}$ is the orientation double cover. Now, since $\tilde{p}\colon \widetilde{\Sigma} \to \Sigma$ given by $\tilde{p}(x, y) = y$ is fully ramified, $\tilde{p}$ has the Birman – Hilden property by Theorem 2.1.

Step 2. We will show that $p\colon \widetilde{N} \to N$ has the Birman – Hilden property. Suppose not. Then there is a diffeomorphism $f : N \to N$ such that f is not isotopic to the identity, and its lift $\tilde{f} : \widetilde{N} \to \widetilde{N}$ is isotopic to a deck transformation.

Let $g : \Sigma \to \Sigma$ and $\tilde{g} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ be the unique orientation preserving lifts f and $\tilde{f}$ to the orientation double covers, respectively. Since f is not isotopic to the identity, g is not isotopic to the identity. On the other hand, the unique orientation preserving lift $\tilde{g}$ of $\tilde{f}$ is isotopic to a deck transformation. So, $\tilde{p}\colon \widetilde{\Sigma} \to \Sigma$ does not have the Birman – Hilden property. By Winarski's result, this is a contradiction, since $\tilde{p}$ is fully ramified.

Theorem 1.1 is proved.

4. Blow up process and the Birman – Hilden property. In this section, we will introduce blow up of coverings and prove Theorem 1.2 which enables us to construct coverings not having the Birman – Hilden property. Furthermore, we prove Theorem 4.1 and we give Theorem 4.2.

4.1. Blow up process. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be a d-fold covering space of orientable surfaces. We delete one disc $D \subset \Sigma$ and denote its lifts (discs) as $D_1, D_2, \ldots, D_d$ in $\widetilde{\Sigma}$ and identify the boundary points of each deleted disc via antipodal map. We denote the resulting covering space $q\colon \widetilde{N} \to N$. This is said to be a blow up process.

***4.2. Proof of Theorem* 1.2.** Let $p\colon \widetilde{\Sigma} \to \Sigma$ and $q\colon \widetilde{N} \to N$ be the coverings given as in Subsection 4.1. We will show if $p\colon \widetilde{\Sigma} \to \Sigma$ does not have the Birman – Hilden property, $q\colon \widetilde{N} \to N$ does not have the Birman – Hilden property.

Let $f\colon \Sigma \to \Sigma$ be a diffeomorphism not isotopic to the identity but it lifts to some $\tilde{f}\colon \widetilde{\Sigma} \to \widetilde{\Sigma}$ which is isotopic to a deck transformation. By isotopy we may assume that f is identity on some disc on which we blow up Σ to get N. Then f induces some diffeomorphism g such that the restriction of g on $N \setminus \mathbb{R}P^2$ is equal to f. Moreover, the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle B} \\ \Sigma & \xrightarrow{f} & \Sigma\,, \end{array}$$

where $B\colon N \to \Sigma$ is the blow up projection. Obviously, g lifts to $\tilde{g}$, which is also a diffeomorphism isotopic to a deck transformation of $\widetilde{N} \to N$. Therefore, $\widetilde{N} \to N$ does not have the Birman – Hilden property.

Theorem 1.2 is proved.

Theorem 4.1. *Let $p\colon \widetilde{S} \to S$ be a finite covering space of surfaces such that $\chi(\widetilde{S}) < 0$. If $p\colon \widetilde{S} \to S$ has the weak curve lifting property, then so does its blow up.*

Proof. Suppose not. Then, there is an inessential (two-sided) simple closed curve γ on the surface S, but γ is an essential (two-sided) simple closed curve on $S\sharp\mathbb{R}P^2$ and all components of the preimage are inessential on $\widetilde{S}\sharp n\mathbb{R}P^2$, where n is the degree of $p\colon \widetilde{S} \to S$. Now, there are two cases to consider:

Case 1. The curve γ bounds a punctured disc, say U, on the surface S. Since γ is an essential simple closed curve on $S\sharp\mathbb{R}P^2$, the point we blow up lies inside the punctured disc bounded by γ. Any component of the preimage of U is still a punctured disc. Each component is a connected cover of U and since the puncture is a branch point, one of the components is at least a two fold cover. After applying the blow up process to the covering space, U becomes a punctured Möbius band and each connected component of the preimage of this Möbius band is a nonorientable surface with one boundary component. Moreover, at least one of them has genus at least two. Let $\tilde{\gamma}$ be the boundary of one such connected component of genus at least two on the surface $\widetilde{S}$. The curve $\tilde{\gamma}$, which is of course a component of the preimage of γ, is an essential curve. To see this note that the surface $\widetilde{S}$ has Euler characteristic at most -1. After the blow up process its genus is increased by at least two. Hence, the blown up nonorientable surface has genus at least five. Therefore, the boundary curve $\tilde{\gamma}$ is essential on $\widetilde{S}\sharp n\mathbb{R}P^2$, which is a contradiction.

Case 2. The curve γ bounds a Möbius band, call again U, on the surface S. Now, any component $\widetilde{U}$ of the preimage of U is either a Möbius band or a cylinder, depending on whether the degree of the covering $\widetilde{U} \to U$ is odd or even. Hence, after the blow up process each component of the preimage of U is a nonorientable surface of genus at least two with one or two boundary components on the surface $\widetilde{S}\sharp n\mathbb{R}P^2$. Finally, an Euler characteristic argument as in the above case finishes the proof.

Theorem 4.1 is proved.

***4.3. An application of Proposition* 3.1.** We deal with the threefold simple branched cover of surfaces $p\colon \widetilde{\Sigma} \to \Sigma$, where $\widetilde{\Sigma}$ is a closed surface of genus g and Σ is a sphere with $2g+4$ branch points. It is known that this covering space does not have the Birman–Hilden property [4, 5]. Also, the example with $g=3$ in Fig. 3 in [8] does not have the Birman–Hilden property since it leaks the weak curve lifting property.

Now, we blow up this covering space and obtain the following covering space as in Fig. 1. As it is indicated in the introduction, the preimages of the branch points which are shown in Fig. 1, are not thought as marked points in $\tilde{N}$. The preimage of the essential simple closed curve c in Fig. 1 is the union of c_1, c_2 and c_3 in $\tilde{N}$ and every c_i is inessential (see Subsection 2.5). Therefore, this covering space does not have the weak curve lifting property and so, it does not have the Birman–Hilden property by Proposition 3.1.

The two figures in the following subsection aim to present that proving a covering space does not have the Birman–Hilden property by using weak curve lifting property may not be easy. On the other hand, Theorem 1.2 will be easier to use instead.

***4.4. An application of Theorem* 1.2.** We consider again the threefold simple branched cover of surfaces $p\colon \widetilde{\Sigma} \to \Sigma$, where $\widetilde{\Sigma}$ is a closed surface of genus g and Σ is a sphere with $2g+4$ branch points and we take $g=2$. Then by blowing up this one twice gives a covering space as in Figs. 2 and 3.

In this situation, the preimage of the essential simple closed curve c in Fig. 2 is the union of c_1, c_2 and c_3 in $\tilde{N}$ and at least one of the components of the preimage of c is essential (c_1 and c_2 are essential, because each c_i bounds a Klein Bottle). Therefore, we could not say immediately that this covering space does not have the weak curve lifting property, so we could not use Proposition 3.1. We need to check whether or not this covering space has the weak curve lifting property for other essential curves.

Winarski described an algorithm to check whether or not a covering space has the weak curve lifting property (Proposition 2 and Theorem 4 in [8]). It follows from the proofs of these results that they are valid for nonorientable surfaces also (for the sake of completeness we include a sketch of her proof):

Theorem 4.2. *Let $p\colon \widetilde{S} \to S$ be a finite covering space of surfaces such that $\chi(\widetilde{S}) < 0$. There is an algorithm to check whether or not p has the weak curve lifting property.*

Sketch of Proof. First we assume that the covering space is unbranched of degree n. Fix base points $x_0 \in S$ and $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{S}$. Clearly, a diffeomorphism of S lifts to $\tilde{S}$ if and only if it preserves the image subgroup $H_0 = p_*(\pi_1(\tilde{S}, \tilde{x}_0))$. Let $\mathrm{LMod}(S, x_0)$ denote the subgroup of elements that have liftable representatives. There is an action of $\mathrm{Mod}(S, x_0)$ on the set of all subgroups $\pi_1\big(\tilde{S}, \tilde{x}_0\big)$ of index n. Denote the orbit of H_0 by $\mathcal{H}$, which is clearly finite. Moreover, from the lifting criteria for maps to the covering spaces we see that the stabilizer of H_0 is $\mathrm{LMod}(S, x_0)$. Denote the homomorphism $P\colon \mathrm{Mod}(S, x_0) \to \Sigma_{\mathcal{H}}$, which describes the restriction of the action to the orbit $\mathcal{H}$.

Now using the finiteness of the group $\Sigma_{\mathcal{H}}$, we choose a finite set F in $\mathrm{Mod}(S, x_0)$ so that for each $H \in \mathcal{H}$, there is some $f \in F$ with $f \cdot H = H_0$. Finally, using the forgetfull homomorphism $\mathrm{Mod}(S, x_0) \to \mathrm{Mod}(S)$ one obtains a set of coset representatives for the subgroup $\mathrm{LMod}(S)$ in $\mathrm{Mod}(S)$. Next we use this algorithm to finish the proof of the theorem for (un)branched coverings.

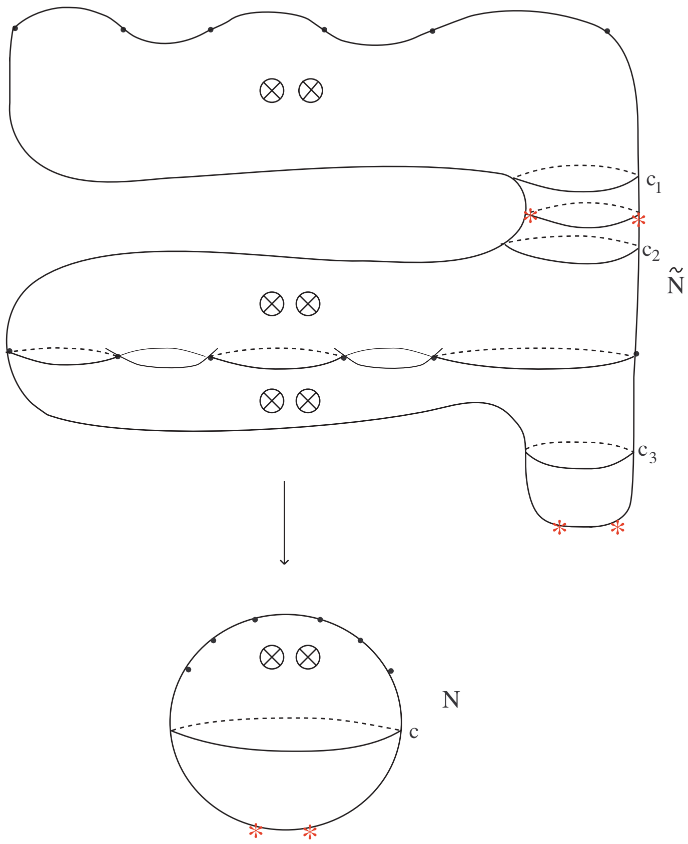

Fig. 2

To do this let S° denote the surface S, where the branched points are removed and let $\tilde{S}^\circ = p^{-1}(S^\circ)$. Let $\mathcal{G}(S)$ denote the set of isotopy classes of two-sided simple closed curves in S and let $\mathcal{M}(S)$ denote the set of isotopy classes of two-sided multicurves in S. The embedding $S^\circ \to S$ clearly takes simple closed curves to simple closed curves. So there is sequence of maps

$$\Psi : \mathcal{G}(S) \to \mathcal{G}(S^\circ) \to \mathcal{M}(S^\circ) \to \mathcal{M}(S),$$

where the first one is a bijection, the second one is given by taking preimage and the last one simply fills in the preimages of the branch points back.

Now if two elements $c, c' \in \mathcal{G}(S)$ are so that $f(c) = c'$ for some $f \in \mathrm{LMod}(S)$, then $\Psi(c)$ and $\Psi(c')$ have the same number of essential and inessential components. Hence, checking the weak curve lifting property amounts to compute $\Psi(c)$ for a set of representatives of $\mathcal{G}(S)/\mathrm{LMod}(S)$. Finally, the proof concludes using the fact that by the classification of surfaces the set $\mathcal{G}(S)/\mathrm{Mod}(S)$ is finite.

Theorem 4.2 is proved.

One should check the weak curve lifting property for essential (two-sided) simple closed curves corresponding to each topological type. Therefore, in order to clarify the construction of this covering space and identify topologically different curves on the surfaces, we need to distinguish dots from stars although they are equivalent topologically.

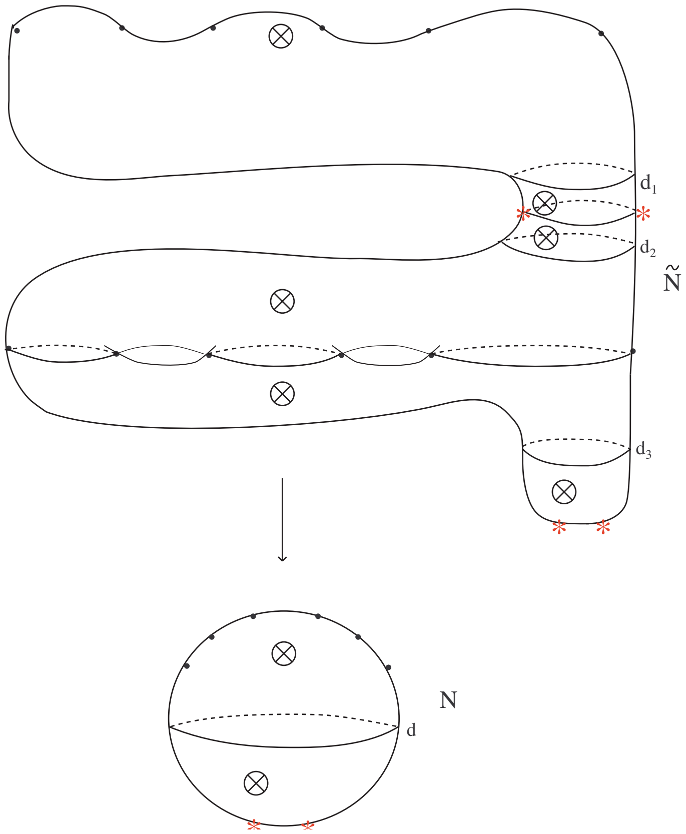

Fig. 3

Now, let us consider another essential simple closed curve d in Fig. 3. Its preimage is the union of d_1, d_2 and d_3 in $\tilde{N}$ and one of the components of the preimage of d is essential (d_2 is essential, because it bounds a nonorientable surface of genus 3).

It turns out that there are 29 topological types one of which is nonseparating and all others are separating for this covering space. One can determine all components of the preimage of every essential curve and one can check whether or not at least one of the components of the preimage of each essential curve is essential. However, this may not always be easy. In this situation, Theorem 1.2 will be easier to use as a criterion to say that this covering space does not have the Birman – Hilden property.

5. Final remarks. Let $p\colon \widetilde{\Sigma} \to \Sigma$ denote a covering (possibly branched) of surfaces. Let $z \in \Sigma$ and $\{\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n\} = p^{-1}(z)$, where z is not a branch point. Let N be the blow up of Σ at z and $\widetilde{N}$ be the blow up of $\widetilde{\Sigma}$ at all $\tilde{z}_i$'s. Then we get a covering $p_B : \widetilde{N} \to N$. In this case, we have the following observations:

1. If $p\colon \widetilde{\Sigma} \to \Sigma$ is a regular covering, then so is $p_B : \widetilde{N} \to N$. Thus, both coverings have the Birman – Hilden property.

2. If $p\colon \widetilde{\Sigma} \to \Sigma$ is a fully ramified, then so is $p_B : \widetilde{N} \to N$. Hence, both coverings have the Birman – Hilden property, by Theorems 2.1 and 1.1.

3. $p\colon \widetilde{\Sigma} \to \Sigma$ is a simple cover(see Subsection 2.4) $\big(\chi(\widetilde{\Sigma}) < 0\big)$ if and only if $p_B : \widetilde{N} \to N$ is a simple cover $\big(\chi(\widetilde{N}) < 0\big)$. If $p\colon \widetilde{\Sigma} \to \Sigma$ is a simple cover $\big(\chi(\widetilde{\Sigma}) < 0\big)$, p does not have the

Birman – Hilden property by Theorem 2 in [8]. If $p_B : \widetilde{N} \to N$ is a simple cover $\big(\chi(\widetilde{N}) < 0\big)$, then p_B does not have the Birman – Hilden property. Because, if $p_B : \widetilde{N} \to N$ is a simple cover, then $p : \widetilde{\Sigma} \to \Sigma$ is a simple cover. By Winarski's result, p does not have the Birman – Hilden property. Since p does not have the Birman – Hilden property, p_B does not have the Birman – Hilden property by Theorem 1.2.

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