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A (p, q)-ANALOGUE OF POLY-EULER POLYNOMIALS AND SOME RELATED POLYNOMIALS

(p, q)-АНАЛОГ ПОЛІЕЙЛЕРІВСЬКИХ ПОЛІНОМІВ ТА ДЕЯКІ СУМІЖНІ ПОЛІНОМИ

We introduce a (p, q)-analogue of the poly-Euler polynomials and numbers by using the (p, q)-polylogarithm function. These new sequences are generalizations of the poly-Euler numbers and polynomials. We give several combinatorial identities and properties of these new polynomials, and also show some relations with (p, q)-poly-Bernoulli polynomials and (p, q)-poly-Cauchy polynomials. The (p, q)-analogues generalize the well-known concept of the q-analogue.

Введено (p, q)-аналоги полiейлерiвських полiномiв i чисел за допомогою (p, q)-полiлогарифмiчної функцiї, якi є узагальненнями полiейлерiвських полiномiв i чисел. Знайдено властивостi цих полiномiв i наведено деякi вiдповiднi комбiнаторнi рiвностi. Також показано зв'язок iз (p, q)-полiномами типу Бернуллi та Кошi. Цi (p, q)-аналоги узагальнюють вiдому концепцiю q-аналогiв.

1. Introduction. The Euler numbers are defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

The sequence $(E_n)_n$ counts the numbers of alternating n-permutations. A n-permutation σ is alternating if the $n-1$ differences $\sigma(i+1) - \sigma(i)$ for $i = 1, 2, \ldots, n-1$ have alternating signs. For example, (1324) and (3241) are alternating permutations (cf. [10]).

The Euler polynomials are given by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Note that $E_n = 2^n E_n(1/2)$.

Many kinds of generalizations of these numbers and polynomials have been presented in the literature (see, e.g., [39]). In particular, we are interested in the poly-Euler numbers and polynomials (cf. [12, 15, 16, 32]).

The poly-Euler polynomials $E_n^{(k)}(x)$ are defined by the following generating function:

$$\frac{2\mathrm{Li}_k(1 - e^{-t})}{1 + e^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}, \quad k \in \mathbb{Z},$$

where

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$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} \tag{1}$$

is the kth polylogarithm function. Note that if $k = 1$, then $\mathrm{Li}_1(t) = -\log(1-t)$, therefore, $E_n^{(1)}(x) = E_{n-1}(x)$ for $n \geq 1$.

It is also possible to define the poly-Bernoulli and poly-Cauchy numbers and polynomials from the kth polylogarithm function. In particular, the poly-Bernoulli numbers $B_n^{(k)}$ were introduced by Kaneko [17] by using the following generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z}.$$

If $k = 1$ we get $B_n^{(1)} = (-1)^n B_n$ for $n \geq 0$, where B_n are the Bernoulli numbers. Remember that the Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The poly-Bernoulli numbers and polynomials have been studied in several papers; among other references, see [3, 4, 7, 8, 21, 22, 25–27].

The poly-Cauchy numbers of the first kind $c_n^{(k)}$ were introduced by the first author in [19]. They are defined as follows:

$$c_n^{(k)} = \underbrace{\int_0^1 \ldots \int_0^1}_{k} (t_1 \ldots t_k)_n \, dt_1 \ldots dt_k,$$

where $(x)_n = x(x-1)\ldots(x-n+1) (n \geq 1)$ with $(x)_0 = 1$. Moreover, its exponential generating function is

$$\mathrm{Lif}_k(\ln(1+t)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z},$$

where

$$\mathrm{Lif}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}$$

is the kth polylogarithm factorial function. For more properties about these numbers see, for example, [8, 20–24]. If $k = 1$, we recover the Cauchy numbers $c_n^{(1)} = c_n$. The Cauchy numbers c_n were introduced in [10] by the generating function

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}.$$

A generalization of the above sequences was done recently in [21], using the kth q-polylogarithm function and the Jackson's integral. In particular, the q-poly-Bernoulli numbers are defined by

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{t^n}{n!}, \qquad k \in \mathbb{Z}, \quad n \geq 0, \quad 0 \leq q < 1,$$

where

$$\mathrm{Li}_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}$$

is the kth q-polylogarithm function (cf. [29]), and $[n]_q = \dfrac{1-q^n}{1-q}$ is the q-integer (cf. [39]). Note that $\lim_{q\to 1}[x]_q = x$, $\lim_{q\to 1} B_{n,q}^{(k)} = B_n^{(k)}$ and $\lim_{q\to 1} \mathrm{Li}_{k,q}(x) = \mathrm{Li}_k(x)$.

The q-poly-Cauchy numbers of the first kind $c_{n,q}^{(k)}$ are defined by using the Jackson's q-integral (cf. [1])

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \ldots \int_0^1}_{k} (t_1 \ldots t_k)_n d_q t_1 \ldots d_q t_k,$$

where

$$\int_0^x f(t) d_q t = (1-q)x \sum_{n=0}^{\infty} f(q^n x) q^n.$$

Moreover, its exponential generating function is

$$\mathrm{Lif}_{k,q}(\ln(1+t)) = \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z},$$

where

$$\mathrm{Lif}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n![n+1]_q^k} \tag{2}$$

is the kth q-polylogarithm factorial function (cf. [18, 21]). Note that $\lim_{q\to 1} c_{n,q}^{(k)} = c_n^{(k)}$ and $\lim_{q\to 1} \mathrm{Lif}_{k,q}(t) = \mathrm{Lif}_k(t)$.

In this paper, we introduce a (p,q)-analogue of the poly-Euler polynomials by

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x) \frac{t^n}{n!}, \quad k \in \mathbb{Z}, \tag{3}$$

with p and q real numbers such that $0 < q < p \leq 1$, and

$$\mathrm{Li}_{k,p,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_{p,q}^k}$$

is an extension of the q-polylogarithm function and we call it the (p,q)-polylogarithm function. The polynomials $E_{n,p,q}^{(k)}(0) := E_{n,p,q}^{(k)}$ are called (p,q)-poly-Euler numbers. The polynomial $[n]_{p,q} = \dfrac{p^n - q^n}{p-q}$ is the n-th (p,q)-integer (cf. [13, 14, 37]), it was introduced in the context of set partition statistics (cf. [40]). Note that $\lim_{p\to 1}[n]_{p,q} = [n]_q$ and $\lim_{p\to 1} \mathrm{Lif}_{k,p,q}(t) = \mathrm{Lif}_{k,q}(t)$.

As we already mentioned the (p, q)-analogues are an extension of the q-analogues, and coincide in the limit when p tends to 1. The (p, q)-calculus was studied in [9], in connection with quantum mechanics. Properties of the (p, q)-analogues of the binomial coefficients were studied in [11]. The (p, q)-analogues of hypergeometric series, special functions, Stirling numbers and their generalizations, Hermite polynomials, Volkenborn integration have been studied before, see, for instance, [2, 14, 30, 31, 33, 34, 36, 38].

The paper is divided in two parts. In Section 2, we show several combinatorial identities of the (p, q)-poly-Euler polynomials. Some of them involving the classical Euler polynomials and another special numbers and polynomials such as the Stirling numbers of the second kind, Bernoulli polynomials of order s, etc. In Section 3, we introduce the (p, q)-poly-Bernoulli polynomials and (p, q)-poly-Cauchy polynomials of both kinds, and we generalize some well-known identities of the classical Bernoulli and Cauchy numbers and polynomials.

2. Some identities of the poly-Euler polynomials. In this section, we give several identities of the (p, q)-poly-Euler polynomials. In particular, Theorem 2 shows a relation between the (p, q)-poly-Euler polynomials and the classical Euler polynomials.

It is possible to give the first values of the (p, q)-polylogarithm function for $k \leq 0$. For example,

$$\mathrm{Li}_{0,p,q}(x) = \frac{x}{1-x},$$

$$\mathrm{Li}_{-1,p,q}(x) = \frac{x}{(1-px)(1-qx)},$$

$$\mathrm{Li}_{-2,p,q}(x) = \frac{x(1+pqx)}{\big(1-p^2x\big)\big(1-q^2x\big)(1-pqx)},$$

$$\mathrm{Li}_{-3,p,q}(x) = \frac{x\left(p^3q^3x^2+2p^2qx+2pq^2x+1\right)}{\left(1-p^3x\right)\left(1-q^3x\right)\left(1-p^2qx\right)\left(1-pq^2x\right)}.$$

In general, the (p, q)-polylogarithm function for $k \leq 0$ is a rational function. Indeed, let k be a nonnegative integer then

$$\mathrm{Li}_{-k,p,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^{-k}} = \sum_{n=1}^{\infty} [n]_{p,q}^{k} x^n = \sum_{n=1}^{\infty} \left(\frac{p^n - q^n}{p-q}\right)^k x^n =$$

$$= \frac{1}{(p-q)^k} \sum_{n=1}^{\infty} \sum_{l=0}^{k} \binom{k}{l} p^{nl} (-q^n)^{k-l} x^n = \frac{1}{(p-q)^k} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \frac{p^l q^{k-l} x}{1 - p^l q^{k-l} x}.$$

Note that from (3) we obtain that $\big\{E_{n,p,q}^{(k)}(x)\big\}_{n\geq 0}$ is an Appel sequence [35]. Therefore, we have the following basic relations.

Theorem 1. *If* $n \geq 0$ *and* $k \in \mathbb{Z}$, *then*

(i) $E_{n,p,q}^{(k)}(x) = \displaystyle\sum_{i=0}^{n} \binom{n}{i} E_{i,p,q}^{(k)} x^{n-i}$,

(ii) $E_{n,p,q}^{(k)}(x+y) = \displaystyle\sum_{i=0}^{n} \binom{n}{i} E_{i,p,q}^{(k)}(x) y^{n-i}$,

(iii) $E^{(k)}_{n,p,q}(mx) = \sum_{i=0}^{n} \binom{n}{i} E^{(k)}_{i,p,q}(x)(m-1)^{n-i}x^{n-i}, \quad m \geq 1,$

(iv) $E^{(k)}_{n,p,q}(x+1) - E^{(k)}_{n,p,q}(x) = \sum_{i=0}^{n-1} \binom{n}{i} E^{(k)}_{i,p,q}(x).$

Theorem 2. *If* $n \geq 1$, *we have*

$$E^{(k)}_{n,p,q}(x) = \sum_{\ell=0}^{\infty} \frac{1}{[\ell+1]^k_{p,q}} \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} (-1)^j E_n(x-j).$$

Proof. From (1) and (3), we get

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = \sum_{\ell=0}^{\infty} \frac{(1-e^{-t})^{\ell+1}}{[\ell+1]^k_{p,q}} \frac{2e^{xt}}{1+e^t} =$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{[\ell+1]^k_{p,q}} \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} (-1)^j \frac{2e^{(x-j)t}}{1+e^t} =$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{[\ell+1]^k_{p,q}} \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} (-1)^j \sum_{n=0}^{\infty} E_n(x-j) \frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we get the desired result.

Theorem 2 is proved.

Theorem 3. *If* $n \geq 1$, *we have*

$$E^{(k)}_{n,p,q}(x) = \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \sum_{j=0}^{i+1} \frac{2(-1)^{\ell-i-j}}{[i+1]^k_{p,q}} \binom{i+1}{j} (\ell-i-j+x)^n.$$

Proof. By using the binomial series, we get

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = 2 \left(\sum_{\ell=0}^{\infty} (-1)^\ell e^{\ell t} \right) \left(\sum_{\ell=0}^{\infty} \frac{(1-e^{-t})^{\ell+1}}{[\ell+1]^k_{p,q}} \right) e^{xt} =$$

$$= 2 \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \frac{(-1)^{\ell-i} e^{(\ell-i)t}}{[i+1]^k_{p,q}} (1-e^{-t})^{i+1} e^{xt} =$$

$$= \left(2 \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \frac{(-1)^{\ell-i} e^{(\ell-i)t}}{[i+1]^k_{p,q}} \right) \left(\sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j e^{-tj} e^{xt} \right) =$$

$$= 2 \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \sum_{j=0}^{i+1} \frac{(-1)^{\ell-i+j} e^{(\ell-i-j+x)t}}{[i+1]^k_{p,q}} \binom{i+1}{j} =$$

$$= 2 \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \sum_{j=0}^{i+1} \frac{(-1)^{\ell-i+j}}{[i+1]^k_{p,q}} \binom{i+1}{j} \sum_{n=0}^{\infty} (\ell-i-j+x)^n \frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{i=0}^{\ell}\sum_{j=0}^{i+1}\frac{2(-1)^{\ell-i+j}}{[i+1]_{p,q}^{k}}\binom{i+1}{j}(\ell-i-j+x)^{n}\frac{t^{n}}{n!}.$$

Comparing the coefficients on both sides, we get the desired result.

Theorem 3 is proved.

2.1. Some relations with other special polynomials. Jolany et al. [15] discovered several combinatorics identities involving generalized poly-Euler polynomials in terms of Stirling numbers of the second kind $S_2(n,k)$, rising factorial functions $(x)^{(m)}$, falling factorial functions $(x)_m$, Bernoulli polynomials $\mathfrak{B}_n^{(s)}(x)$ of order s, and Frobenius – Euler functions $H_n^{(s)}(x;u)$. We will give similar expressions in terms of (p,q)-poly-Euler polynomials.

Remember that the Stirling numbers of the second kind are defined by

$$\frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{x^n}{n!}. \tag{4}$$

Theorem 4. *We have the following identity*:

$$E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{\infty}\sum_{i=\ell}^{n}\binom{n}{i}S_2(i,\ell)E_{n-i,p,q}^{(k)}(-\ell)(x)^{(\ell)}, \tag{5}$$

where

$$(x)^{(m)} = x(x+1)\ldots(x+m-1), \qquad m\geq 1, \quad \textit{with} \quad (x)^{(0)}=1.$$

Proof. From (3), (4), and by the binomial series

$$\frac{1}{(1-x)^c} = \sum_{n=0}^{\infty}\binom{c+n-1}{n}x^n,$$

we get

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{xt} = \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}(1-(1-e^{-t}))^{-x} =$$

$$= \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}\sum_{\ell=0}^{\infty}\binom{x+\ell-1}{\ell}(1-e^{-t})^{\ell} =$$

$$= \sum_{\ell=0}^{\infty}\frac{(x)^{(\ell)}}{\ell!}(1-e^{-t})^{\ell}\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} =$$

$$= \sum_{\ell=0}^{\infty}(x)^{(\ell)}\frac{(e^t-1)^{\ell}}{\ell!}\left(\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{-t\ell}\right) =$$

$$= \sum_{\ell=0}^{\infty}(x)^{(\ell)}\left(\sum_{n=0}^{\infty}S_2(n,\ell)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(-\ell)\frac{t^n}{n!}\right) =$$

$$= \sum_{\ell=0}^{\infty}(x)^{(\ell)}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}S_2(i,\ell)E_{n-i,p,q}^{(k)}(-\ell)\right)\frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\sum_{i=\ell}^{n}\binom{n}{i}S_2(i,\ell)E_{n-i,p,q}^{(k)}(-\ell)(x)^{(\ell)}\right)\frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we have (5). Note that we use the following relation:

$$\binom{x+\ell-1}{s} = \frac{(x)^{(\ell)}}{s!}.$$

Theorem 4 is proved.

Theorem 5. *We have the following identity*:

$$E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{\infty}\sum_{i=\ell}^{n}\binom{n}{i}S_2(i,\ell)E_{n-i,p,q}^{(k)}(x)_\ell, \tag{6}$$

where

$$(x)_m = x(x-1)\ldots(x-m+1), \qquad m\geq 1, \quad \text{with} \quad (x)_0 = 1.$$

Proof. From (3) and (4), we obtain

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{xt} = \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}((e^t-1)+1)^x =$$

$$= \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}\sum_{\ell=0}^{\infty}\binom{x}{\ell}(e^t-1)^\ell =$$

$$= \sum_{\ell=0}^{\infty}\frac{(x)_\ell}{\ell!}(e^t-1)^\ell\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} =$$

$$= \sum_{\ell=0}^{\infty}(x)_\ell\left(\sum_{n=0}^{\infty}S_2(n,\ell)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}\frac{t^n}{n!}\right) =$$

$$= \sum_{\ell=0}^{\infty}(x)_\ell\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}S_2(i,\ell)E_{n-i,p,q}^{(k)}\right)\frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\sum_{i=\ell}^{n}\binom{n}{i}S_2(i,\ell)E_{n-i,p,q}^{(k)}(x)_\ell\right)\frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we have (6). Note that we use the following relation:

$$\binom{x}{s} = \frac{(x)_s}{s!}.$$

Theorem 5 is proved.

The Bernoulli polynomials $\mathfrak{B}_n^{(s)}(x)$ of order s are defined by

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty}\mathfrak{B}_n^{(s)}(x)\frac{t^n}{n!}. \tag{7}$$

It is clear that if $s=1$ we recover the classical Bernoulli polynomials. For some explicit formulae of these polynomials see, for example, [28].

Theorem 6. *We have the following identity*:

$$E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} S_2(\ell+s,s) \sum_{i=0}^{n-\ell} \frac{\binom{n-\ell}{i}}{\binom{\ell+s}{s}} \mathfrak{B}_i^{(s)}(x) E_{n-\ell-i,p,q}^{(k)}. \tag{8}$$

Proof. From (3) and (7), we get

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = \frac{(e^t-1)^s}{s!} \frac{t^s e^{xt}}{(e^t-1)^s} \left(\sum_{n=0}^{\infty} E_{n,p,q}^{(k)} \frac{t^n}{n!} \right) \frac{s!}{t^s} =$$

$$= \left(\sum_{n=0}^{\infty} S_2(n+s,s) \frac{t^{n+s}}{(n+s)!} \right) \left(\sum_{n=0}^{\infty} \mathfrak{B}_n^{(s)}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} E_{n,p,q}^{(k)} \frac{t^n}{n!} \right) \frac{s!}{t^s} =$$

$$= \left(\sum_{n=0}^{\infty} S_2(n+s,s) \frac{t^{n+s}}{(n+s)!} \right) \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} \mathfrak{B}_i^{(s)}(x) E_{n-i,p,q}^{(k)} \right) \frac{t^n}{n!} \frac{s!}{t^s} =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} S_2(\ell+s,s) \frac{t^{\ell+s}}{(\ell+s)!} \sum_{i=0}^{n-\ell} \binom{n-\ell}{i} \mathfrak{B}_i^{(s)}(x) E_{n-\ell-i,p,q}^{(k)} \frac{t^{n-\ell}}{(n-\ell)!} \right) \frac{s!}{t^s} =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \binom{n}{\ell} S_2(\ell+s,s) \sum_{i=0}^{n-\ell} \frac{\binom{n-\ell}{i}}{\binom{\ell+s}{s}} \mathfrak{B}_i^{(s)}(x) E_{n-\ell-i,p,q}^{(k)} \right) \frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we obtain (8).

Theorem 6 is proved.

The Frobenius – Euler functions $H_n^{(s)}(x;u)$ are defined by

$$\left(\frac{1-u}{e^t-u} \right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x;u) \frac{t^n}{n!}. \tag{9}$$

Theorem 7. *We have the following identity*:

$$E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{(1-u)^s} \sum_{i=0}^{s} \binom{s}{i} (-u)^{s-i} H_\ell^{(s)}(x;u) E_{n-\ell,p,q}^{(k)}(i). \tag{10}$$

Proof. From (3) and (9), we get

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = \frac{(1-u)^s}{(e^t-u)^s} e^{xt} \frac{(e^t-u)^s}{(1-u)^s} \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} =$$

$$= \frac{1}{(1-u)^s} \left(\sum_{n=0}^{\infty} H_n^{(s)}(x;u) \frac{t^n}{n!} \right) \sum_{i=0}^{s} \binom{s}{i} e^{ti} (-u)^{s-i} \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t} =$$

$$= \frac{1}{(1-u)^s}\left(\sum_{n=0}^{\infty} H_n^{(s)}(x;u)\frac{t^n}{n!}\right)\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(i)\frac{t^n}{n!} =$$

$$= \frac{1}{(1-u)^s}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}\left(\sum_{n=0}^{\infty} H_n^{(s)}(x;u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(i)\frac{t^n}{n!}\right) =$$

$$= \frac{1}{(1-u)^s}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}H_\ell^{(s)}(x;u)E_{n-\ell,p,q}^{(k)}(i)\right)\frac{t^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^s}\sum_{\ell=0}^{n}\binom{n}{\ell}\sum_{i=0}^{s}\binom{s}{i}(-u)^{s-i}H_\ell^{(s)}(x;u)E_{n-\ell,p,q}^{(k)}(i)\right)\frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we have (10).

Theorem 7 is proved.

3. The (p, q)-poly-Bernoulli polynomials and the (p, q)-poly-Cauchy polynomials. In this section, we introduce the (p, q)-poly-Bernoulli polynomials by means of the (p, q)-polylogarithm function and the (p, q)-poly-Cauchy polynomials by using the (p, q)-integral. In general it is not difficult to extend the results of [21].

The (p, q)-derivative of the function f is defined by (cf. [5, 13])

$$D_{p,q}f(x) = \begin{cases} \dfrac{f(px)-f(qx)}{(p-q)x}, & \text{if} \quad x \neq 0, \\ f'(0), & \text{if} \quad x = 0. \end{cases}$$

In particular, if $p \to 1$ we obtain the q-derivative [1]. The (p, q)-integral of the function f is defined by

$$\int_0^x f(t)d_{p,q}t = \begin{cases} (q-p)x\sum_{n=0}^{\infty}\dfrac{p^n}{q^{n+1}}f\left(\dfrac{p^n}{q^{n+1}}x\right), & \text{if} \quad |p/q| < 1, \\ (p-q)x\sum_{n=0}^{\infty}\dfrac{q^n}{p^{n+1}}f\left(\dfrac{q^n}{p^{n+1}}x\right), & \text{if} \quad |p/q| > 1. \end{cases}$$

For example,

$$\int_0^{\ell} t^{\ell}d_{p,q}t = \frac{1}{[\ell+1]_{p,q}}.$$

We introduce the (p, q)-poly-Bernoulli polynomials by

$$\frac{\mathrm{Li}_{k,p,q}(1-e^{-t})}{1-e^{-t}}e^{-xt} = \sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(x)\frac{t^n}{n!}, \quad k \in \mathbb{Z}.$$

In particular, $\lim_{p\to 1} B_{n,p,q}^{(k)}(x) = B_{n,q}^{(k)}(x)$, which are the q-poly-Bernoulli polynomials studied recently in [21].

The following theorem related the (p, q)-poly-Bernoulli polynomials and (p, q)-poly-Euler polynomials.

Theorem 8. *If $n \geq 1$, we have*

$$E_{n,p,q}^{(k)}(x) + E_{n,p,q}^{(k)}(x+1) = 2B_{n,p,q}^{(k)}(-x) - 2B_{n,p,q}^{(k)}(1-x).$$

Proof. From the equality

$$\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1+e^t}(1+e^t)e^{xt}=\frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{1-e^{-t}}(1-e^{-t})e^{xt},$$

we obtain

$$\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(x+1)\frac{t^n}{n!}=2\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(-x)\frac{t^n}{n!}-2\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(1-x)\frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we get the desired result.

Theorem 8 is proved.

The weighted Stirling numbers of the second kind, $S_2(n,m,x)$, were defined by Carlitz [6] as follows:

$$\frac{e^{xt}(e^t-1)^m}{m!}=\sum_{n=m}^{\infty}S_2(n,m,x)\frac{t^n}{n!}.$$

Theorem 9. *If* $n\geq 1$, *we have*

$$B_{n,p,q}^{(k)}(x)=\sum_{m=0}^{n}\frac{(-1)^{m+n}m!}{[m+1]_{p,q}^k}S_2(n,m,x).$$

Proof. We obtain

$$\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(x)\frac{t^n}{n!}=\frac{\mathrm{Li}_{p,q}(1-e^{-t})}{1-e^{-t}}e^{-xt}=\sum_{m=0}^{\infty}\frac{(1-e^{-t})^m}{[m+1]_{p,q}^k}e^{-xt}=$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m m!}{[m+1]_{p,q}^k}\frac{(e^{-t}-1)^m}{m!}e^{-xt}=\sum_{m=0}^{\infty}\frac{(-1)^m m!}{[m+1]_{p,q}^k}\sum_{n=m}^{\infty}S_2(n,m,x)\frac{(-t)^n}{n!}=$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(-1)^{m+n}m!}{[m+1]_{p,q}^k}S_2(n,m,x)\right)\frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we get the desired result.

Theorem 9 is proved.

The (p,q)-poly-Cauchy polynomials of the first kind are defined by

$$C_{n,p,q}^{(k)}(x)=\underbrace{\int_0^1\ldots\int_0^1}_{k}(t_1\ldots t_k-x)_n d_{p,q}t_1\ldots d_{p,q}t_k. \qquad (11)$$

Note that $\lim_{p\to 1}C_{n,p,q}^{(k)}(x)=C_{n,q}^{(k)}(x)$, i.e., we obtain the q-poly-Cauchy polynomials [18, 21].

Remember that the (unsigned) Stirling numbers of the first kind are defined by

$$\frac{(\ln(1+x))^m}{m!}=\sum_{n=m}^{\infty}(-1)^{n-m}S_1(n,m)\frac{x^n}{n!}. \qquad (12)$$

Moreover, they satisfy (cf. [10])

$$x^{(n)} = x(x+1)\dots(x+n-1) = \sum_{m=0}^{n} S_1(n,m)x^m. \tag{13}$$

The weighted Stirling numbers of the first kind, $S_1(n,m,x)$, are defined by [6]

$$\frac{(1-t)^{-x}(-\ln(1-t))^m}{m!} = \sum_{n=m}^{\infty} S_1(n,m,x)\frac{t^n}{n!}.$$

Theorem 10. *If $n \geq 1$, we have*

$$C^{(k)}_{n,p,q}(x) = \sum_{m=0}^{n}(-1)^{n-m}S_1(n,m)\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{(-x)^{\ell}}{[m-\ell+1]^k_{p,q}} = \tag{14}$$

$$= \sum_{m=0}^{n} S_1(n,m,x)\frac{(-1)^{n-m}}{[m+1]^k_{p,q}}. \tag{15}$$

Proof. By (11), (13) and $(x)_n = (-1)^n(-x)^{(n)}$, we obtain

$$C^{(k)}_{n,p,q}(x) = \sum_{m=0}^{n}(-1)^{n-m}S_1(n,m)\underbrace{\int_0^1\dots\int_0^1}_{k}(t_1\dots t_k - x)^m d_{p,q}t_1\dots d_{p,q}t_k =$$

$$= \sum_{m=0}^{n}(-1)^{n-m}S_1(n,m)\sum_{\ell=0}^{m}\binom{m}{\ell}(-x)^{m-\ell}\underbrace{\int_0^1\dots\int_0^1}_{k} t_1^{\ell}\dots t_k^{\ell} d_{p,q}t_1\dots d_{p,q}t_k =$$

$$= \sum_{m=0}^{n}(-1)^{n-m}S_1(n,m)\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{(-x)^{m-\ell}}{[\ell+1]^k_{p,q}} =$$

$$= \sum_{m=0}^{n}(-1)^{n-m}S_1(n,m)\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{(-x)^{\ell}}{[m-\ell+1]^k_{p,q}}.$$

Comparing the coefficients on both sides, we get (14). Finally, from the relation [6] (Eq. (5.2))

$$S_1(n,m,x) = \sum_{i=0}^{n}\binom{m+i}{i}x^i S_1(n,m+i),$$

we have

$$C^{(k)}_{n,p,q}(x) = \sum_{m=0}^{n}(-1)^{n-m}S_1(n,m)\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{(-x)^{\ell}}{[m-\ell+1]^k_{p,q}} =$$

$$= \sum_{\ell=0}^{n}\sum_{m=\ell}^{n}(-1)^{n-m}S_1(n,m)\binom{m}{\ell}\frac{(-x)^{\ell}}{[m-\ell+1]^k_{p,q}} =$$

$$= \sum_{\ell=0}^{n} \sum_{m=\ell}^{n+\ell} (-1)^{n-m} S_1(n,m) \binom{m}{\ell} \frac{(-x)^\ell}{[m-\ell+1]_{p,q}^k} =$$

$$= \sum_{\ell=0}^{n} \sum_{m=0}^{n} (-1)^{n-m+\ell} S_1(n,m+\ell) \binom{m+\ell}{\ell} \frac{(-x)^\ell}{[m+1]_{p,q}^k} =$$

$$= \sum_{m=0}^{n} \frac{(-1)^{n-m}}{[m+1]_{p,q}^k} \sum_{\ell=0}^{m} \binom{m+\ell}{\ell} S_1(n,m+\ell) x^\ell = \sum_{m=0}^{n} \frac{(-1)^{n-m}}{[m+1]_{p,q}^k} S_1(n,m,x).$$

Theorem 10 is proved.

It is not difficult to give a (p,q)-analogue of (2).

Theorem 11. *The exponential generating function of the (p,q)-poly-Cauchy polynomials $C_{n,p,q}^{(k)}(x)$ is*

$$\frac{\mathrm{Lif}_{k,p,q}\left(\ln(1+t)\right)}{(1+t)^x} = \sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x) \frac{t^n}{n!}, \tag{16}$$

where

$$\mathrm{Lif}_{k,p,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n![n+1]_{p,q}^k}$$

is the kth (p,q)-polylogarithm factorial function.

Proof. From Theorem 10, we have

$$\sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^\ell}{[m-\ell+1]_{p,q}^k} \frac{t^n}{n!} =$$

$$= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (-1)^{n-m} S_1(n,m) \frac{t^n}{n!} \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^\ell}{[m-\ell+1]_{p,q}^k} =$$

$$= \sum_{m=0}^{\infty} \frac{(\ln(1+t))^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^\ell}{[m-\ell+1]_{p,q}^k} =$$

$$= \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{\ell!} \sum_{m=\ell}^{\infty} \frac{(\ln(1+t))^m}{(m-\ell)![m-\ell+1]_{p,q}^k} = \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{\ell!} \sum_{n=0}^{\infty} \frac{(\ln(1+t))^{n+\ell}}{n![n+1]_{p,q}^k} =$$

$$= \frac{1}{(1+t)^x} \sum_{n=0}^{\infty} \frac{(\ln(1+t))^n}{n![n+1]_{p,q}^k} = \frac{\mathrm{Lif}_{k,p,q}\left(\ln(1+t)\right)}{(1+t)^x}.$$

Theorem 11 is proved.

Similarly, we can defined the (p,q)-poly-Cauchy polynomials of the second kind by

$$\widehat{C}_{n,p,q}^{(k)}(x) = \underbrace{\int_0^1 \ldots \int_0^1}_{k} (-t_1 \ldots t_k + x)_n d_{p,q}t_1 \ldots d_{p,q}t_k.$$

We can find analogous expressions to (14)–(16).

Theorem 12. *If* $n \geq 1,$ *we have*

$$\widehat{C}^{(k)}_{n,p,q}(x) = (-1)^n \sum_{m=0}^{n} S_1(n,m) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^\ell}{[m-\ell+1]^k_{p,q}} =$$

$$= (-1)^n \sum_{m=0}^{n} S_1\left(n,m,-x\right) \frac{1}{[m+1]^k_{p,q}}.$$

Moreover, the exponential generating function of the (p,q)*-poly-Cauchy polynomials* $\widehat{C}^{(k)}_{n,p,q}(x)$ *is*

$$(1+t)^x \mathrm{Lif}_{k,p,q}\left(-\ln(1+t)\right) = \sum_{n=0}^{\infty} \widehat{C}^{(k)}_{n,p,q}(x) \frac{t^n}{n!}.$$

3.1. Some relations between (p,q)***-poly-Bernoulli polynomials and*** (p,q)***-poly-Cauchy polynomials.*** The weighted Stirling numbers satisfy the following orthogonality relation [6]:

$$\sum_{\ell=m}^{n} (-1)^{n-\ell} S_2(n,\ell,x) S_1(\ell,m,x) = \sum_{\ell=m}^{n} (-1)^{\ell-m} S_1(n,\ell,x) S_2(\ell,m,x) = \delta_{m,n},$$

where $\delta_{m,n} = 1$ if $m = n$ and 0 otherwise. From above relations we obtain the inverse relation

$$f_n = \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m,x) g_m \iff g_n = \sum_{m=0}^{n} S_2(n,m,x) f_m.$$

Theorem 13. *The* (p,q)*-poly-Bernoulli polynomials and* (p,q)*-poly-Cauchy polynomials of both kinds satisfy the following relations*:

$$\sum_{m=0}^{n} S_1(n,m,x) B^{(k)}_{m,p,q}(x) = \frac{n!}{[n+1]^k_{p,q}}, \tag{17}$$

$$\sum_{m=0}^{n} S_2(n,m,x) C^{(k)}_{m,p,q}(x) = \frac{1}{[n+1]^k_{p,q}}, \tag{18}$$

$$\sum_{m=0}^{n} S_2(n,m,-x) \widehat{C}^{(k)}_{m,p,q}(x) = \frac{(-1)^n}{[n+1]^k_{p,q}}. \tag{19}$$

Proof. From Theorem 9 and the inverse relation for the weighted Stirling numbers with

$$f_m = \frac{(-1)^m m!}{[m+1]^k_{p,q}}, \qquad g_n = (-1)^n B^{(k)}_{n,p,q}(x),$$

we obtain the identity (17). The remaining relations can be verified in a similar way by using Theorems 10 and 12.

Theorem 13 is proved.

Note that if $p \to 1$ we obtain Theorem 6 in [21].

Theorem 14. *The (p,q)-poly-Bernoulli polynomials and (p,q)-poly-Cauchy polynomials of both kinds satisfy the following relations*:

$$B_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n}\sum_{m=0}^{n}(-1)^{n-m}m!S_2(n,m,x)S_2(m,\ell,y)C_{\ell,p,q}^{(k)}(y), \tag{20}$$

$$B_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n}\sum_{m=0}^{n}(-1)^{n}m!S_2(n,m,x)S_2(m,\ell,-y)\widehat{C}_{\ell,p,q}^{(k)}(y), \tag{21}$$

$$C_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n}\sum_{m=0}^{n}\frac{(-1)^{n-m}}{m!}S_1(n,m,x)S_1(m,\ell,y)B_{\ell,p,q}^{(k)}(y), \tag{22}$$

$$\widehat{C}_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n}\sum_{m=0}^{n}\frac{(-1)^{n}}{m!}S_1(n,m,-x)S_1(m,\ell,y)B_{\ell,p,q}^{(k)}(y). \tag{23}$$

Proof. We only show the proof of (22). The proofs of the remaining identities are similar. From equations (15) and (17), we have

$$\sum_{\ell=0}^{n}\sum_{m=0}^{n}\frac{(-1)^{n-m}}{m!}S_1(n,m,x)S_1(m,\ell,y)B_{\ell,p,q}^{(k)}(y) =$$

$$= \sum_{m=0}^{n}\frac{(-1)^{n-m}}{m!}S_1(n,m,x)\sum_{\ell=0}^{m}S1(m,\ell,y)B_{\ell,p,q}^{(k)}(y) =$$

$$= \sum_{m=0}^{n}\frac{(-1)^{n-m}}{m!}S_1(n,m,x)\frac{m!}{[m+1]_{p,q}^{k}} = C_{n,p,q}^{(k)}(x).$$

Theorem 14 is proved.

Finally, we show some relations between (p,q)-poly-Cauchy polynomials of both kinds.

Theorem 15. *If $n \geq 1$, we have*

$$(-1)^n\frac{C_{n,p,q}^{(k)}(x)}{n!} = \sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\widehat{C}_{m,p,q}^{(k)}(x)}{m!}, \tag{24}$$

$$(-1)^n\frac{\widehat{C}_{n,p,q}^{(k)}(x)}{n!} = \sum_{m=1}^{n}\binom{n-1}{m-1}\frac{C_{m,p,q}^{(k)}(x)}{m!}. \tag{25}$$

Proof. From definition of the (p,q)-poly-Cauchy polynomials of the first kind, we get

$$(-1)^n\frac{C_{n,p,q}^{(k)}(x)}{n!} = (-1)^n\underbrace{\int_0^1\ldots\int_0^1}_{k}\frac{(t_1\ldots t_k - x)_n}{n!}d_{p,q}t_1\ldots d_{p,q}t_k =$$

$$= (-1)^n\underbrace{\int_0^1\ldots\int_0^1}_{k}\binom{t_1\ldots t_k - x}{n}d_{p,q}t_1\ldots d_{p,q}t_k =$$

$$=\underbrace{\int_0^1\ldots\int_0^1}_{k}\binom{x-t_1\ldots t_k+n-1}{n}d_{p,q}t_1\ldots d_{p,q}t_k.$$

By using the Vandermonde convolution

$$\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k}=\binom{r+s}{n}$$

with $r=x-t_1\ldots t_k$ and $s=n-1$, we obtain

$$(-1)^n\frac{C_{n,p,q}^{(k)}(x)}{n!}=\underbrace{\int_0^1\ldots\int_0^1}_{k}\sum_{\ell=0}^{n}\binom{x-t_1\ldots t_k}{\ell}\binom{n-1}{n-\ell}d_{p,q}t_1\ldots d_{p,q}t_k=$$

$$=\sum_{\ell=0}^{n}\binom{n-1}{n-\ell}\underbrace{\int_0^1\ldots\int_0^1}_{k}\binom{x-t_1\ldots t_k}{\ell}d_{p,q}t_1\ldots d_{p,q}t_k=$$

$$=\sum_{\ell=0}^{n}\binom{n-1}{n-\ell}\frac{1}{\ell!}\underbrace{\int_0^1\ldots\int_0^1}_{k}(-t_1\ldots t_k+x)_\ell d_{p,q}t_1\ldots d_{p,q}t_k=\sum_{\ell=0}^{n}\binom{n-1}{n-\ell}\frac{\widehat{C}_{\ell,p,q}^{(k)}(x)}{\ell!}.$$

The proof of (25) is similar.

Theorem 15 is proved.

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