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NEAR-ISOMETRIES OF THE UNIT SPHERE
БЛИЗЬКІ ІЗОМЕТРІЇ СФЕРИ ОБ'ЄДНАННЯ

We approximate ε-isometries of the unit sphere in ℓ_2^n and ℓ_∞^n by linear isometries.

Наведено наближення ε-ізометрій одиничної сфери в ℓ_2^n і ℓ_∞^n лінійними ізометріями.

1. Introduction. *Notation.* Throughout the paper X and Y denote real normed spaces. The sphere and closed ball with center z and radius r are denoted by $S(z, r)$ and $B(z, r)$; we also write $S(0, r) = S(r)$ and $B(0, r) = B(r)$. The unit sphere and ball are denoted by S and B (or S_E and B_E when we need to specify the space). For a point x in $\mathbb{R}^n$, x_i denotes its ith coordinate in the standard basis $\{e_i\}_{i=1}^n$.

A local version of the classical Mazur – Ulam theorem asserts that a local isometry f, which maps an open connected subset of X onto an open subset of Y, is the restriction of an affine isometry of X onto Y (see, for example, [1, p. 341]). This classical result was generalized in several directions. One of them is the study of the isometric extension problem posed by D. Tingley [8]: Let T be a surjective isometry between the spheres of X and Y. Is T necessarily the restriction of a linear isometry between X and Y? There are a number of publications devoted to Tingley's problem (see [2] for a survey of corresponding results) and, in particular, the problem is solved in positive for many concrete classical Banach spaces.

When distances are known only imprecisely, it is natural to study how close f is to be an isometry. There are various different useful concepts of an approximate isometry, and one may then ask whether such a mapping, which only nearly preserves distances, can be well approximated by a true isometry, especially by an affine isometry (see [1], Chapters 14 and 15, and surveys [6] and [7] for more complete exposition and literature on this subject).

Definition. *Let A be a subset of X and $\varepsilon \geq 0$. A map $f : A \to Y$ is called an ε-isometry if*

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \leq \varepsilon \tag{1}$$

for all $x, y \in A$.

The author [9, 11] has presented sharp results on approximation of ε-isometries of balls in ℓ_2^n and ℓ_∞^n.

In the present paper we study approximation of ε-isometries of spheres in ℓ_2^n and ℓ_∞^n. We give the following results, proceeding the way of [9, 11].

Theorem 1. *Let $f\colon S_{\ell_2^n} \to S_{\ell_2^n}$ be an ε-isometry. Then there is a linear isometry U of ℓ_2^n such that*

$$\|f(x) - Ux\| \le C\log(n+1)\varepsilon, \qquad x \in S_{\ell_2^n}, \tag{2}$$

for some absolute constant C.

The upper bound in (2) *is sharp.*

Theorem 2. *There are absolute constants C and c with the following property: Let $0 < \varepsilon < c$ and $f\colon S_{\ell_\infty^n} \to S_{\ell_\infty^n}$ be an ε-isometry. Then there is a unique linear isometry U of ℓ_∞^n such that*

$$\|f(x) - Ux\| \le C\varepsilon, \qquad x \in S_{\ell_\infty^n}. \tag{3}$$

2. Proofs. We need the following lemma, which is proven, in fact, by the proof of Lemma 6 in [9].

Lemma 1. *Let A be a subset of X and $f\colon A \to Y$ be an ε-isometry. Then there is a continuous 5ε-isometry $f_1\colon A \to Y$ such that $\|f_1(x) - f(x)\| \le 2\varepsilon$ for every $x \in A$.*

(The word "open" is redundant in the statement of [9], Lemma 6.)

***Proof of Theorem* 1.** By Lemma 1, we can assume that f is continuous. We will use the following statement that follows from [10] (Theorem II).

Proposition 1. *Let $f\colon B_{\ell_2^n} \to B_{\ell_2^n}$ be a continuous map satisfying*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \le \varepsilon.$$

Then there are an absolute constant C and a linear isometry U such that

$$\|f(x) - Ux\| \le C\log(n+1)\varepsilon, \qquad x \in B_{\ell_2^n}.$$

Define $\tilde f\colon B \to B$ by

$$\tilde f(x) = \begin{cases} 0, & x = 0,\\ \|x\| f\left(\dfrac{x}{\|x\|}\right), & \text{otherwise.}\end{cases}$$

Then

$$2\left|\langle \tilde f(x), \tilde f(y)\rangle - \langle x, y\rangle\right| =$$

$$= 2\|x\|\|y\|\left|\left\langle f\left(\frac{x}{\|x\|}\right), f\left(\frac{y}{\|y\|}\right)\right\rangle - \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right| =$$

$$= \|x\|\|y\|\left|\left(\left\|f\left(\frac{x}{\|x\|}\right)\right\|^2 - 2\left\langle f\left(\frac{x}{\|x\|}\right), f\left(\frac{y}{\|y\|}\right)\right\rangle + \left\|f\left(\frac{y}{\|y\|}\right)\right\|^2\right) -\right.$$

$$\left.- \left(\left\|\frac{x}{\|x\|}\right\|^2 - 2\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle + \left\|\frac{y}{\|y\|}\right\|^2\right)\right| =$$

$$= \|x\|\|y\|\left|\left\|f\left(\frac{x}{\|x\|}\right) - f\left(\frac{y}{\|y\|}\right)\right\|^2 - \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right| =$$

$$= \|x\|\|y\| \left| \left\| f\left(\frac{x}{\|x\|}\right) - f\left(\frac{y}{\|y\|}\right) \right\| - \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \right| \times$$

$$\times \left| \left\| f\left(\frac{x}{\|x\|}\right) - f\left(\frac{y}{\|y\|}\right) \right\| + \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \right| \leq 4\varepsilon.$$

The result follows by Proposition 1.

***Sharpness of the estimate in* (2):**

Let n be an even natural number, say $n = 2m$. Following Matouškova [4], identify $\mathbb{R}^n$ with $\mathbb{C}^m$. For $z \in \mathbb{C}$ set $\varphi_\varepsilon(z) = ze^{i\varepsilon \log |z|}$ (if $z = 0$ then $\varphi_\varepsilon(z) = 0$). Now set

$$f_\varepsilon(z_1, \ldots, z_m) = (\varphi_\varepsilon(z_1), \ldots, \varphi_\varepsilon(z_m)).$$

Then f_ε is an ε-isometry of $B_{\ell_2^n}$ onto $B_{\ell_2^n}$ and, for every $0 < t \leq 1$, $f_\varepsilon(tS_{\ell_2^n}) = tS_{\ell_2^n}$.

Kalton in the proof of [3] (Proposition 2.1) actually proved that if $0 < \varepsilon < \dfrac{2\pi}{\log m}$ then, for any affine isometry U of ℓ_2^n, we have

$$\max_{x \in S_{\ell_2^n}} \|f_\varepsilon(x) - U(x)\| \geq \sin\left(\frac{1}{4}\varepsilon \log m\right).$$

***Proof of Theorem* 2.** We set $c = 1/210$ and fix an ε-isometry f, satisfying the conditions of Theorem 2.

Lemma 2. *There is a permutation π of $\{1, \ldots, n\}$ such that, for every $i \leq n$,*

$$|f_{\pi(i)}(e_i) - f_{\pi(i)}(-e_i)| \geq 2 - \varepsilon$$

and

$$|f_k(e_i) - f_k(-e_i)| \leq 1 + \varepsilon, \qquad k \neq \pi(i).$$

Proof. Set, for each $i \leq n$,

$$A_i := \{j \leq n : |f_j(e_i) - f_j(-e_i)| > 1 + \varepsilon\}.$$

By (1), we have, for every $i \leq n$,

$$\|f(e_i) - f(-e_i)\| \geq 2 - \varepsilon > 1 + \varepsilon. \tag{4}$$

Thus A_i is nonempty.

Now we show that $A_i \cap A_k = \varnothing$ for $i \neq k$, which implies that all A_i are disjoint singletons. Assume to the contrary that there is $j \in A_i \cap A_k$, i.e.,

$$|f_j(e_i) - f_j(-e_i)| > 1 + \varepsilon \qquad \text{and} \qquad |f_j(e_k) - f_j(-e_k)| > 1 + \varepsilon.$$

Let $\theta_i, \theta_k \in \{-1, 1\}$ be such that

$$|f_j(e_i) - f_j(-e_i)| + |f_j(e_k) - f_j(-e_k)| = f_j(\theta_i e_i) - f_j(-\theta_i e_i) + f_j(\theta_k e_k) - f_j(-\theta_k e_k).$$

Then $f_j(\theta_i e_i) - f_j(-\theta_i e_i) + f_j(\theta_k e_k) - f_j(-\theta_k e_k) > 2(1 + \varepsilon)$.

On the other hand, since $\|e_i \pm e_k\| = 1$, we have $\|f(\theta_i e_i) - f(\theta_k e_k)\| \le 1+\varepsilon$ for every $\theta_i, \theta_k \in \{-1, 1\}$. In particular,

$$|f_j(\theta_i e_i) - f_j(-\theta_k e_k)| + |f_j(-\theta_i e_i) - f_j(\theta_k e_k)| \le 2(1+\varepsilon);$$

a contradiction.

It follows that if $\{j\} \ne A_i$, then $|f_j(e_i) - f_j(-e_i)| \le 1+\varepsilon$ by the definition of A_i; and if $\{j\} = A_i$, then $|f_j(e_i) - f_j(-e_i)| \ge 2-\varepsilon$ by (4). So, the desired permutation π is defined by $\pi(i) \in A_i$.

Lemma 3. *For every* $i \le n$, $|f_{\pi(i)}(\pm e_i)| \ge 1-\varepsilon$, $f_{\pi(i)}(e_i)$ *and* $f_{\pi(i)}(-e_i)$ *have opposite signs.*

Proof. By the definition of π, we get

$$|f_{\pi(i)}(e_i)| \ge 2 - \varepsilon - |f_{\pi(i)}(-e_i)| \ge 1-\varepsilon.$$

The final comment follows directly from $|f_{\pi(i)}(e_i) - f_{\pi(i)}(-e_i)| \ge 2-\varepsilon$ and $|f_{\pi(i)}(\pm e_i)| \le 1$.

It follows that there is a map $s\colon \{1,\ldots,n\} \to \{-1,1\}$ such that

$$\operatorname{sgn} f_{\pi(i)}(e_i) = s(i) \qquad \text{and} \qquad \operatorname{sgn} f_{\pi(i)}(-e_i) = -s(i), \qquad i \le n. \tag{5}$$

Denote by H_i the hyperplane $\{x\colon x_i = 1\}$ and by H_{-i} the hyperplane $\{x\colon x_i = -1\}$. Denote by S_i and S_{-i} the following $(n-1)$-dimensional faces $S_i = S \cap H_i$, $S_{-i} = S \cap H_{-i}$.

Lemma 4. *For every* $i \le n$, $f(e_i) \in S_{s(i)\pi(i)}$ *and* $f(-e_i) \in S_{-s(i)\pi(i)}$. *Moreover, for any* $k \ne i$ $f(e_i), f(-e_i) \notin S_{\pm\pi(k)}$.

Proof. Let $f(e_i) \in S_j$ and $f(-e_i) \in S_k$ for some $-n \le j, k \le n$.

Assume to the contrary that $j \ne \pm\pi(i)$. Then $j = s\pi(l)$ for some $s = \pm 1$ and $l \ne i$. Assume without loss of generality $s = s(l) = 1$. By Lemma 3, $f_j(-e_l) \le -1+\varepsilon$. Hence, $|f_j(-e_l) - f_j(e_i)| \ge 2-\varepsilon$, while $\|e_l + e_i\| = 1$ is a contradiction. Thus, $j \in \{-\pi(i), \pi(i)\}$. Similarly, $k \in \{-\pi(i), \pi(i)\}$.

By (5), the result follows.

It follows that $f_{\pi(i)}(e_i) = s(i)$ and $f_{\pi(i)}(-e_i) = -s(i)$.

Lemma 5. *Let* $x \in S_{\pm i}$. *Then* $f(x) \notin S_{\mp s(i)\pi(i)}$.

Proof. Assume without loss of generality $x \in S_i$. Then $\|x - e_i\| \le 1$, $\operatorname{dist}\big(f(e_i), S_{-s(i)\pi(i)}\big) = \operatorname{dist}\big(S_{s(i)\pi(i)}, S_{-s(i)\pi(i)}\big) = 2$ and the result follows.

Lemma 6. *For every* $i \le n$, $\|f(e_i) - s(i)e_{\pi(i)}\| \le \varepsilon$ *and* $\|f(-e_i) + s(i)e_{\pi(i)}\| \le \varepsilon$.

Proof. We prove only the first estimation; the second one can be proved following the same path. Assume to the contrary that $\|f(e_i) - s(i)e_{\pi(i)}\| > \varepsilon$. Then there is $k \ne i$ such that $|f_{\pi(k)}(e_i)| > \varepsilon$. It follows by Lemma 4 that either

$$|f_{\pi(k)}(e_i) - f_{\pi(k)}(e_k)| > 1+\varepsilon \quad \text{or} \quad |f_{\pi(k)}(e_i) - f_{\pi(k)}(-e_k)| > 1+\varepsilon.$$

But this contradicts $\|e_i - e_k\| = 1$.

Lemma 7. *For every* $-n \le i \le n$, *there exists a linear isometry* $U_i\colon \ell_\infty^n \to \ell_\infty^n$ *such that* $U_i e_i = s(i)e_{\pi(i)}$ *and*

$$\|f(x) - U_i x\| \le 100\varepsilon, \qquad x \in S_i.$$

Proof. We will use the following statement (see [11], Proposition 2).

Proposition 2. *Let $0 < \varepsilon < 1/6$. Let $f\colon B_{\ell_\infty^n} \to B_{\ell_\infty^n}$ be a continuous ε-isometry with $f(0) = 0$. Then there is a unique linear isometry U of ℓ_∞^n such that*

$$\|f(x) - U_f x\| \le 2\varepsilon, \qquad x \in \overline{B}(1-2\varepsilon).$$

Assume without loss of generality $i \ge 1$. As $\|f(e_i) - s(i)e_{\pi(i)}\| \le \varepsilon$, $f\left(B_{H_i}(1-3\varepsilon)\right) \subset \subset B_{H_{s(i)\pi(i)}}(f(e_i), 1-2\varepsilon) \subset S_{s(i)\pi(i)}$. Denote by V the linear isometry of ℓ_∞^n defined by

$$V\left(\sum_{k\notin\{i,\pi(i)\}}(x_k e_k) + x_i e_i + x_{\pi(i)}e_{\pi(i)}\right) =$$

$$= \sum_{k\notin\{i,s(i)\pi(i)\}}(x_k e_k) + s(i)x_{\pi(i)}e_i + x_i e_{\pi(i)}.$$

Denote $F_1 = V \circ f$. Then $\|F_1(e_i) - e_i\| \le \varepsilon$ and F_1 is an ε-isometry such that

$$F_1\left(B_{H_i}(1-3\varepsilon)\right) \subset B_{H_i}(F_1(e_i), 1-2\varepsilon) \subset S_i.$$

We consider now H_i as an $(n-1)$-dimensional normed space with the origin $\varnothing_{H_i} = e_i$. Then $B_{H_i} = S_i$ and $\|F_1(0)\| \le \varepsilon$.

Define a map $F_2\colon B_{H_i} \to B_{H_i}$ by $F_2(x) = F_1((1-3\varepsilon)x) - F_1(0)$. Then F_2 is an 7ε-isometry with $F_2(0) = 0$. By Lemma 1, there is a continuous 35ε-isometry $F_3\colon B_{H_i} \to B_{H_i}$ such that $\|F_3(x) - F_2(x)\| \le 14\varepsilon$ for every $x \in B_{H_i}$. By the choice of ε, F_3 satisfies the conditions of Proposition 2. Hence there is a linear (in H_i) isometry U such that

$$\|F_3(x) - Ux\| \le 70\varepsilon, \qquad x \in B_{H_i}(1-2\varepsilon).$$

Since $F_1(x) = F_2\left(\dfrac{x}{1-3\varepsilon}\right) + F_1(0)$, we have, on $B_{H_i}(1-5\varepsilon)$,

$$\|F_1(x) - Ux\| \le$$

$$\le \left\|F_2\left(\frac{x}{1-3\varepsilon}\right) - F_3\left(\frac{x}{1-3\varepsilon}\right)\right\| + \left\|F_3\left(\frac{x}{1-3\varepsilon}\right) - U\left(\frac{x}{1-3\varepsilon}\right)\right\| +$$

$$+\frac{3\varepsilon}{1-3\varepsilon}\|Ux\| + \|F_1(0)\| < 89\varepsilon.$$

Let $x \in B_{H_i} \setminus B_{H_i}(1-5\varepsilon)$. Then

$$\|F_1(x) - Ux\| \le$$

$$\le \left\|F_1(x) - F_1\left(\frac{(1-5\varepsilon)x}{\|x\|}\right)\right\| + \left\|Ux - U\frac{(1-5\varepsilon)x}{\|x\|}\right\| +$$

$$+\left\|F_1\left(\frac{(1-5\varepsilon)x}{\|x\|}\right) - U\frac{(1-5\varepsilon)x}{\|x\|}\right\| <$$

$$< 2(\|x\| - 1 + 5\varepsilon) + 90\varepsilon \le 100\varepsilon.$$

Obviously, U is the restriction on H_i of the linear isometry U' of ℓ_∞^n defined by

$$U'\left(\sum_{k\neq i}(x_k e_k) + x_i e_i\right) = \sum_{k\neq i} U(x_k e_k) + x_i e_i.$$

Thus, $U_i = V^{-1}U'$ is a desired isometry.

Define a linear isometry U by

$$U\left(\sum_i x_i e_i\right) = \sum_i s(i) x_i e_{\pi(i)}.$$

Lemma 8. *For every $-n \le i \le n$, $U_i = U$.*

Proof. Suppose that $U_i \neq U$ for some i. Then there is $j \neq |i|$ such that $U_i e_j \neq U e_j$. Assume without loss of generality $i, j \ge 1$. Then there are $k \notin \{\pi(i), \pi(j)\}$ and $s \in \{-1, 1\}$ such that $U_i e_j = s e_k$. Hence,

$$\|U_i(e_i + e_j) - U_j(e_i + e_j)\| = \|s(i)e_{\pi(i)} + s e_k - U_j e_i - s(j) e_{\pi(j)}\| \ge 1.$$

On the other hand, by Lemma 7

$$\|U_i(e_i + e_j) - U_j(e_i + e_j)\| \le$$

$$\le \|U_i(e_i + e_j) - f(e_i + e_j)\| + \|f(e_i + e_j) - U_j(e_i + e_j)\| \le 200\varepsilon < 1;$$

a contradiction.

Thus, U satisfies (3) with $C = 100$.

Uniqueness: Suppose that U' is another linear isometry of ℓ_∞^n satisfying (3) with $C = 100$. Then

$$\|f(e_i) - U e_i\| \le 100\varepsilon \quad \text{and} \quad \|f(e_i) - U' e_i\| \le 100\varepsilon \quad \text{for every} \quad i \le n.$$

Thus, $\|U' e_i - U e_i\| \le 200\varepsilon < 1$, which means $U' e_i = U e_i$.

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