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M. Boujelben (Univ. de Sfax, Tunisia)

FIRST COHOMOLOGY SPACE OF THE ORTHOSYMPLECTIC LIE SUPERALGEBRA $\mathfrak{osp}(n|2)$ IN THE LIE SUPERALGEBRA OF SUPERPSEUDODIFFERENTIAL OPERATORS

ПРОСТІР ПЕРШОЇ КОГОМОЛОГІЇ ОРТОСИМПЛЕКТИЧНОЇ СУПЕРАЛГЕБРИ ЛІ $\mathfrak{osp}(n|2)$ У СУПЕРАЛГЕБРІ ЛІ СУПЕРПСЕВДОДИФЕРЕНЦІАЛЬНИХ ОПЕРАТОРІВ

We investigate the first cohomology space associated with the embedding of the Lie orthosymplectic superalgebra $\mathfrak{osp}(n|2)$ on the $(1,n)$-dimensional superspace $\mathbb{R}^{1|n}$ in the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}(n)$ (for $n \geq 4$) of superpseudodifferential operators with smooth coefficients. Following Ovsienko and Roger, we give explicit expressions of the basis cocycles. This work is the simplest generalization of a result by Basdouri [*First space cohomology of the orthosymplectic Lie superalgebra in the Lie superalgebra of superpseudodifferential operators*, Algebras and Representation Theory, **16**, 35 – 50 (2013)].

Вивчається простір першої когомології, пов'язаний з вкладенням ортосимплектичної супералгебри Лі $\mathfrak{osp}(n|2)$ на $(1,n)$-вимірному суперпросторі $\mathbb{R}^{1|n}$ у супералгебрі Лі $\mathcal{S}\Psi\mathcal{DO}(n)$ (для $n \geq 4$) суперпсевдодиференціальних операторів з гладкими коефіцієнтами. Наслідуючи Овсієнка та Роджера, ми наводимо точні вирази для базису коциклів. Ця робота є найпростішим узагальненням результату Basdouri [*First space cohomology of the orthosymplectic Lie superalgebra in the Lie superalgebra of superpseudodifferential operators*, Algebras and Representation Theory, **16**, 35 – 50 (2013)].

1. Introduction. The procedure of contraction is opposite to deformation. This procedure is important in physics because it explains, in terms of Lie algebras, why some theories arise as a limit regime of more "exact" theories. Motivated by the need to relate the symmetries underlying Einstein's mechanics and Newtonian mechanics, Inönü and Wigner introduced the concept of contraction, which consists in multiplying the generators of the symmetry by "contraction parameters", such that when these parameters reach some singularity point, one obtains a "different" Lie algebra with the same dimension [13]. A similar procedure had been mentioned previously by Segal [16]. The method has been generalized a few years later by Saletan [17]. Another physical example is the contraction of the de Sitter algebras to the Poincaré algebra, in the limit of large (universe) radius. These examples suggest that deformations are likely to be more useful than contractions in the investigation of fundamental theories [10].

In the 1960, deformation theory of Lie algebras began with the works of Gerstenhaber and, Nijenhuis and Richardson. Recently, multiparameter deformations of Lie (super)algebras and their modules were intensively studied.

To study the formal and polynomial deformations of the natural embedding of the Lie algebra $\mathfrak{vect}(S^1)$ of smooth vector fields on the circle S^1 into the Lie algebra $\Psi\mathcal{DO}(S^1)$ of pseudodifferential operators, Ovsienko and Roger [15] calculate the first cohomology space $\mathrm{H}^1(\mathfrak{vect}(S^1),\ \Psi\mathcal{DO}(S^1))$, where the action is given by the standard embedding. The graded space $\mathrm{Gr}(\Psi\mathcal{DO}(S^1))$ associated with the natural filtration given by order of pseudodifferential operators coincides with $\mathcal{P}$ the Lie – Poisson algebra of symbols of pseudodifferential operators (formal Laurent series in the symbol ξ

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of $\dfrac{\partial}{\partial x}$ with coefficients in the space $C^\infty(S^1)$ of smooth functions on S^1). Since the $\mathrm{Vect}(S^1)$-module $\mathcal{P}$ is isomorphic to a direct sum of $\mathrm{Vect}(S^1)$-modules of tensor densities on S^1, the space $\mathrm{H}^1(\mathrm{Vect}(S^1),\ \mathcal{P})$ can be deduced from the cohomology of $\mathrm{Vect}(S^1)$ with coefficients in tensor densities computed by Feigin and Fuchs [11, 12]. The method of Ovsienko and Roger goes over from the graded space $\mathcal{P}$ to the filtered space $\Psi\mathcal{DO}(S^1)$ using the spectral sequence.

In paper [1, 2], using the same methods as in the paper [15] the authors computed $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO}(1))$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(2))$. The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2); \mathcal{SP}(n))$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2); \mathcal{S}\Psi\mathcal{DO}(n))$ was calculated in [4] for $0 \le n \le 2$ and for $n = 3$ in [3].

In this paper, we restrict ourselves to the cases $n \ge 4$ and we restrict the action to the orthosymplectic Lie (super)algebra $\mathfrak{osp}(n|2)$ and we consider the spaces $\mathcal{SP}(n)$ as $\mathfrak{osp}(n|2)$-modules. We compute the cohomology spaces $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}(n))$ and $\mathrm{H}^1\,(\mathfrak{osp}(n|2), \mathcal{S}\Psi\mathcal{DO}(n))$. We show that these cohomology spaces are nontrivial. These cohomology spaces are closely related to the deformation theory (see, e.g., [6, 7, 9, 10, 14, 15]. These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{osp}(n|2)$-modules. We hope to be able to describe in the future all the deformations of these modules $\mathcal{S}\Psi\mathcal{DO}(n)$.

2. Definitions and notations. ***2.1. The Lie superalgebra of contact vector fields on*** $\mathbb{R}^{1|n}$**.** Let $\mathbb{R}^{1|n}$ be the superspace with coordinates $(x, \theta_1, \ldots, \theta_n)$, where x is an even indeterminate and $\theta_1, \ldots, \theta_n$ are odd indeterminates: $\theta_i\theta_j = -\theta_j\theta_i$. This superspace is equipped with the standard contact structure given by the distribution $D = \langle \overline{\eta}_1, \ldots, \overline{\eta}_n \rangle$ generated by the vector fields $\overline{\eta}_i = \partial_{\theta_i} - \theta_i \partial_x$. That is, the distribution D is the kernel of the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i.$$

Consider the superspace $C^\infty(\mathbb{R}^{1|n})$ which is the space of functions F of the form

$$F = \sum_{1 \le i_1 < \ldots < i_k \le n} f_{i_1,\ldots,i_k}(x)\theta_{i_1} \ldots \theta_{i_k}, \quad \text{where} \quad f_{i_1,\ldots,i_k} \in C^\infty(\mathbb{R}). \tag{2.1}$$

Of course, even (resp., odd) elements in $C^\infty(\mathbb{R}^{1|n})$ are the functions (2.1) for which the summation is only over even (resp., odd) integer k. Denote by $p(F)$ the parity of a homogeneous function F. On $C^\infty(\mathbb{R}^{1|n})$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{p(F)} \sum_{i=1}^{n} \overline{\eta}_i(F)\overline{\eta}_i(G), \tag{2.2}$$

where the superscript $'$ stands for $\dfrac{\partial}{\partial x}$. Consider the superspace $\mathcal{K}(n)$ of contact vector fields on $\mathbb{R}^{1|n}$. That is, $\mathcal{K}(n)$ is the superspace of vector fields on $\mathbb{R}^{1|n}$ preserving the distribution $\langle \overline{\eta}_1, \ldots, \overline{\eta}_n \rangle$:

$$\mathcal{K}(n) = \big\{ X \in \mathrm{Vect}(\mathbb{R}^{1|n}) \mid [X, \overline{\eta}_i] = F_X \overline{\eta}_i \quad \text{for some } F_X \in C^\infty(\mathbb{R}^{1|n}) \big\}.$$

The Lie superalgebra $\mathcal{K}(n)$ is spanned by the vector fields of the form

$$X_F = F\partial_x - \frac{1}{2}(-1)^{p(F)} \sum_{i=1}^{n} \overline{\eta}_i(F)\overline{\eta}_i, \quad \text{where} \quad F \in C^\infty(\mathbb{R}^{1|n}).$$

The vector field X_F has the same parity as F. The bracket in $\mathcal{K}(n)$ can be written as

$$[X_F,\, X_G] = X_{\{F,G\}}.$$

For every contact vector fields X_F, one define a one-parameter family of first-order differential operators on $C^\infty(\mathbb{R}^{1|n})$:

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F', \quad \lambda \in \mathbb{R}.$$

We easily check that

$$[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}] = \mathfrak{L}^\lambda_{X_{\{F,G\}}}.$$

We thus obtain a one-parameter family of $\mathcal{K}(n)$-modules on $C^\infty(\mathbb{R}^{1|n})$ that we denote $\mathfrak{F}^n_\lambda$, the space of all weighted densities on $C^\infty(\mathbb{R}^{1|n})$ of weight λ with respect to α_n:

$$\mathfrak{F}^n_\lambda = \left\{ F\alpha_n^\lambda \mid F \in C^\infty(\mathbb{R}^{1|n}) \right\}.$$

In particular, we have $\mathfrak{F}^0_\lambda = \mathcal{F}_\lambda$. Obviously the adjoint $\mathcal{K}(n)$-module is isomorphic to the space of weighted densities on $C^\infty(\mathbb{R}^{1|n})$ of weight -1.

The orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2)$ can be realized as a subalgebra of $\mathcal{K}(n)$:

$$\mathfrak{osp}(n|2) = \mathrm{Span}(X_1,\, X_x,\, X_{x^2},\, X_{x\theta_i},\, X_{\theta_i},\, X_{\theta_i\theta_j}), \quad 1 \le i,j \le n.$$

We easily see that $\mathfrak{osp}(n-1|2)$ is a subalgebra of $\mathfrak{osp}(n|2)$:

$$\mathfrak{osp}(n-1|2) = \big\{ X_F \in \mathfrak{osp}(n|2) \mid \partial_{\theta_n} F = 0 \big\}.$$

Note also that, for any $i \in \{1,2,\ldots,n-1\}$, $\mathfrak{osp}(n-1|2)$ is isomorphic to

$$\mathfrak{osp}(n-1|2)_i = \big\{ X_F \in \mathfrak{osp}(n|2) \mid \partial_{\theta_i} F = 0 \big\}.$$

Therefore, the spaces of weighted densities $\mathfrak{F}^n_\lambda$ are also $\mathfrak{osp}(n-1|2)$-modules. In [5], it was proved that, as $\mathfrak{osp}(n-1|2)$-modules, we have

$$\mathfrak{F}^n_\lambda \simeq \mathfrak{F}^{n-1}_\lambda \oplus \Pi\left(\mathfrak{F}^{n-1}_{\lambda+\frac{1}{2}}\right), \tag{2.3}$$

where Π is the change of parity operator.

As $\mathfrak{osp}(n-1|2)_i$-isomorphism

$$\mathfrak{osp}(n|2) \simeq \mathfrak{osp}(n-1|2)_i \oplus \Pi(\mathcal{H}_i),$$

where $\mathcal{H}_i$ is the subspace of $\mathfrak{F}^i_{-\frac{1}{2}}$ spanned by $\{\theta_i\alpha_1^{-\frac{1}{2}},\, x\alpha_1^{-\frac{1}{2}},\, \alpha_1^{-\frac{1}{2}}\}$, where $i = 1,2,\ldots,n-1$. To be more precise, any element X_F is decomposed into $X_F = X_{F_i} + X_{F_{n-i}\theta_{n-i}}$ where $\partial_{\theta_{n-i}} F_i = \partial_{\theta_{n-i}} F_{n-i} = 0$, and then $X_{F_i} \in \mathfrak{osp}(n-1|2)_i$ and $X_{F_{n-i}\theta_{n-i}}$ can be identified to $\Pi(F_{n-i}\alpha_1^{-\frac{1}{2}}) \in \Pi(\mathcal{H}_i)$. Moreover, we can see easily that

$$[\mathfrak{osp}(n-1|2)_i, \Pi(\mathcal{H}_i)] \subset \Pi(\mathcal{H}_i) \quad \text{and} \quad [\Pi(\mathcal{H}_i), \Pi(\mathcal{H}_i)] \subset \mathfrak{osp}(n-1|2)_i.$$

2.2. Superpseudodifferential operators on $\mathbb{R}^{1|n}$. The superspace of the supercommutative algebra of superpseudodifferential symbols on $\mathbb{R}^{1|n}$ with its natural multiplication is spanned by the series

$$\mathcal{SP}(n) =$$

$$= \left\{ F = \sum_{k=-M}^{\infty} \sum_{\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)} a_{k,\varepsilon}(x,\theta)\xi^{-k}\bar{\theta}_1^{\varepsilon_1}\ldots\bar{\theta}_n^{\varepsilon_n} : \quad a_{k,\varepsilon} \in C^\infty(\mathbb{R}^{1|n});\ \varepsilon_i = 0,\,1;\ M \in \mathbb{N} \right\},$$

where ξ corresponds to ∂_x and $\bar{\theta}_i$ corresponds to ∂_{θ_i} $(p(\bar{\theta}_i) = 1)$.

The space $\mathcal{SP}(n)$ has a structure of the Poisson – Lie superalgebra given by the following bracket:

$$\{F,\ G\} = \frac{\partial(F)}{\partial\xi}\frac{\partial(G)}{\partial x} - \frac{\partial(F)}{\partial x}\frac{\partial(G)}{\partial\xi} - (-1)^{p(F)}\sum_{i=1}^{n}\Big(\frac{\partial(F)}{\partial\theta_i}\frac{\partial(G)}{\partial\bar{\theta}_i} + \frac{\partial(F)}{\partial\bar{\theta}_i}\frac{\partial(G)}{\partial\theta_i}\Big).$$

It endows $\mathcal{SP}(n)$ with a Lie superalgebra structure (still denoted $\mathcal{SP}(n)$).

The space $\mathcal{SP}(n)$ is $\mathbb{Z}$-graded where the degrees of x and θ are equal to 0 and the degrees of ξ and $\overline{\theta}$ are equal to 1. A homogeneous element of degree m has the following form:

$$A_m = F_0\xi^m + \sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n} F_{i_1\ldots i_k}\xi^{m-k}\overline{\theta}_{i_1}\ldots\overline{\theta}_{i_k}, \text{ where } F_0,\ F_{i_1\ldots i_k} \in C^\infty(\mathbb{R}^{1|n}).$$

We will denote $\mathcal{SP}_m(n)$ the space of homogeneous elements of degree $-m$.

This definition endows the space $\mathcal{SP}(n)$ with a $\mathbb{Z}$-grading:

$$\mathcal{SP}(n) = \widetilde{\bigoplus_{m\in\mathbb{Z}}}\mathcal{SP}_m(n),$$

where $\widetilde{\bigoplus_{m\in\mathbb{Z}}} = (\bigoplus_{m<0})\bigoplus\prod_{m\ge 0}$ and

$$\mathcal{SP}_m(n) =$$

$$= \Big\{F\xi^{-m} + G_1\xi^{-m-1}\bar{\theta}_1 + G_2\xi^{-m-1}\bar{\theta}_2 + \ldots + H_{1,2}\xi^{-m-2}\bar{\theta}_1\bar{\theta}_2 + \ldots \mid F, G_i, H_{i,j} \in C^\infty(\mathbb{R}^{1|n})\Big\}$$

is the homogeneous subspace of degree $-m$.

The associative superalgebra of superpseudodifferential operators $\mathcal{S}\Psi\mathcal{DO}(n)$ on $\mathbb{R}^{1|n}$ has the same underlying vector space as $\mathcal{SP}(n)$ by the multiplication is now defined by the following rule:

$$F \circ G = \sum_{k\ge 0,\,\nu_i=0,\,1} \frac{(-1)^{\nu_i(p(F)+1)}}{k!}(\partial_\xi^k\partial_{\bar{\theta}_i}^{\nu_i}F)(\partial_x^k\partial_{\theta_i}^{\nu_i}G).$$

This composition rule induces the supercommutator defined by

$$[F,\ G] = F \circ G - (-1)^{p(F)p(G)}G \circ F.$$

Of course, the case $n = 0$ corresponds to the classical setting: $\mathcal{K}(0) = \mathrm{Vect}(\mathbb{R})$ and the corresponding orthosymplectic Lie algebra $\mathfrak{osp}(0|2)$ is nothing but the classical Lie algebra $\mathfrak{sl}(2)$, which is isomorphic to the Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$ generated by

$$\mathfrak{sl}(2) = \mathrm{Span}\left(\frac{d}{dx}, \, x\frac{d}{dx}, \, x^2\frac{d}{dx}\right)$$

and $\mathfrak{F}^0_\lambda$ are the classical λ-densities, usually denoted

$$\mathcal{F}_\lambda = \left\{ f(dx)^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}.$$

$\mathcal{SP}(0)$ is the classical spaces of symbols, usually denoted

$$\mathcal{P} = \left\{ F(x,\xi), \, F(x,\xi) = \sum_{k\in\mathbb{Z}} f_k(x)\xi^k \right\},$$

and $\mathcal{S}\Psi\mathcal{DO}(0)$ is the classical associative algebra of pseudodifferential operators, usually denoted $\Psi\mathcal{DO}$.

2.3. The structure of $\mathcal{SP}(n)$ as a $\mathfrak{osp}(n|2)$-module. The natural embedding of $\mathfrak{osp}(n|2)$ into $\mathcal{SP}(n)$ defined by

$$\pi(X_F) = F\xi + \frac{(-1)^{p(F)+1}}{2} \sum_{i=1}^{n} \bar{\eta}_i(F)\bar{\zeta}_i, \quad \text{where} \quad \bar{\zeta}_i = \bar{\theta}_i - \theta_i\xi,$$

and $\pi(X_F) = F\xi$ for $n = 0$, induces an $\mathfrak{osp}(n|2)$-module structure on $\mathcal{SP}(n)$. Setting $\deg x = \deg\theta_i = 0, \ \deg\xi = \deg\bar{\theta}_i = 1$ for all i.

Each element of $\mathcal{S}\Psi\mathcal{DO}(m)$ can be expressed as

$$A = \sum_{k\in\mathbb{Z}} (F_k + G^1_k\xi^{-1}\bar{\theta}_1 + \ldots + H^{1,2}_k\xi^{-2}\bar{\theta}_1\bar{\theta}_2 + \ldots)\xi^{-k},$$

where $F_k, \, G^i_k, \, H^{i,j}_k \in C^\infty(\mathbb{R}^{1|n})$. We define the *order* of A to be

$$\mathrm{ord}(A) = \sup\{k \mid F_k \neq 0 \text{ or } G^i_k \neq 0 \text{ or } H^{i,j}_k \neq 0\}.$$

This definition of order equips $\mathcal{S}\Psi\mathcal{DO}(n)$ with a decreasing filtration as follows: set

$$\mathbf{F}_m = \{A \in \mathcal{S}\Psi\mathcal{DO}(n), \, \mathrm{ord}(A) \leq -m\},$$

where $m \in \mathbb{Z}$. So, one has

$$\ldots \subset \mathbf{F}_{m+1} \subset \mathbf{F}_m \subset \ldots.$$

This filtration is compatible with the multiplication and the Poisson bracket, that is, for $A \in \mathbf{F}_p$ and $B \in \mathbf{F}_q$, one has $A \circ B \in \mathbf{F}_{p+q}$ and $\{A, B\} \in \mathbf{F}_{p+q-1}$. This filtration makes $\mathcal{S}\Psi\mathcal{DO}(n)$ an associative filtered superalgebra. Moreover, this filtration is compatible with the natural $\mathfrak{osp}(n|2)$-action on $\mathcal{S}\Psi\mathcal{DO}(n)$. Indeed, if $X_F \in \mathfrak{osp}(n|2)$ and $A \in \mathbf{F}_m$, then

$$X_F.A = [X_F, A] \in \mathbf{F}_m.$$

The induced $\mathfrak{osp}(n|2)$-module structure on the quotient $\mathbf{F}_m/\mathbf{F}_{m+1}$ is isomorphic to that the $\mathfrak{osp}(n|2)$-module $\mathcal{SP}_m(n)$. Therefore,

$$\mathcal{SP}(n) \simeq \widetilde{\bigoplus_{m\in\mathbb{Z}}} \mathbf{F}_m/\mathbf{F}_{m+1}.$$

Therefore, $\mathcal{SP}(n)$ is $\mathfrak{osp}(n-1|2)_i$-module.

For $i \in \{1,2,\ldots,n-1\}$, let $\mathfrak{F}_\lambda^{n-1,i}$ be the $\mathfrak{osp}(n-1|2)_i$-module of weighted densities of weight λ on $\mathbb{R}^{1|n-1}$.

3. Theory of cohomology. Let us first recall some fundamental concepts from cohomology theory (see, e.g., [12]).

Let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be a Lie superalgebra acting on a superspace $V = V_{\bar 0} \oplus V_{\bar 1}$. The space of k-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^k(\mathfrak{g}, V) := \mathrm{Hom}(\Lambda^k\mathfrak{g}; V).$$

The *coboundary operator* $\delta_k : C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{k-1} = 0$. The kernel of δ_k, denoted $Z^k(\mathfrak{g}, V)$, is the space of k-*cocycles*, among them, the elements in the range of δ_{k-1} are called k-*coboundaries*. We denote $B^k(\mathfrak{g}, V)$ the space of k-coboundaries. By definition, the kth cohomology space is the quotient space

$$\mathrm{H}^k(\mathfrak{g}, V) = Z^k(\mathfrak{g}, V)/B^k(\mathfrak{g}, V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 and 1: for $v \in C^0(\mathfrak{g}, V) = V$, $\delta v(x) := (-1)^{p(x)p(v)} x \cdot v$, for $\Upsilon \in C^1(\mathfrak{g}, V)$,

$$\delta(\Upsilon)(x,\, y) := (-1)^{p(x)p(\Upsilon)} x \cdot \Upsilon(y) - (-1)^{p(y)(p(x)+p(\Upsilon))} y \cdot \Upsilon(x) - \Upsilon([x,\, y]).$$

The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2); \mathcal{SP}(n))$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2); \mathcal{S\Psi DO}(n))$ for $0 \le n \le 3$ was calculated in [4] for $0 \le n \le 2$ and for $n = 3$ in [3].

In this paper, we study the differential cohomology spaces $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}(n))$ and $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{S\Psi DO}(n))$ for $n \ge 4$.

We recall that the space $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{S\Psi DO}(n))$ is equal to the space $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{P})$, and this two spaces are spanned by the same generators.

Proposition 3.1 [2]. 1. *As a* $\mathfrak{osp}(n-1|2)_i$*-module,* $i \in \{1,2,\ldots,n-1\}$, *we have*

$$\mathcal{SP}_m(n) \simeq \mathfrak{F}^n_m \oplus \Pi(\mathfrak{F}^n_{m+\frac12} \oplus \mathfrak{F}^n_{m+\frac12}) \oplus \mathfrak{F}^n_{m+1} \quad \textit{for} \ \ m = 0, -1.$$

2. *For* $m \neq 0, -1$:

a) *the following subspace of* $\mathcal{SP}_m(n)$:

$$\mathcal{SP}_{m,\ i}(n) =$$

$$= \left\{ B_F^{(m,i)} = F\theta_{n-i}\bar\theta_{n-i}\xi^{-m-1} + \theta_{n-i}\left(\bar\eta_{n-i} - \frac{1}{2}\bar\eta_i\right)(F)\bar\zeta_i\bar\zeta_{n-i}\xi^{-m-2} \mid F \in C^\infty(\mathbb{R}^{1|n-1}) \right\}$$

is a $\mathfrak{osp}(n-1|2)_i$*- module,* $i = 1,2,\ldots,n-1$, *isomorphic to* $\mathfrak{F}^n_{m+1}$;

b) *as a* $\mathfrak{osp}(n-1|2)_i$*-module we get*

$$\mathcal{SP}_m(n)/\mathcal{SP}_{m,\ i}(n) \simeq \mathfrak{F}^n_m \oplus \Pi(\mathfrak{F}^n_{m+\frac12} \oplus \mathfrak{F}^n_{m+\frac12}), \quad i = 1,2,\ldots,n-1.$$

To prove Proposition 3.1, we need the following result (see [2]).

The space $\mathrm{H}^1(\mathfrak{osp}(n|2),\ \mathcal{SP}(n))$ inherits the grading (2.2) of $\mathcal{SP}(n)$, so it suffices to compute it in each degree.

4. The space $\mathbf{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2);\mathcal{SP}(n))$ for $n \geq 4$. In this section, we will compute the first differential cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2);\mathcal{SP}(n))$ for $n \geq 4$. Our main result is the following theorem.

Theorem 4.1.

$$\mathrm{H}^1(\mathfrak{osp}(n|2),\mathcal{SP}(n)) \simeq \mathbb{R}^4.$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(n|2),\mathcal{SP}(n))$ *are spanned by the cohomology classes of the* 1-*cocycles* $\Lambda_1,\ \Lambda_2,\ \Lambda_3$ *and* Λ_4:

$$\Lambda_1(X_F) = F',$$

$$\Lambda_2(X_F) = F'\xi^{-1}\bar{\zeta}_1\ldots\bar{\zeta}_n,$$

$$\Lambda_3(X_F) = \Big(\bar{\eta}_1(F')\bar{\zeta}_1 + \ldots + \bar{\eta}_n(F')\bar{\zeta}_n\Big)\xi^{-1},$$

$$\Lambda_4(X_F) = F''\xi^{-2}\bar{\zeta}_1\ldots\bar{\zeta}_n.$$

We know that any element $\Upsilon \in Z^1(\mathfrak{osp}(n|2),\mathcal{SP}_m(n))$ is decomposed into $\Upsilon = \Upsilon' + \Upsilon''$ where $\Upsilon' \in Z^1(\mathfrak{osp}(n-1|2)_i,\mathcal{SP}_m(n))$ and $\Upsilon'' \in \mathrm{Hom}(\Pi(\mathcal{H}_i),\mathcal{SP}_m(n))$.

To prove the Theorem 4.1 we need first to proof the following lemma and propositions.

The first cohomology space $\mathrm{H}^1(\mathfrak{osp}(n|2),\mathfrak{F}^n_\lambda)$ was computed in [8]. The result is the following.

Theorem 4.2. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2);\mathfrak{F}^n_\lambda)$ *has the following structure*:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2);\mathfrak{F}^n_\lambda) \simeq \begin{cases} \mathbb{R}^2, & \textit{if} \quad n=2 \ \textit{and} \ \lambda=0, \\ \mathbb{R}, & \textit{if} \quad \begin{cases} n=0 \ \textit{and} \ \lambda = 0,\ 1, \\ n=1 \ \textit{and} \ \lambda=0,\ \dfrac{1}{2}, \\ n\geq 3 \ \textit{and} \ \lambda=0, \end{cases} \\ 0 & \textit{otherwise}. \end{cases}$$

Moreover, basis for nontrivial cohomology spaces are given in the following table:

$(n,\ \lambda)$	1-cocycles
$(n,\ 0)$	$\Upsilon^n_\lambda(X_F) = F'$
$(0,\ 1)$	$\Upsilon^0_1(X_F) = F''dx^1$
$\left(1,\ \dfrac{1}{2}\right)$	$\Upsilon^1_{\frac{1}{2}}(X_F) = \bar{\eta}_1(F')\alpha_1^{\frac{1}{2}}$
$(2,\ 0)$	$\Lambda^2_0(X_F) = \bar{\eta}_1\bar{\eta}_2(F)$

Proposition 4.1. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2)_i;\mathfrak{F}^n_\lambda)$ *has the following structure*:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2)_i;\mathfrak{F}^n_\lambda) \simeq \begin{cases} \mathbb{R}, & \textit{if} \ \lambda = 0, \\ \mathbb{R}, & \textit{if} \ \lambda = -\dfrac{1}{2}, \\ 0 & \textit{otherwise}. \end{cases}$$

Moreover, basis for nontrivial cohomology spaces are given in the following:

$$C_1(X_F) = F', \qquad \text{if} \quad \lambda = 0,$$
$$C_2(X_F) = F'\theta_{n-i}, \quad \text{if} \quad \lambda = -\frac{1}{2}. \tag{4.1}$$

Proof. From the isomorphism (2.3), we have

$$\mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathfrak{F}^n_\lambda) \simeq \mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathfrak{F}^{n-1}_\lambda) \oplus \mathrm{H}^1_{\text{diff}}\left(\mathfrak{osp}(n-1|2)_i; \Pi\left(\mathfrak{F}^{n-1}_{\lambda+\frac{1}{2}}\right)\right).$$

By using the Theorem 4.2, we get

$$\mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathfrak{F}^{n-1}_\lambda) \simeq \mathbb{R}, \ \text{ if } \ \lambda = 0,$$

and

$$\mathrm{H}^1_{\text{diff}}\left(\mathfrak{osp}(n-1|2)_i; \Pi\left(\mathfrak{F}^{n-1}_{\lambda+\frac{1}{2}}\right)\right) \simeq \mathbb{R}, \ \text{ if } \ \lambda = -\frac{1}{2}.$$

Proposition 4.1 is proved.

The proof of Theorem 4.1 need the following lemma.

Lemma 4.1. *The* 1*-cocycle* $\Upsilon \in \mathrm{Z}^1(\mathfrak{osp}(n|2), \mathcal{SP}_m(n))$, $m \in \mathbb{Z}$, *is a coboundary if and only if* $\Upsilon_{|_{\mathfrak{osp}(n-1|2)_i}}$, $1 \le i \le n$, *is a coboundary.*

Proof. It is easy to see that if Υ is a couboundary for $\mathfrak{osp}(n|2)$ then $\Upsilon_{|_{\mathfrak{osp}(n-1|2)_i}}$ is a coboundary over $\Upsilon_{|_{\mathfrak{osp}(n-1|2)_i}}$, $1 \le i \le n$. Now assume that $\Upsilon_{|_{\mathfrak{osp}(n-1|2)_i}}$, $1 \le i \le n$, is a coboundary over $\Upsilon_{|_{\mathfrak{osp}(n-1|2)_i}}$, $1 \le i \le n$, that is, there exists $A \in \mathcal{SP}_m(n)$ such that, for all $X_{F_i} \in \mathfrak{osp}(n-1|2)_i$,

$$\Upsilon(X_{F_i}) = \{X_{F_i}\ ,\ A\}.$$

Using the condition of a 1-cocycle, we have

$$\Upsilon(X_{\theta_i\theta_j}) = \{X_{\theta_i\theta_j}\ ,\ A\}.$$

We prove that $\Upsilon(X_F) = \{X_F, A\}$ for any $X_F \in \mathfrak{osp}(n|2)$, and, therefore, Υ is a coboundary of $\mathfrak{osp}(n|2)$.

***4.1. Proof of Theorem* 4.1.** According to Lemma 4.1, the restriction of any nontrivial 1-cocycle of $\mathfrak{osp}(n|2)$ with coefficients in $\mathcal{SP}_m(n)$ to $\mathfrak{osp}(n-1|2)_i$ is a nontrivial 1-cocycle.

We see that if $m \neq 0, -1$, and by Lemma 4.1, the corresponding cohomology $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}_m(n))$ vanishes.

If $m \in \{0, -1\}$, from the Proposition 3.1, we have

$$\mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathcal{SP}_m(n)) \simeq \mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathfrak{F}^n_m) \oplus \mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \Pi(\mathfrak{F}^n_{m+\frac{1}{2}}) \oplus$$
$$\oplus\, \mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \Pi(\mathfrak{F}^n_{m+\frac{1}{2}}) \oplus \mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathfrak{F}^n_{m+1}).$$

By using the Proposition 4.1, we obtain

$$\mathrm{H}^1_{\text{diff}}(\mathfrak{osp}(n-1|2)_i; \mathcal{SP}_m(n)) \simeq \begin{cases} \mathbb{R}^3, & \text{if } \ m = -1, \\ \mathbb{R}, & \text{if } \ m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Case 1: $m=-1$, the space $\mathrm{H}^1(\mathfrak{osp}(n-1|2)_i, \mathcal{SP}_{-1}(n))$ is spanned by the following 1-cocycles:

$$\Phi^i_1(X_F)=\psi^i_{-1,1}(C_1(X_F)),$$

$$\Phi^i_2(X_F)=\psi^i_{-1,\frac{1}{2}}(\Pi(C_2(X_F))),$$

$$\Phi^i_3(X_F)=\widetilde{\psi}^i_{-1,\frac{1}{2}}(\Pi(C_2(X_F))).$$

Case 2: $m=0$, the space $\mathrm{H}^1(\mathfrak{osp}(n-1|2)_i, \mathcal{SP}_{0}(n))$ is spanned by the following 1-cocycle:

$$\Phi^i_4(X_F)=\psi^i_{0,0}(C_1(X_F)),$$

where the cocycles C_1 and C_2 are defined by the formulae (4.1) and $\psi^i_{m,\,j}$, $\widetilde{\psi}^i_{m,\,j}$ are as in [2].

Now, any a nontrivial 1-cocycle of $\mathfrak{osp}(n|2)$ with coefficients in $\mathcal{SP}_m(n)$ can be decomposed as $\Upsilon=(\Upsilon',\ \Upsilon'')$ and

$$\Upsilon' : \mathfrak{osp}(n-1|2)_i \longrightarrow \mathcal{SP}_m(n),$$

$$\Upsilon'' : \Pi(\mathcal{H}_i) \longrightarrow \mathcal{SP}_m(n),$$

where Υ', Υ'' are linear maps.

The space $\mathrm{H}^1(\mathfrak{osp}(n-1|2)_i, \mathcal{SP}_n(2)), i=1,\ 2,\ldots,n,$ determines the linear maps Υ'. Then $\Upsilon'=\Phi^i$. More precisely, we get:

case 1: $m=0,\ \Upsilon'=\alpha_1\Phi^i_4,$

case 2: $m=-1,\ \Upsilon'=\alpha_2\Phi^i_1+\alpha_3\Phi^i_2+\alpha_4\Phi^i_3$, where the coefficients α_k are constants.

In each case, the 1-cocycle conditions determines Υ''. We obtain, for $m=0,\ \Upsilon_0=\alpha_1\Lambda_1$ and $m=-1,\ \Upsilon_{-1}=\alpha_2\Lambda_2+\alpha_3\Lambda_3+\alpha_4\Lambda_4$.

Thus, the space $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}_0(n))$ is spanned by the nontrivial cocycle Λ_1 and the space $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}_{-1}(n))$ is generated by the nontrivial cocycles: Λ_2, Λ_3 and Λ_4.

Theorem 4.1 is proved.

5. Cohomology of $\mathfrak{osp}(n|2)$ in $\mathcal{S}\Psi\mathcal{DO}(n)$. ***5.1. The spectral sequence for a filtered module over a Lie superalgebra*** **[15].** The reader should refer to [15], for details on homological algebra used to construct spectral sequences. We will merely quote the results for a filtered module M with decreasing filtration $\{M_n\}_{n\in\mathbb{Z}}$ over a Lie (super)algebra $\mathfrak{g}$ so that $M_{n+1}\subset M_n$, $\cup_{n\in\mathbb{Z}}M_n=M$ and $\mathfrak{g}M_n\subset M_n$.

Consider the natural filtration induced on the space of cochains by setting:

$$F^n(C^*(\mathfrak{g},\ M))=C^*(\mathfrak{g},\ M_n),$$

then we have

$$dF^n(C^*(\mathfrak{g},\ M))\subset F^n(C^*(\mathfrak{g},\ M))\ \ \text{(i.e., the filtration is preserved by } d\text{)},$$

$$F^{n+1}(C^*(\mathfrak{g},\ M))\subset F^n(C^*(\mathfrak{g},\ M))\ \text{(i.e., the filtration is decreasing)}.$$

Then there is a spectral sequence $(E^{*,*}_r, d_r)$ for $r\in\mathbb{N}$ with d_r of degree $(r,1-r)$ and

$$E^{p,q}_0=F^p(C^{p+q}(\mathfrak{g},\ M))/F^{p+1}(C^{p+q}(\mathfrak{g},\ M)),\quad E^{p,q}_1=H^{p+q}(\mathfrak{g},\ \mathrm{Grad}^p(M)).$$

To simplify the notations, we have to replace $F^n(C^*(\mathfrak{g},\ M))$ by F^nC^*. We define

$$Z_r^{p,q} = F^pC^{p+q}\bigcap d^{-1}(F^{p+r}C^{p+q+1}),$$

$$B_r^{p,q} = F^pC^{p+q}\bigcap d(F^{p-r}C^{p+q-1}),$$

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}).$$

The differential d maps $Z_r^{p,q}$ into $Z_r^{p+r,q-r+1}$, and hence includes a homomorphism

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

The spectral sequence converges to $H^*(C,d)$, that is,

$$E_\infty^{p,q} \simeq F^pH^{p+q}(C,d)/F^{p+1}H^{p+q}(C,d),$$

where $F^pH^*(C,d)$ is the image of the map $H^*(F^pC,d) \to H^*(C,d)$ induced by the inclusion $F^pC \to C$.

5.2. Computing $\mathbf{H}^1(\mathfrak{osp}(n|2),\mathcal{S}\Psi\mathcal{DO}(n))$**.** Now we can check the behavior of the cocycles $\Lambda_1,\ldots,\Lambda_4$ under the successive differentials of the spectral sequence. The cocycle Λ_1 belongs to $E_1^{0.1}$ and this cocycles $\Lambda_2,\ \Lambda_3,\ \Lambda_4$ belong to $E_1^{-1.2}$. Consider a cocycle in $\mathcal{SP}(n)$, by compute its differential as if it were with values in $\mathcal{S}\Psi\mathcal{DO}(n)$ and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image under d_1. The higher order differentials d_r can be calculated by iteration of this procedure, the space $E_r^{p+r,q-r+1}$ contains the subspace coming from $\mathrm{H}^{p+q+1}(\mathfrak{osp}(n|2);\mathrm{Grad}^{p+1}(\mathcal{S}\Psi\mathcal{DO}(n)))$. It is now easy to see that the cocycles $\Lambda_1,\ldots,\Lambda_4$ will survive in the same form, we obtain the following corollary.

Corollary **5.1.**

$$\mathrm{H}^1(\mathfrak{osp}(n|2),\mathcal{S}\Psi\mathcal{DO}(n)) \simeq \mathbb{R}^4.$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(n|2),\mathcal{S}\Psi\mathcal{DO}(n))$ *are spanned by the cohomology classes of the* 1-*cocycles* $\Delta_1,\ \Delta_2,\ \Delta_3$ *and* Δ_4:

$$\Delta_1(X_F) = F',$$

$$\Delta_2(X_F) = F'\xi^{-1}\bar{\zeta}_1\ldots\bar{\zeta}_n,$$

$$\Delta_3(X_F) = \Big(\bar{\eta}_1(F')\bar{\zeta}_1 + \ldots + \bar{\eta}_n(F')\bar{\zeta}_n\Big)\xi^{-1},$$

$$\Delta_4(X_F) = F''\xi^{-2}\bar{\zeta}_1\ldots\bar{\zeta}_n.$$

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