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S. S. Dragomir[1] (College Engineering and Sci., Victoria Univ., Melbourne, Australia; School Comput. Sci. and Appl. Math., Univ. Witwatersrand, Johannesburg, South Africa)

TWO POINTS AND nTH DERIVATIVES NORM INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

НЕРІВНОСТІ ДЛЯ НОРМИ ДВОХ ТОЧОК І n-Ї ПОХІДНОЇ ДЛЯ АНАЛІТИЧНИХ ФУНКЦІЙ У БАНАХОВИХ АЛГЕБРАХ

Let $\mathcal{B}$ be a unital Banach algebra, let $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$, let $\alpha,\ \beta \in G$, and let $f : G \to \mathbb{C}$ be analytic on G. By using the analytic functional calculus, we obtain among others the following result:

$$\left\| f(a) - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k!}\left[f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k f^{(k)}(\beta)(\beta - a)^k \right] \right\| \le$$

$$\le \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1} + \|\beta - a\|^{n+1} \right] \times$$

$$\times \max\left\{ \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|,\ \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| \right\}.$$

Some examples for the exponential function on Banach algebras are also given.

Нехай $\mathcal{B}$ — унітальна алгебра Банаха, $a \in \mathcal{B}$, G — опукла область в $\mathbb{C}$ з $\sigma(a) \subset G$, $\alpha,\ \beta \in G$, а $f : G \to \mathbb{C}$ є аналітичною на G. Використовуючи аналітичне функціональне числення, ми отримуємо серед інших такий результат:

$$\left\| f(a) - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k!}\left[f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k f^{(k)}(\beta)(\beta - a)^k \right] \right\| \le$$

$$\le \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1} + \|\beta - a\|^{n+1} \right] \times$$

$$\times \max\left\{ \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|,\ \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| \right\}.$$

Наведено також деякі приклади для експоненціальних функцій на алгебрах Банаха.

1. Introduction. Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further, $\|ab\| \le \|a\|\|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\dfrac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If $a, b \in \mathrm{Inv}(\mathcal{B})$ then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

[1] e-mail: sever.dragomir@vu.edu.au.

(i) if $a \in \mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B} \colon \|1 - b\| < 1\} \subset \mathrm{Inv}(\mathcal{B})$;
(iii) $\mathrm{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) the map $\mathrm{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \big\{z \in \mathbb{C} \colon z - a \in \mathrm{Inv}(\mathcal{B})\big\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a \colon \rho(a) \to \mathrm{Inv}(\mathcal{B})$, $R_a(z) := (z - a)^{-1}$. For each $z, w \in \rho(a)$, we have the identity

$$R_a(w) - R_a(z) = (z - w)R_a(z)R_a(w).$$

We also get that

$$\sigma(a) \subset \big\{z \in \mathbb{C} \colon |z| \le \|a\|\big\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\big\{|z| \colon z \in \sigma(a)\big\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then:
(i) the resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) for any *bounded linear functionals* $\lambda \colon \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) the spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) for each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \dfrac{1}{2\pi i}\displaystyle\int_{|\xi|=r} \xi^n(\xi - a)^{-1}d\xi$;
(v) we have $\nu(a) = \lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f \colon G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_{\delta} f(\xi)(\xi - a)^{-1}d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \mathrm{ins}(\delta)$, the inside of δ.

It is well-known (see, for instance, [6, p. 201 – 204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if $f,\ g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f + g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [6, p. 201 – 203].

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*

(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*

(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*

(d) *If $f(z) = z$ for all z, then $f(a) = a$.*

(e) *If $f,\ f_1, \ldots, f_n, \ldots$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\left\| f_n(a) - f(a) \right\| \to 0$ as $n \to \infty$.*

(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [4, 5, 9–15].

By using the analytic functional calculus in Banach algebra $\mathcal{B}$ and function $f \in \mathfrak{Hol}(a)$, we establish in this paper some norm error estimates in approximation the element $f(a)$ by some simpler expressions such as

$$(1-\lambda)f(\alpha) + \lambda f(\beta) + \sum_{k=1}^{n} \frac{1}{k!}\Big[(1-\lambda)f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k \lambda f^{(k)}(\beta)(\beta-a)^k\Big],$$

$$\frac{1}{\beta-\alpha}[(\beta-a)f(\alpha) + (a-\alpha)f(\beta)] + \frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\times$$

$$\times \sum_{k=1}^{n} \frac{1}{k!}\Big\{(a-\alpha)^{k-1} f^{(k)}(\alpha) + (-1)^k (\beta-a)^{k-1} f^{(k)}(\beta)\Big\}$$

and

$$\frac{1}{\beta-\alpha}[(a-\alpha)f(\alpha) + (\beta-a)f(\beta)] +$$

$$+ \frac{1}{\beta-\alpha}\sum_{k=1}^{n} \frac{1}{k!}\Big\{(a-\alpha)^{k+1} f^{(k)}(\alpha) + (-1)^k (\beta-a)^{k+1} f^{(k)}(\beta)\Big\},$$

where $\alpha, \beta \in D$ and $\lambda \in \mathbb{C}$.

2. Scalar identities. Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $\xi,\ \alpha \in D$. Then we have the following Taylor expansion with integral remainder:

$$f(\xi) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\alpha)(\xi-\alpha)^k +$$

$$+ \frac{1}{n!}(\xi-\alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + s\xi](1-s)^n ds \qquad (2.1)$$

for $n \geq 0$ (see, for instance, [24]).

Consider the function $f(\xi) = \mathrm{Log}(\xi)$, where $\mathrm{Log}(\xi) = \ln|\xi| + i\,\mathrm{Arg}(\xi)$ and $\mathrm{Arg}(\xi)$ is such that $-\pi < \mathrm{Arg}(\xi) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{\alpha + i\beta \colon \alpha \leq 0,\ \beta = 0\}$ and

$$f^{(k)}(\xi) = \frac{(-1)^{k-1}(k-1)!}{\xi^k}, \quad k \geq 1, \quad \xi \in \mathbb{C}_\ell.$$

By using the representation (2.1), we have

$$\mathrm{Log}(\xi) = \mathrm{Log}(\alpha) + \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{\xi - \alpha}{\alpha}\right)^k +$$

$$+ (-1)^n (\xi - \alpha)^{n+1} \int_0^1 \frac{(1-s)^n ds}{[(1-s)\alpha + s\xi]^{n+1}}$$

for all $\xi, \alpha \in \mathbb{C}_\ell$ with $(1-s)\alpha + s\xi \in \mathbb{C}_\ell$ for $s \in [0,1]$.

Consider the complex exponential function $f(\xi) = \exp(\xi)$, then by (2.1) we get

$$\exp(\xi) = \sum_{k=0}^{n} \frac{1}{k!} (\xi - \alpha)^k \exp(\alpha) +$$

$$+ \frac{1}{n!} (\xi - \alpha)^{n+1} \int_0^1 (1-s)^n \exp[(1-s)\alpha + s\xi] ds$$

for all $\xi, \alpha \in \mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions, see [1 – 3, 16 – 23].

Lemma 1. *Let $f \colon D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and ξ, α, $\beta \in D$. Then, for all $\lambda \in \mathbb{C}$ and $n \geq 1$, we have*

$$f(\xi) = (1-\lambda) f(\alpha) + \lambda f(\beta) +$$

$$+ \sum_{k=1}^{n} \frac{1}{k!} \Big[(1-\lambda) f^{(k)}(\alpha) (\xi - \alpha)^k + (-1)^k \lambda f^{(k)}(\beta) (\beta - \xi)^k \Big] + S_{n,\lambda}(\xi, \alpha, \beta), \tag{2.2}$$

where the remainder $S_{n,\lambda}(\xi, \alpha, \beta)$ is given by

$$S_{n,\lambda}(\xi, \alpha, \beta) := \frac{1}{n!} \left[(1-\lambda)(\xi - \alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + s\xi](1-s)^n ds \, + \right.$$

$$\left. + (-1)^{n+1} \lambda (\beta - \xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\xi + s\beta] s^n \, ds \right]. \tag{2.3}$$

Proof. If we replace in (2.1) α by β, then we get

$$f(\xi) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\alpha)(\xi - \beta)^k +$$

$$+ \frac{1}{n!}(\xi - \beta)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\beta + s\xi](1-s)^n ds =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\alpha)(\beta - \xi)^k +$$

$$+ \frac{(-1)^{n+1}}{n!}(\beta - \xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\beta + s\xi](1-s)^n ds =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\alpha)(\beta - \xi)^k +$$

$$+ \frac{(-1)^{n+1}}{n!}(\beta - \xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\xi + s\beta]s^n \, ds. \tag{2.4}$$

Assume that $\lambda \neq 1, 0$. If we multiply (2.1) by $1 - \lambda$ and (2.4) by λ we get the desired representation (2.2) with the remainder $S_{n,\lambda}(\xi, \alpha, \beta)$ given by (2.3).

If either $\lambda = 1$ or $\lambda = 0$, then the theorem also holds by the use of Taylor's usual expansion.

Lemma 1 is proved.

***Remark* 1.** We observe that for $n = 0$ the representation from Lemma 1 becomes

$$f(\xi) = (1 - \lambda)f(\alpha) + \lambda f(\beta) + S_\lambda(\xi, \alpha, \beta), \tag{2.5}$$

where the remainder $S_\lambda(\xi, \alpha, \beta)$ is given by

$$S_\lambda(\xi, \alpha, \beta) := (1 - \lambda)(\xi - \alpha) \int_0^1 f'((1-s)\alpha + s\xi)ds -$$

$$- \lambda \, (\beta - \xi) \int_0^1 f'((1-s)\xi + s\beta)ds.$$

***Corollary* 1.** *With the assumptions in Lemma* 1 *we have, for each distinct* $\xi, \alpha, \beta \in D$ *with* $\beta \neq \alpha$,

$$f(\xi) = \frac{1}{\beta - \alpha}[(\beta - \xi)f(\alpha) + (\xi - \alpha)f(\beta)] + \frac{(\beta - \xi)(\xi - \alpha)}{\beta - \alpha} \times$$

$$\times \sum_{k=1}^{n} \frac{1}{k!} \Big\{ (\xi - \alpha)^{k-1} f^{(k)}(\alpha) + (-1)^k (\beta - \xi)^{k-1} f^{(k)}(\beta) \Big\} + L_n(\xi, \alpha, \beta), \tag{2.6}$$

where

$$L_n(\xi,\alpha,\beta) := \frac{(\beta-\xi)(\xi-\alpha)}{n!(\beta-\alpha)}\left[(\xi-\alpha)^n\int\limits_0^1 f^{(n+1)}((1-s)\alpha+s\xi)(1-s)^n ds+\right.$$

$$\left.+(-1)^{n+1}(\beta-\xi)^n\int\limits_0^1 f^{(n+1)}((1-s)\xi+s\beta)s^n\,ds\right]$$

and

$$f(\xi)=\frac{1}{\beta-\alpha}\big[(\xi-\alpha)f(\alpha)+(\beta-\xi)f(\beta)\big]+$$

$$+\frac{1}{\beta-\alpha}\sum_{k=1}^{n}\frac{1}{k!}\Big\{(\xi-\alpha)^{k+1}f^{(k)}(\alpha)+(-1)^k(\beta-\xi)^{k+1}f^{(k)}(\beta)\Big\}+P_n(\xi,\alpha,\beta), \qquad (2.7)$$

where

$$P_n(\xi,\alpha,\beta) := \frac{1}{n!(\beta-\alpha)}\left[(\xi-\alpha)^{n+2}\int\limits_0^1 f^{(n+1)}((1-s)\alpha+s\xi)(1-s)^n ds\;+\right.$$

$$\left.+\,(-1)^{n+1}(\beta-\xi)^{n+2}\int\limits_0^1 f^{(n+1)}((1-s)\xi+s\beta)s^n\,ds\right],$$

respectively.

The proof is obvious, by choosing $\lambda=(\xi-\alpha)/(\beta-\alpha)$ and $\lambda=(\beta-\xi)/(\beta-\alpha)$, respectively, in Lemma 1. The details are omitted.

The case $n=0$ produces the following simple identities for each distinct $\xi,\alpha,\beta\in D$:

$$f(\xi)=\frac{1}{\beta-\alpha}\big[(\beta-\xi)f(\alpha)+(\xi-\alpha)f(\beta)\big]+L(\xi,\alpha,\beta),$$

where

$$L(\xi,\alpha,\beta) := \frac{(\beta-\xi)(\xi-\alpha)}{\beta-\alpha}\left[\int\limits_0^1 f'((1-s)\alpha+s\xi)ds-\int\limits_0^1 f'((1-s)\xi+s\beta)ds\right]$$

and

$$f(\xi)=\frac{1}{\beta-\alpha}[(\xi-\alpha)f(\alpha)+(\beta-\xi)f(\beta)]+P(\xi,\alpha,\beta),$$

where

$$P(\xi,\alpha,\beta) := \frac{1}{\beta-\alpha}\left[(\xi-\alpha)^2\int\limits_0^1 f'((1-s)\alpha+s\xi)ds-(\beta-\xi)^2\int\limits_0^1 f'((1-s)\xi+s\beta)ds\right].$$

3. Identities in Banach algebras. We have the following two point representation of an analytic function on Banach algebras.

Theorem 2. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\alpha, \beta \in G$. If $f : G \to \mathbb{C}$ is analytic on G, then, for all $\lambda \in \mathbb{C}$ and $n \geq 1$, we have*

$$f(a) = (1-\lambda) f(\alpha) + \lambda f(\beta) +$$

$$+ \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k \lambda f^{(k)}(\beta)(\beta - a)^k \right] + S_{n,\lambda}(a, \alpha, \beta), \tag{3.1}$$

where the remainder $S_{n,\lambda}(a, \alpha, \beta)$ is given by

$$S_{n,\lambda}(a, \alpha, \beta) := \frac{1}{n!} \left[(1-\lambda)(a-\alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + sa](1-s)^n ds \, + \right.$$

$$\left. + \, (-1)^{n+1} \lambda (\beta - a)^{n+1} \int_0^1 f^{(n+1)}[(1-s)a + s\beta] s^n \, ds \right]. \tag{3.2}$$

In particular, for $\lambda = \dfrac{1}{2}$, we obtain the trapezoid type identity

$$f(a) = \frac{f(\alpha) + f(\beta)}{2} +$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k!} \left[f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k f^{(k)}(\beta)(\beta - a)^k \right] + T_n(a, \alpha, \beta), \tag{3.3}$$

where the remainder $T_n(a, \alpha, \beta)$ is given by

$$T_n(a, \alpha, \beta) := \frac{1}{2n!} \left[(a-\alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + sa](1-s)^n ds \, + \right.$$

$$\left. + \, (-1)^{n+1} (\beta - a)^{n+1} \int_0^1 f^{(n+1)}[(1-s)a + s\beta] s^n \, ds \right].$$

Proof. Assume that $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \subset \operatorname{ins}(\delta)$. By using the analytic functional calculus (1.1) and Lemma 1, we get

$$\frac{1}{2\pi i} \int_\delta f(\xi)(\xi - a)^{-1} d\xi = [(1-\lambda) f(\alpha) + \lambda f(\beta)] \frac{1}{2\pi i} \int_\delta (\xi - a)^{-1} d\xi +$$

$$+ \sum_{k=1}^{n} \frac{1}{k!} (1-\lambda) f^{(k)}(\alpha) \frac{1}{2\pi i} \int_\delta (\xi - \alpha)^k (\xi - a)^{-1} d\xi +$$

$$+\sum_{k=1}^{n}\frac{1}{k!}(-1)^k\lambda f^{(k)}(\beta)\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^k(\xi-a)^{-1}d\xi+$$

$$+\frac{1}{n!}(1-\lambda)\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)\alpha+s\xi](1-s)^n ds\right)(\xi-a)^{-1}d\xi+$$

$$+\frac{1}{n!}(-1)^{n+1}\lambda\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)\xi+s\beta]s^n\,ds\right)(\xi-a)^{-1}d\xi=$$

$$=\frac{1}{n!}(1-\lambda)\int_0^1\left(\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi](\xi-a)^{-1}d\xi\right)(1-s)^n ds+$$

$$+\frac{1}{n!}(-1)^{n+1}\lambda\int_0^1\left(\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^{n+1}f^{(n+1)}[(1-s)\xi+s\beta](\xi-a)^{-1}d\xi\right)s^n\,ds, \tag{3.4}$$

where for the last equality we used Fubini's theorem.

By using the functional calculus for the analytic functions

$$G\ni\xi\mapsto(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi]\in\mathbb{C}$$

and

$$G\ni\xi\mapsto(\beta-\xi)^{n+1}f^{(n+1)}[(1-s)\xi+s\beta]\in\mathbb{C},$$

we have

$$\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi](\xi-a)^{-1}d\xi=$$

$$=(a-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+sa]$$

and

$$\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^{n+1}f^{(n+1)}[(1-s)\xi+s\beta](\xi-a)^{-1}d\xi=$$

$$=(\beta-a)^{n+1}f^{(n+1)}[(1-s)a+s\beta].$$

Therefore,

$$\int_0^1\left(\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi](\xi-a)^{-1}d\xi\right)(1-s)^n ds=$$

$$= \int_0^1 (a-\alpha)^{n+1} f^{(n+1)}[(1-s)\alpha + sa](1-s)^n ds =$$

$$= (a-\alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + sa](1-s)^n ds$$

and

$$\int_0^1 \left(\frac{1}{2\pi i} \int_\delta (\beta-\xi)^{n+1} f^{(n+1)}[(1-s)\xi + s\beta](\xi - a)^{-1} d\xi \right) s^n \, ds =$$

$$= \int_0^1 (\beta - a)^{n+1} f^{(n+1)}[(1-s)a + s\beta] s^n \, ds =$$

$$= (\beta - a)^{n+1} \int_0^1 f^{(n+1)}[(1-s)a + s\beta] s^n \, ds.$$

Since

$$\frac{1}{2\pi i} \int_\delta (\xi - a)^{-1} d\xi = 1, \qquad \frac{1}{2\pi i} \int_\delta (\xi - \alpha)^k (\xi - a)^{-1} d\xi = (a-\alpha)^k$$

and

$$\frac{1}{2\pi i} \int_\delta (\beta - \xi)^k (\xi - a)^{-1} d\xi = (\beta - a)^k$$

for $k = 1, \ldots, n,$ hence by (3.4) we get the representation (3.1) with the remainder (3.2).

Theorem 2 is proved.

***Remark* 2.** Withe the assumptions from Theorem 2 and by using the scalar identity (2.5) we have, for $n = 0$, that

$$f(a) = (1-\lambda) f(\alpha) + \lambda f(\beta) + S_\lambda(a, \alpha, \beta), \tag{3.5}$$

where the remainder $S_\lambda(a, \alpha, \beta)$ is given by

$$S_\lambda(a, \alpha, \beta) := (1-\lambda)(a-\alpha) \int_0^1 f'((1-s)\alpha + sa) ds -$$

$$- \lambda \, (\beta - a) \int_0^1 f'\big((1-s)a + s\beta\big) ds.$$

In particular, we have

$$f(a)=\frac{f(\alpha)+f(\beta)}{2}+T(a,\alpha,\beta), \tag{3.6}$$

where the remainder $T(a,\alpha,\beta)$ is given by

$$T(a,\alpha,\beta):=\frac{1}{2}\left[(a-\alpha)\int\limits_0^1 f'\big((1-s)\alpha+sa\big)ds-(\beta-a)\int\limits_0^1 f'\big((1-s)a+s\beta\big)ds\right].$$

We also have the following theorem.

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and α, $\beta\in G$ with $\alpha\neq\beta$. If $f:G\to\mathbb{C}$ is analytic on G, then*

$$f(a)=\frac{1}{\beta-\alpha}\big[f(\alpha)(\beta-a)+f(\beta)(a-\alpha)\big]+\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\times$$

$$\times\sum_{k=1}^{n}\frac{1}{k!}\Big\{f^{(k)}(\alpha)(a-\alpha)^{k-1}+(-1)^k f^{(k)}(\beta)(\beta-a)^{k-1}\Big\}+L_n(a,\alpha,\beta), \tag{3.7}$$

where

$$L_n(a,\alpha,\beta):=\frac{(\beta-a)(a-\alpha)}{n!(\beta-\alpha)}\left[(a-\alpha)^n\int\limits_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n ds\ +\right.$$

$$\left.+\ (-1)^{n+1}(\beta-a)^n\int\limits_0^1 f^{(n+1)}((1-s)a+s\beta)s^n\,ds\right]$$

and

$$f(a)=\frac{1}{\beta-\alpha}\big[f(\alpha)(a-\alpha)+f(\beta)(\beta-a)\big]+$$

$$+\frac{1}{\beta-\alpha}\sum_{k=1}^{n}\frac{1}{k!}\Big\{f^{(k)}(\alpha)(a-\alpha)^{k+1}+(-1)^k f^{(k)}(\beta)(\beta-a)^{k+1}\Big\}+P_n(a,\alpha,\beta), \tag{3.8}$$

where

$$P_n(a,\alpha,\beta):=\frac{1}{n!(\beta-\alpha)}\left[(a-\alpha)^{n+2}\int\limits_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n ds+\right.$$

$$\left.+(-1)^{n+1}(\beta-a)^{n+2}\int\limits_0^1 f^{(n+1)}((1-s)a+s\beta)s^n\,ds\right].$$

The proof follows in a similar way to the one from Theorem 2 by utilising the functional calculus for analytic functions (1.1) and the scalar identities (2.6) and (2.7).

The case $n = 0$ produces the following simple identities for each distinct $\alpha,\ \beta \in G$:

$$f(a) = \frac{1}{\beta - \alpha}\big[f(\alpha)(\beta - a) + f(\beta)(a - \alpha)\big] + L(a, \alpha, \beta),$$

where

$$L(a, \alpha, \beta) := \frac{(\beta - a)(a - \alpha)}{\beta - \alpha}\left[\int\limits_0^1 f'((1 - s)\alpha + sa)ds - \int\limits_0^1 f'\big((1 - s)a + s\beta\big)ds\right],$$

and

$$f(a) = \frac{1}{\beta - \alpha}\big[f(\alpha)(a - \alpha) + f(\beta)(\beta - a)\big] + P(a, \alpha, \beta),$$

where

$$P(a, \alpha, \beta) := \frac{1}{\beta - \alpha}\left[(a - \alpha)^2\int\limits_0^1 f'((1 - s)\alpha + sa)ds - (\beta - a)^2\int\limits_0^1 f'\big((1 - s)a + s\beta\big)ds\right].$$

4. Norm inequalities. The following result providing norm error estimates, holds.

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\alpha, \beta \in G$. If $f : G \to \mathbb{C}$ is analytic on G, then, for all $\lambda \in \mathbb{C}$ and $n \geq 1$, we have the representation* (2.3) *and the remainder $S_{n,\lambda}(a, \alpha, \beta)$ satisfies the norm inequalities*

$$\|S_{n,\lambda}(a, \alpha, \beta)\| \leq \frac{1}{n!}\left[|1 - \lambda|\|a - \alpha\|^{n+1}\int\limits_0^1 \left\|f^{(n+1)}[(1 - s)\alpha + sa]\right\|(1 - s)^n ds\ + \right.$$

$$\left. +\ |\lambda|\|\beta - a\|^{n+1}\int\limits_0^1 \left\|f^{(n+1)}[(1 - s)a + s\beta]\right\|s^n\, ds\right] \leq$$

$$\leq \frac{1}{n!}|1 - \lambda|\|a - \alpha\|^{n+1}\begin{cases} \dfrac{1}{n + 1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1 - s)\alpha + sa]\right\|, \\ \dfrac{1}{(nq + 1)^{1/q}}\left(\displaystyle\int_0^1 \left\|f^{(n+1)}[(1 - s)\alpha + sa]\right\|^p ds\right)^{1/p} \\ \text{for } \ p, q > 1, \ \text{ where } \ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \displaystyle\int_0^1 \left\|f^{(n+1)}[(1 - s)\alpha + sa]\right\|ds \end{cases} +$$

$$+\ \frac{1}{n!}|\lambda|\|\beta - a\|^{n+1}\begin{cases} \dfrac{1}{n + 1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1 - s)a + s\beta]\right\|, \\ \dfrac{1}{(nq + 1)^{1/q}}\left(\displaystyle\int_0^1 \left\|f^{(n+1)}[(1 - s)a + s\beta]\right\|^p ds\right)^{1/p} \\ \text{for } \ p, q > 1, \ \text{ where } \ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \displaystyle\int_0^1 \left\|f^{(n+1)}[(1 - s)a + s\beta]\right\|ds. \end{cases} \tag{4.1}$$

In particular, we get the representation (3.3) *and the remainder satisfies the norm inequalities*

$$\|T_n(a,\alpha,\beta)\| \le \frac{1}{2n!}\left[\|a-\alpha\|^{n+1}\int_0^1 \left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|(1-s)^n ds\ + \right.$$

$$\left. + \|\beta-a\|^{n+1}\int_0^1 \left\|f^{(n+1)}[(1-s)a+s\beta]\right\| s^n\, ds\right] \le$$

$$\le \frac{1}{2n!}\|a-\alpha\|^{n+1}\begin{cases} \dfrac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|, \\ \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1 \left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|^p ds\right)^{1/p} \\ \textit{for}\ \ p,q>1,\ \ \textit{where}\ \ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \displaystyle\int_0^1 \left\|f^{(n+1)}[(1-s)\alpha+sa]\right\| ds \end{cases} +$$

$$+ \frac{1}{2n!}\|\beta-a\|^{n+1}\begin{cases} \dfrac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|, \\ \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1 \left\|f^{(n+1)}[(1-s)a+s\beta]\right\|^p ds\right)^{1/p} \\ \textit{for}\ \ p,q>1,\ \ \textit{where}\ \ \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \displaystyle\int_0^1 \left\|f^{(n+1)}[(1-s)a+s\beta]\right\| ds. \end{cases}$$

Proof. By using the representation (2.4), we obtain

$$\|S_{n,\lambda}(a,\alpha,\beta)\| \le \frac{1}{n!}\left[|1-\lambda|\left\|(a-\alpha)^{n+1}\int_0^1 f^{(n+1)}[(1-s)\alpha+sa](1-s)^n ds\right\| + \right.$$

$$\left. + |\lambda|\left\|(\beta-a)^{n+1}\int_0^1 f^{(n+1)}[(1-s)a+s\beta]s^n\, ds\right\|\right] \le$$

$$\le \frac{1}{n!}\left[|1-\lambda|\|(a-\alpha)^{n+1}\|\left\|\int_0^1 f^{(n+1)}[(1-s)\alpha+sa](1-s)^n ds\right\| + \right.$$

$$\left. + |\lambda|\|(\beta-a)^{n+1}\|\left\|\int_0^1 f^{(n+1)}[(1-s)a+s\beta]s^n\, ds\right\|\right] \le$$

$$\le \frac{1}{n!}\left[|1-\lambda|\|a-\alpha\|^{n+1}\int_0^1 \left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|(1-s)^n ds\ + \right.$$

$$+ |\lambda| \|\beta - a\|^{n+1} \int_0^1 \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| s^n \, ds \Bigg] =: A. \tag{4.2}$$

This proves the first inequality in (4.1).

By using Hölder's integral inequality, we have

$$\int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| (1-s)^n ds \le$$

$$\le \begin{cases} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| \int_0^1 (1-s)^n ds, \\ \left(\int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|^p ds \right)^{1/p} \left(\int_0^1 (1-s)^{qn} ds \right)^{1/q} \\ \text{for } p, q > 1, \text{ where } \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \sup_{s\in[0,1]} (1-s)^n \int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| ds \end{cases} =$$

$$= \begin{cases} \dfrac{1}{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|, \\ \dfrac{1}{(nq+1)^{1/q}} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|^p ds \right)^{1/p} \\ \text{for } p, q > 1, \text{ where } \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| ds. \end{cases}$$

Similarly,

$$\int_0^1 \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| s^n \, ds \le$$

$$\le \begin{cases} \dfrac{1}{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + s\beta] \right\|, \\ \dfrac{1}{(nq+1)^{1/q}} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)a + s\beta] \right\|^p ds \right)^{1/p} \\ \text{for } p, q > 1, \text{ where } \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \int_0^1 \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| ds. \end{cases}$$

Therefore,

$$A \leq \frac{1}{n!}|1-\lambda|\|a-\alpha\|^{n+1}\begin{cases}\dfrac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|,\\ \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|^p ds\right)^{1/p}\\ \text{for } p,q>1, \text{ where } \dfrac{1}{p}+\dfrac{1}{q}=1,\\ \displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|ds\end{cases} +$$

$$+\frac{1}{n!}|\lambda|\|\beta-a\|^{n+1}\begin{cases}\dfrac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|,\\ \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|^p ds\right)^{1/p}\\ \text{for } p,q>1, \text{ where } \dfrac{1}{p}+\dfrac{1}{q}=1,\\ \displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|ds.\end{cases}$$

By using (4.2) we get the second part of (4.1).

Theorem 4 is proved.

***Remark* 3.** In the case $n=0$ we have the representations (3.5) and (3.6) and the remainders $S_\lambda(a,\alpha,\beta)$ and $T(a,\alpha,\beta)$ satisfy the bounds

$$\|S(a,\alpha,\beta)\| \leq |1-\lambda|\|a-\alpha\|\int_0^1\left\|f'[(1-s)\alpha+sa]\right\|ds +$$

$$+|\lambda|\|\beta-a\|\int_0^1\left\|f'[(1-s)a+s\beta]\right\|ds \leq$$

$$\leq |1-\lambda|\|a-\alpha\|\begin{cases}\sup_{s\in[0,1]}\|f'[(1-s)\alpha+sa]\|,\\ \left(\displaystyle\int_0^1\|f'[(1-s)\alpha+sa]\|^p ds\right)^{1/p}\\ \text{for } p,q>1, \text{ where } \dfrac{1}{p}+\dfrac{1}{q}=1,\end{cases} +$$

$$+|\lambda|\|\beta-a\|\begin{cases}\sup_{s\in[0,1]}\|f'[(1-s)a+s\beta]\|,\\ \left(\displaystyle\int_0^1\|f[(1-s)a+s\beta]\|^p ds\right)^{1/p}\\ \text{for } p,q>1, \text{ where } \dfrac{1}{p}+\dfrac{1}{q}=1,\end{cases}$$

and

$$\|T(a,\alpha,\beta)\| \le \frac{1}{2}\left[\|a-\alpha\| \int\limits_0^1 \|f[(1-s)\alpha + sa]\|ds \, + \right.$$

$$\left. + \, \|\beta - a\| \int\limits_0^1 \left\|f'[(1-s)a + s\beta]\right\| ds \right] \le$$

$$\le \frac{1}{2}\|a-\alpha\| \begin{cases} \sup_{s\in[0,1]} \|f'[(1-s)\alpha + sa]\|, \\ \left(\int_0^1 \left\|f'[(1-s)\alpha + sa]\right\|^p ds\right)^{1/p} \\ \text{for } \ p,q>1, \ \text{ where } \ \dfrac{1}{p}+\dfrac{1}{q}=1, \end{cases} +$$

$$+\frac{1}{2}\|\beta - a\| \begin{cases} \sup_{s\in[0,1]} \|f'[(1-s)a + s\beta]\|, \\ \left(\int_0^1 \left\|f'[(1-s)a + s\beta]\right\|^p ds\right)^{1/p} \\ \text{for } \ p,q>1, \ \text{ where } \ \dfrac{1}{p}+\dfrac{1}{q}=1. \end{cases}$$

***Corollary* 2.** *With the assumptions of Theorem* 4 *we have*

$$\|S_{n,\lambda}(a,\alpha,\beta)\| \le \frac{1}{(n+1)!}\left[|1-\lambda|\|a-\alpha\|^{n+1} + |\lambda|\|\beta-a\|^{n+1}\right]\times$$

$$\times \max\left\{\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha + sa]\right\|, \ \sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a + s\beta]\right\|\right\}$$

and, *in particular*,

$$\|T_n(a,\alpha,\beta)\| \le \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1} + \|\beta-a\|^{n+1}\right]\times$$

$$\times \max\left\{\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha + sa]\right\|, \ \sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a + s\beta]\right\|\right\}.$$

We have the following theorem.

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\alpha, \beta \in G$ with $\alpha \neq \beta$. If $f: G \to \mathbb{C}$ is analytic on G, then, for $n \ge 1$, we have the representations* (3.7) *and* (3.8) *and the remainders $L_n(a,\alpha,\beta)$ and $P_n(a,\alpha,\beta)$ satisfy the norm inequalities*

$$\big\|L_n(a,\alpha,\beta)\big\| \le \frac{1}{n!|\beta-\alpha|}\big\|(\beta-a)(a-\alpha)\big\|\times$$

$$\times\left[\|a-\alpha\|^n\int\limits_0^1\Big\|f^{(n+1)}\big((1-s)\alpha+sa\big)\Big\|(1-s)^n ds+\right.$$

$$\left.+\|\beta-a\|^n\int\limits_0^1\Big\|f^{(n+1)}\big((1-s)a+s\beta\big)\Big\|s^n\,ds\right]\le$$

$$\le\frac{1}{n!|\beta-\alpha|}\big\|(\beta-a)(a-\alpha)\big\|\times$$

$$\times\left[\|a-\alpha\|^n\begin{cases}\dfrac{1}{n+1}\sup_{s\in[0,1]}\big\|f^{(n+1)}[(1-s)\alpha+sa]\big\|,\\[2mm] \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1\Big\|f^{(n+1)}[(1-s)\alpha+sa]\Big\|^p ds\right)^{1/p}\\ \textit{for}\;\; p,q>1,\;\; \textit{where}\;\; \dfrac{1}{p}+\dfrac{1}{q}=1,\\[2mm] \displaystyle\int_0^1\Big\|f^{(n+1)}[(1-s)\alpha+sa]\Big\|ds\end{cases}\right. +$$

$$\left.+\|\beta-a\|^n\begin{cases}\dfrac{1}{n+1}\sup_{s\in[0,1]}\big\|f^{(n+1)}[(1-s)a+s\beta]\big\|,\\[2mm] \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1\Big\|f^{(n+1)}[(1-s)a+s\beta]\Big\|^p ds\right)^{1/p}\\ \textit{for}\;\; p,q>1,\;\; \textit{where}\;\; \dfrac{1}{p}+\dfrac{1}{q}=1,\\[2mm] \displaystyle\int_0^1\Big\|f^{(n+1)}[(1-s)a+s\beta]\Big\|ds\end{cases}\right] \qquad (4.3)$$

and

$$\big\|P_n(a,\alpha,\beta)\big\| \le \frac{1}{n!|\beta-\alpha|}\left[\|a-\alpha\|^{n+2}\int\limits_0^1\Big\|f^{(n+1)}((1-s)\alpha+sa)\Big\|(1-s)^n ds+\right.$$

$$\left.+\|\beta-a\|^{n+2}\int\limits_0^1\Big\|f^{(n+1)}\big((1-s)a+s\beta\big)\Big\|s^n\,ds\right]\le$$

$$\leq \frac{1}{n!|\beta-\alpha|}\left[\|a-\alpha\|^{n+2}\begin{cases}\dfrac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|,\\ \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|^p ds\right)^{1/p}\\ \textit{for } p,q>1, \textit{ where } \dfrac{1}{p}+\dfrac{1}{q}=1,\\ \displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|ds\end{cases}\right. +$$

$$+\left.\|\beta-a\|^{n+2}\begin{cases}\dfrac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|,\\ \dfrac{1}{(nq+1)^{1/q}}\left(\displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|^p ds\right)^{1/p}\\ \textit{for } p,q>1, \textit{ where } \dfrac{1}{p}+\dfrac{1}{q}=1,\\ \displaystyle\int_0^1\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|ds\end{cases}\right]. \tag{4.4}$$

Proof. From Theorem 3 we have

$$\left\|L_n(a,\alpha,\beta)\right\| \leq \frac{1}{n!|\beta-\alpha|}\left\|(\beta-a)(a-\alpha)\right\|\times$$

$$\times\left[\left\|(a-\alpha)^n\int_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n ds\right\|+\right.$$

$$\left.+\left\|(\beta-a)^n\int_0^1 f^{(n+1)}\big((1-s)a+s\beta\big)s^n\,ds\right\|\right]\leq$$

$$\leq \frac{1}{n!|\beta-\alpha|}\left\|(\beta-a)(a-\alpha)\right\|\left[\|(a-\alpha)^n\|\left\|\int_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n ds\right\|+\right.$$

$$\left.+\|(\beta-a)^n\|\left\|\int_0^1 f^{(n+1)}\big((1-s)a+s\beta\big)s^n\,ds\right\|\right]\leq$$

$$\leq \frac{1}{n!|\beta-\alpha|}\left\|(\beta-a)(a-\alpha)\right\|\left[\|a-\alpha\|^n\int_0^1\left\|f^{(n+1)}\big((1-s)\alpha+sa\big)\right\|(1-s)^n ds+\right.$$

$$\left.+\|\beta-a\|^n\int_0^1\left\|f^{(n+1)}\big((1-s)a+s\beta\big)\right\|s^n\,ds\right]=:B,$$

which proves the first inequality in (4.3). The second part follows by Hölder’s integral inequality.

The inequality (4.4) can be proved in a similar way.

Theorem 5 is proved.

***Remark* 4.** In the case $n = 0$ we get

$$f(a) = \frac{1}{\beta - \alpha}[f(\alpha)(\beta - a) + f(\beta)(a - \alpha)] + L(a, \alpha, \beta),$$

where

$$\big\|L(a, \alpha, \beta)\big\| \le \frac{1}{|\beta - \alpha|}\big\|(\beta - a)(a - \alpha)\big\|\times$$

$$\times \int_0^1 \big\|f'((1 - s)\alpha + sa)ds - f'(sa + (1 - s)\beta)\big\|ds \tag{4.5}$$

and

$$f(a) = \frac{1}{\beta - \alpha}\big[f(\alpha)(a - \alpha) + f(\beta)(\beta - a)\big] + P(a, \alpha, \beta),$$

where

$$\big\|P(a, \alpha, \beta)\big\| \le \frac{1}{|\beta - \alpha|}\times$$

$$\times\left[\|a - \alpha\|^2 \int_0^1 \big\|f'((1 - s)\alpha + sa)\big\|ds + \|\beta - a\|^2 \int_0^1 \big\|f'\big((1 - s)a + s\beta\big)\big\|ds\right].$$

Moreover, if there exist $L_a > 0$ such that

$$\big\|f'((1 - s)\alpha + sa)ds - f'(sa + (1 - s)\beta)\big\| \le (1 - \alpha)L_a|\alpha - \beta|$$

for all $s \in [0, 1]$, then by (4.5) we get

$$\big\|L(a, \alpha, \beta)\big\| \le \frac{1}{2}L_a\big\|(\beta - a)(a - \alpha)\big\|.$$

5. Examples for exponential function. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$. Consider the exponential function $f(z) = \exp(z)$, $z \in \mathbb{C}$ and put

$$E_{a,z} := \sup_{s\in[0,1]} \big\|\exp[(1 - s)a + sz]\big\| < \infty, \quad n \ge 0.$$

Observe that

$$\exp((1 - t)\lambda + ta) = \exp[(1 - t)\lambda] \exp(ta),$$

which gives

$$\big\| \exp((1 - t)\lambda + ta)\big\| = |\exp[(1 - t)\lambda]|\| \exp(ta)\| =$$

$$= \exp[(1-t)\operatorname{Re}\lambda]\|\exp(ta)\| \le$$

$$\le \exp[(1-t)\operatorname{Re}\lambda]\exp(t\|a\|) = \exp\left[(1-t)\operatorname{Re}\lambda + t\|a\|\right] \le$$

$$\le \exp(\max\{\operatorname{Re}\lambda, \|a\|\})$$

for any $t \in [0,1],\ \lambda \in \mathbb{C}.$

Therefore,

$$E_{a,z} \le \exp\big(\max\{\operatorname{Re} z, \|a\|\}\big).$$

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and α, $\beta \in G$. If $f : G \to \mathbb{C}$ is analytic on G, then by the inequality (4.1) we have

$$\|T_n(a,\alpha,\beta)\| \le \frac{1}{2(n+1)!}\|a-\alpha\|^{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa]\right\| +$$

$$+\frac{1}{2(n+1)!}\|\beta-a\|^{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + s\beta]\right\|. \tag{5.1}$$

If we apply the inequality (5.1) for the exponential function, then we get the norm inequality

$$\left\|\exp a - \frac{\exp\alpha + \exp\beta}{2} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k!}\left[\exp(\alpha)(a-\alpha)^k + (-1)^k\exp(\beta)(\beta-a)^k\right]\right\| \le$$

$$\le \frac{1}{2(n+1)!}\Big[\|a-\alpha\|^{n+1}\exp(\max\{\operatorname{Re}\alpha, \|a\|\}) +$$

$$+\|\beta-a\|^{n+1}\exp\big(\max\{\operatorname{Re}\beta, \|a\|\}\big)\Big] \le$$

$$\le \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1} + \|\beta-a\|^{n+1}\right]\exp(\max\{\operatorname{Re}\alpha, \operatorname{Re}\beta, \|a\|\}).$$

By using the inequality (4.3), we have, for $\alpha \ne \beta$, that

$$\big\|L_n(a,\alpha,\beta)\big\| \le \frac{1}{(n+1)!|\beta-\alpha|}\big\|(\beta-a)(a-\alpha)\big\|\times$$

$$\times\left[\|a-\alpha\|^n \sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha + sa]\right\| +\right.$$

$$\left. +\|\beta-a\|^n \sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a + s\beta]\right\|\right]. \tag{5.2}$$

If we apply the inequality (5.2) for the exponential function, we obtain

$$\left\|\exp a - \frac{1}{\beta-\alpha}[\exp(\alpha)(\beta-a) + \exp(\beta)(a-\alpha)] - \right.$$

$$-\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\sum_{k=1}^{n}\frac{1}{k!}\Big\{\exp(\alpha)(a-\alpha)^{k-1}+(-1)^k\exp(\beta)(\beta-a)^{k-1}\Big\}\Bigg\|\leq$$

$$\leq\frac{1}{(n+1)!|\beta-\alpha|}\big\|(\beta-a)(a-\alpha)\big\|\times$$

$$\times\big[\|a-\alpha\|^n\exp(\max\{\operatorname{Re}\alpha,\|a\|\})+\|\beta-a\|^n\exp\big(\max\{\operatorname{Re}\beta,\|a\|\}\big)\big]\leq$$

$$\leq\frac{1}{(n+1)!|\beta-\alpha|}\big\|(\beta-a)(a-\alpha)\big\|\times$$

$$\times\big[\|a-\alpha\|^n+\|\beta-a\|^n\big]\exp(\max\{\operatorname{Re}\alpha,\operatorname{Re}\beta,\|a\|\}).$$

By using the inequality (4.4), we get

$$\big\|P_n(a,\alpha,\beta)\big\|\leq\frac{1}{(n+1)!|\beta-\alpha|}\times$$

$$\times\Bigg[\|a-\alpha\|^{n+2}\sup_{s\in[0,1]}\Big\|f^{(n+1)}[(1-s)\alpha+sa]\Big\|+$$

$$+\|\beta-a\|^{n+2}\sup_{s\in[0,1]}\Big\|f^{(n+1)}[(1-s)a+s\beta]\Big\|\Bigg] \tag{5.3}$$

for $\alpha\neq\beta$.

By writing this inequality (5.3) for the exponential function, we have

$$\Bigg\|\exp(a)-\frac{1}{\beta-\alpha}\sum_{k=0}^{n}\frac{1}{k!}\Big\{\exp(\alpha)(a-\alpha)^{k+1}+(-1)^k\exp(\beta)(\beta-a)^{k+1}\Big\}\Bigg\|\leq$$

$$\leq\frac{1}{(n+1)!|\beta-\alpha|}\Big[\|a-\alpha\|^{n+2}\exp(\max\{\operatorname{Re}\alpha,\|a\|\})+$$

$$+\|\beta-a\|^{n+2}\exp\big(\max\{\operatorname{Re}\beta,\|a\|\}\big)\Big]\leq$$

$$\leq\frac{1}{(n+1)!|\beta-\alpha|}\big[\|a-\alpha\|^{n+2}+\|\beta-a\|^{n+2}\big]\exp(\max\{\operatorname{Re}\alpha,\operatorname{Re}\beta,\|a\|\}).$$

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