DOI: 10.37863/umzh.v74i8.6152

UDC 517.9

H. Damak[1] (Univ. Sfax, Tunisia)

COMPENSATOR DESIGN VIA THE SEPARATION PRINCIPLE FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS

РОЗРОБКА КОМПЕНСАТОРА ЗА ПРИНЦИПОМ ПОДІЛУ ДЛЯ КЛАСУ НАПІВЛІНІЙНИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ

We establish a compensator design via the separation principle in the practical sense for a class of semilinear evolution equations in Hilbert spaces. Under a restriction imposed on the perturbation, which is bounded by an integrable function, we propose a nonlinear time-varying practical Luenberger observer to estimate the states of the system and prove that the Luenberger observer based on the linear controller stabilizes the system. An illustrative example is given to demonstrate the applicability of our theoretical results.

Встановлено схему компенсатора через принцип поділу в практичному сенсі для класу напівлінійних еволюційних рівнянь в гільбертових просторах. За умов обмеження на збурення, яке обмежене інтегровною функцією, запропоновано застосовувати нелінійного змінного у часі практичного спостерігача Люенбергера для оцінки станів системи та доведено, що спостерігач Люенбергера на основі лінійного регулятора стабілізує систему. Наведено наочний приклад, що демонструє можливість застосування наших теоретичних результатів.

1. Introduction. Feedback compensator design of partial differential equations has been attracting a lot of attention (see [4, 6, 7, 10, 11, 24]) over the past few decades, due to its wide potential applications in heat exchanger [18], chemical engineering [5], and flexible mechanical systems [19]. The theory of compensator design is a straightforward extension of the finite dimensional theory and has been used as a starting point in many control designs for distributed parameter systems, see [1, 17, 20, 22, 25]. Alternative direct state-space finite-dimensional compensator designs can be found in [4, 9]. For extensions to systems with unbounded input and output operators (see [7, 10]), and for a comparison of various finite-dimensional control designs (see [8]). The authors in [24] give an observer-based output feedback design for linear parabolic partial differential equation (PDE) with local piecewise control and pointwise observation in space. Moreover, the problem of compensator design for linear systems in Hilbert spaces can be solved (see [6, 11]), but if the system contains some nonlinearities as a disturbances or perturbations, the problem remains a difficult task. However, the problem of practical stabilization of the infinite-dimension time-varying control systems in Hilbert spaces has been presented in [12]. In finite dimensions one simple way of designing a compensator is to first construct a state feedback stabilizer and an observer for the system and then to combine the two to design a compensator using a feedback of the observer instead of the state. This is the so-called separation principle (see [2, 3, 13, 15, 16]). In the present paper, the result is obtained in the Hilbert space setting and, hence, can be regarded as a contribution to this class of problems. Give a compensator design based on analysis results for cascaded systems, as done for instance in [15, 16, 21] used a nonlinear time-varying practical observer to estimate the system states for a class of nonlinear systems.

[1] e-mail: hanen.damak@yahoo.fr.

The main contribution of this paper is the study of the problem of practical feedback stabilization for a class of semilinear evolution equations and the design of a Luenberger observer. We refer the reader to [11, 15, 16, 24] and the references therein for more information on this direction. A compensator design is given under a restriction about the perturbed term that the perturbation is bounded by an integrable function where the nominal system is supposed to be exponentially stabilizable by a linear feedback law. We show, how under the assumptions of stabilizability and detectability of the pairs (A, B) and (A, C), we can construct a stabilizing feedback law and a Luenberger observer. A practical approach is obtained.

This paper is organized as follows. The system description, notations and some preliminary results are presented in Section 2. The required assumptions and the statement of the main results are provided in Section 3. In Section 4, an example of application of the result is given. The paper is concluded in Section 5.

2. Preliminaries and system description. Throughout this paper we adopt the following notations: $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, H denotes a Hilbert space with the norm $\|.\|$ and the inner product $\langle ..\rangle$. $L(X)$ (respectively, $L(X,Y)$) denotes the space of all linear bounded operators S mapping X into X (respectively, X into Y) endowed with the norm

$$\|S\| = \sup\big\{\|Sx\| : x \in X,\ \ \|x\| \leq 1\big\}.$$

The domain and the adjoint of an operator A are denoted by $D(A)$ and A^*, respectively. I denotes the identity operator. $C([0,\infty), H)$ denotes the space of all continuous functions from $[0,\infty)$ to H.

In this paper, we consider the controlled system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) + G(t,x), \quad t \geq 0, \\ &y = Cx, \end{aligned} \tag{1}$$

where $x \in H$ is the system state, $u \in U$ is the control input, $y \in Y$ is the measured output. H, U and Y are assumed to be Hilbert spaces. Further, the operator $A\colon D(A) \subset H \to H$ is the generator of a C_0-semigroup over H with a domain of definition $D(A)$, $B \in L(U,H)$ and $C \in L(H,Y)$.

Given an initial condition

$$x(t_0) = x_0 \in H.$$

Let $x(t) = x(t, t_0, x_0, u)$ denote the state of a system (1) at moment $t \geq t_0 \geq 0$ associated with an initial condition $x_0 \in H$ at $t = t_0$ and input $u \in U$.

We consider mild solutions of (1), i.e., solutions of the integral equation

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)[Bu(s) + G(s, x(s))]ds,$$

belonging to the class $C([t_0, t_0+\delta], H)$ for certain $\delta > 0$. Here $\{S(t),\ t \geq 0\}$ is a C_0-semigroup on a Hilbert space H with an infinitesimal generator $A\colon D(A) \subset H \to H,\ Ax = \lim_{t \longrightarrow 0^+} \dfrac{S(t)x - x}{t}$, whose domain of definition $D(A)$ consists of those $x \in H$, for which this limits exists.

Definition 1. *We call $G\colon \mathbb{R}_+ \times H \longrightarrow H$ locally Lipschitz continuous in x, uniformly in t on bounded intervals if for every $\tilde{t} \geq 0$ and constant $c \geq 0$, there is a constant $L(c, \tilde{t}\,)$, such that*

$$\big\|G(t,u) - G(t,v)\big\| \leq L(c,\tilde{t}\,)\|u - v\|$$

holds for all $u, v \in H$, with $\|u\| \leq c$, $\|v\| \leq c$, and $t \in [0, \tilde{t}\,]$.

We will use the following assumption concerning nonlinearity G throughout this paper.

$(\mathcal{H}_1)$ We assume that $G\colon \mathbb{R}_+ \times H \longrightarrow H$ is continuous in t and locally Lipschitz continuous in x, uniformly in t on bounded intervals and there exists a function ϕ such that

$$\|G(t,x)\| \leq \phi(t) \quad \forall t \geq 0, \tag{2}$$

with

$$\int\limits_0^{+\infty} \phi(s)ds \leq M_\phi < +\infty.$$

The corresponding system without perturbations, called the nominal system, is described by

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \quad t \geq 0. \tag{3}$$

Next, we recall the definition of the generator of an exponentially stable semigroup as well as that of the exponential stabilizability and detectability (see [11] for details).

Definition 2. *The operator A generates an exponentially stable semigroup $S(t)$ if the initial value problem* (3) *has a unique solution $x(t) = S(t)x_0$ and $\|S(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$ with some positive numbers M and α.*

The α is called the decay rate.

If $S(t)$ is exponentially stable, then the solution to the abstract Cauchy problem (3) tends to zero exponentially as $t \longrightarrow \infty$.

An important criterion for exponential stability is the following.

Lemma 1. *The C_0-semigroup $S(t)$ on H is exponentially stable if and only if, for every $x \in H$, there exists a positive constant γ_x such that*

$$\int\limits_0^{\infty} \|S(t)x\|^2 dt \leq \gamma_x.$$

Definition 3. *The pair $\{A, B\}$ is said to be exponentially stabilizable if there exists a feedback operator $D \in L(H, U)$ such that the operator $A + BD$ generates an exponentially stable semigroup S_{BD}.*

Definition 4. *The pair $\{A, C\}$ is said to be exponentially detectable if there exists an output injection operator $L \in L(Y, U)$ such that the operator $A + LC$ generates an exponentially stable semigroup S_{LC}.*

To study stability properties of (1) with respect to external inputs, we use the notion of practical stabilizability.

Definition 5. *System* (1) *is practically stabilizable if there exists a continuous feedback control $u\colon X \rightarrow U$ such that system* (1) *with $u(t) = u(x(t))$ satisfies the following properties*:

(i) *for any initial condition* $x_0 \in H$, *there exists a unique mild solution* $x(t)$ *defined on* $\mathbb{R}_+$;
(ii) *there exist positive scalars* ω, k, *and* r *such that the solution of the system* (1) *satisfies*

$$\|x(t)\| \leq k\|x_0\|e^{-\omega(t-t_0)} + r \quad \forall t \geq t_0 \geq 0.$$

When (i) and (ii) are satisfied for (1), we say that (1) with $u(t) = u(x(t))$ is globally uniformly practically exponentially stable.

***Remark* 1.** We deal with the practical stabilizability of (1) whose the origin is not necessarily an equilibrium point. In the case of infinite-dimensional space, the practical stabilizability is studied by [12] of a class of time-varying control systems having a time-varying linear part.

In what follows, we shall that $V : X \to \mathbb{R}_+$ is a Lyapunov function.

Definition 6. *The Lie derivative of* V *corresponding to the input* u *is defined by*

$$\dot{V}(x) = \limsup_{t \to 0^+} \frac{1}{t}\big(V(x(t, x, u) - V(x)\big).$$

Recall that a self-adjoint operator $\mathcal{P} \in L(H)$ is called positive if $\langle \mathcal{P}x, x\rangle > 0$ holds for all $x \in H \backslash \{0\}$. A positive operator $\mathcal{P} \in L(H)$ is called coercive if there exists $k > 0$ such that $\langle \mathcal{P}x, x\rangle \geq k\|x\|^2 \;\; \forall x \in D(\mathcal{P})$.

Proposition 1 (see [11, 14])**.** *Suppose that* A *is the infinitesimal generator of the* C_0*-semigroup* $S(t)$ *on the Hilbert space* H. *Then* $S(t)$ *is exponentially stable if and only if there exists a coercive positive self-adjoint operator* $\mathcal{P} \in L(H)$ *such that*

$$\langle Ax, \mathcal{P}x\rangle + \langle \mathcal{P}x, Ax\rangle = -\langle x, x\rangle \quad \forall x \in D(A). \tag{4}$$

Equation (4) is called a Lyapunov equation.

The following technical lemma will be needed in ours investigations.

Lemma 2 (generalized Gronwall – Bellman inequality [27])**.** *Let* β, $\rho \colon \mathbb{R}_+ \to \mathbb{R}$ *be continuous functions and* $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ *a function such that*

$$\dot{\varphi}(t) \leq \beta(t)\varphi(t) + \rho(t) \quad \forall t \geq t_0.$$

Then, for any $t \geq t_0 \geq 0$, *we have the inequality*

$$\varphi(t) \leq \varphi(t_0)\exp\left(\int_{t_0}^{t} \beta(v)dv\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} \beta(v)dv\right)\rho(s)ds.$$

3. Main results. ***3.1. Practical stabilization.*** We consider the nonlinear system (1) satisfying the following assumptions.

($\mathcal{H}_2$) The pair $\{A, B\}$ is exponentially stabilizable, there exists a constant operator $D \in L(X, U)$ such that a sufficient condition specially related to operator $A_D = A + BD$ is presented in [11] as the following:

there exists a coercive positive self-adjoint operator $\mathcal{P}_1$

$$\mu I \leq \mathcal{P}_1 \leq \|\mathcal{P}_1\|I,$$

where $\mu > 0$, which satisfies

$$A_D^*\mathcal{P}_1 + \mathcal{P}_1 A_D = -I.$$

We start with the following result which assures the global existence and uniqueness of mild solutions of (1).

Lemma 3. *Suppose that* $(\mathcal{H}_1)$ *holds. Then the system* (1) *possesses a unique mild solution* $x \in C([0,\infty), H)$ *for any* $x_0 \in H$.

Proof. The local Lipschitz of G suffices to assure that the mild solution of (1) exists and is unique, according to a classical existence and uniqueness theorem (Theorem 1.4 in [23]).

Using the fact that $u(t) = Dx(t)$, we have

$$\|x(t)\| \le M(\|x_0\| + M_\phi) + M\left(\int_{t_0}^{t} \|B\|\|D\|\|x\|\right) ds. \tag{5}$$

By applying Gronwall inequality (see [26, p. 42], Lemma 2.7) to inequality (5), any solution of this equation is uniformly bounded

$$\|x(t)\| \le M(\|x_0\| + M_\phi)e^{M\|B\|\|D\|\delta},$$

where $M = \sup\big\{\|S(t-s)\|\colon 0 \le t_0 \le s \le t \le t+\delta\big\}$ on an arbitrary time interval $[t_0, t_0+\delta]$. Then, using Theorem 1.4 in [23], we have $t_0 + \delta = \infty$, and so the corresponding $x \in C([0,\infty), H)$ is a mild solution of (1).

The lemma is proved.

Next, sufficient conditions are presented to guarantee the practical stabilizability of a perturbed control system using the Gronwall – Bellman inequality and Lyapunov direct method.

Theorem 1. *If assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *are fulfilled, then the system* (1) *in closed-loop with the linear feedback* $u(t) = Dx$ *is globally uniformly practically exponentially stable.*

Proof. Let $x(t)$ be the solution of system (1). We consider the Lyapunov function

$$W(x) = \langle \mathcal{P}_1 x, x\rangle.$$

The Lie derivative of W in t along the solution of the system (1) in the closed-loop with the controller $u(t) = Dx$ leads to

$$\dot W(x) = \langle \mathcal{P}_1 \dot x, x\rangle + \langle \mathcal{P}_1 x, \dot x\rangle =$$
$$= \big\langle \mathcal{P}_1[(A+BD)x + G(t,x)], x\big\rangle + \big\langle \mathcal{P}_1 x, [(A+BD)x + G(t,x)]\big\rangle.$$

By using $(\mathcal{H}_2)$ with the help of Cauchy – Schwartz inequality, we obtain

$$\dot W(x) \le -\langle x, x\rangle + 2\|\mathcal{P}_1\|\|G(t,x)\|\|x\|.$$

It follows from (2) that

$$\dot W(x) \le -\frac{1}{\|\mathcal{P}_1\|}W(x) + 2\frac{\|\mathcal{P}_1\|}{\sqrt{\mu}}\phi(t)\sqrt{W(x)}.$$

Let

$$\vartheta(t) = \sqrt{W(x(t))}.$$

The derivative of ϑ is given by

$$\dot{\vartheta}(t) = \frac{\dot{W}(x(t))}{2\sqrt{W(x)}},$$

which implies that

$$\dot{\vartheta}(t) \leq -\frac{1}{2\|\mathcal{P}_1\|}\vartheta(t) + \frac{\|\mathcal{P}_1\|}{\sqrt{\mu}}\phi(t).$$

According to Lemma 2, we have

$$\vartheta(t) \leq \vartheta(t_0)e^{-\frac{1}{2\|\mathcal{P}_1\|}(t-t_0)} + \frac{\|\mathcal{P}_1\|}{\sqrt{\mu}}\int\limits_{t_0}^{t}\phi(s)e^{-\frac{1}{2\|\mathcal{P}_1\|}(t-s)}ds.$$

Using $(\mathcal{H}_1)$, we get

$$\vartheta(t) \leq \vartheta(t_0)e^{-\frac{1}{2\|\mathcal{P}_1\|}(t-t_0)} + \frac{\|\mathcal{P}_1\|}{\sqrt{\mu}}M_\phi.$$

We deduce that

$$\|x(t)\| \leq \sqrt{\frac{\|\mathcal{P}_1\|}{\mu}}\|x_0\|e^{-\frac{1}{2\|\mathcal{P}_1\|}(t-t_0)} + \frac{\|\mathcal{P}_1\|}{\mu}M_\phi.$$

Consequently, the system (1) in closed-loop with the linear feedback $u(t) = Dx$ is globally uniformly practically exponentially stable.

Theorem 1 is proved.

3.2. Practical Luenberger observer. In this subsection, we use the measurements to estimate the full state (the construction of a Luenberger observer) and to apply state feedback on the estimated state. The Luenberger observer is a dynamical system that is expected to reconstruct the states of the system. Our objective is to design a state reconstructor for the system (1), such that the practical global exponential stability of the resulting error system can be guaranteed.

We shall introduce the following assumptions:

$(\mathcal{H}_3)$ The pair $\{A, C\}$ is exponentially detectable, there exists a constant operator $L \in L(Y, H)$, such that a sufficient condition specially related to operator $A_L = A + LC$ is presented in [11] as the following:

there exists a coercive positive self-adjoint operator $\mathcal{P}_2$

$$\nu I \leq \mathcal{P}_2 \leq \|\mathcal{P}_2\| I,$$

where $\nu > 0$, which satisfies

$$A_L^*\mathcal{P}_2 + \mathcal{P}_2 A_L = -I.$$

To design a Luenberger observer, we shall consider the system

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G(t, \hat{x}) + L(\hat{y}(t) - y(t)), \quad t \geq 0,$$
$$\hat{y}(t) = C\hat{x}(t), \tag{6}$$

where $\hat{x}$ is the Luenberger observer with output injection $L \in L(Y, H)$.

Define estimation error e as $e = \hat{x} - x$, which is governed by

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A + LC)e(t) + G(t, \hat{x}(t)) - G(t, x(t)), \tag{7}$$

where $e_0 = \hat{x}_0 - x_0$.

The following lemma provides sufficient conditions for the global solution of (7).

Lemma 4. *Under assumption* $(\mathcal{H}_1)$, *the system* (7) *possesses a unique mild solution* $e \in \in C([0, \infty), H)$ *for any* $e_0 \in H$.

Proof. It is known from [23] that for every initial state $e_0 \in H$, system (7) has a unique mild solution given by

$$e(t) = S_{LC}(t - t_0)e_0 + \int_{t_0}^{t} S_{LC}(t - s)\big[G(s, \hat{x}(s)) - G(s, x(s))\big]ds, \quad t_0 \le t \le t_0 + \delta, \quad \delta > 0,$$

where S_{LC} is the C_0-semigroup of A_L.

From the above equation, we get

$$\|e(t)\| \le N\|e_0\| + N\left(2\int_{t_0}^{t} \phi(s)ds\right),$$

where $N = \sup\big\{\|S(t - s)\| : 0 \le t_0 \le s \le t \le t + \delta\big\}$ on an arbitrary time interval $[t_0, t_0 + \delta]$. Then, using Theorem 1.4 in [23], we have $t_0 + \delta = \infty$, and so the system (7) has a unique mild solution e which exists for all $t \ge t_0$.

By arguing in exactly the same way as in Theorem 1, we prove that the output injection L can be chosen in such a way that system (6) is a practical exponential Luenberger observer for system (1).

Theorem 2. *Under assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_3)$, *the system* (6) *is a practical exponential Luenberger observer for the system* (1).

Proof. Let $e(t)$ be the solution of system (7). We consider the Lyapunov function candidate

$$Z(e) = \langle \mathcal{P}_2 e, e\rangle.$$

The Lie derivative of Z along the trajectories of system (7) is given by

$$\dot{Z}(e) = \langle \mathcal{P}_2\dot{e}, e\rangle + \langle \mathcal{P}_2 e, \dot{e}\rangle =$$

$$= \langle \mathcal{P}_2[(A + LC)e + G(t, \hat{x}) - G(t, x)], e\rangle +$$

$$+\langle \mathcal{P}_2 e, [(A + LC)e + G(t, \hat{x}) - G(t, x)]\rangle.$$

By using $(\mathcal{H}_3)$ with the help of Cauchy – Schwartz inequality, we have

$$\dot{Z}(e) \le -\langle e, e\rangle + 2\|\mathcal{P}_2\|\|G(t, \hat{x}) - G(t, x)\|\|e\|.$$

It follows from (2) that

$$\dot{Z}(e) \le -\|e\|^2 + 4\frac{\|\mathcal{P}_2\|}{\sqrt{\nu}}\phi(t)\sqrt{Z(e)}.$$

Let

$$\theta(t) = \sqrt{Z(e(t))}.$$

The derivative of θ is given by

$$\dot{\theta}(t) = \frac{\dot{Z}(e(t))}{2\sqrt{Z(e)}},$$

which implies that

$$\dot{\theta}(t) \leq -\frac{1}{2\|\mathcal{P}_2\|}\theta(t) + \frac{2\|\mathcal{P}_2\|\phi(t)}{\sqrt{\nu}}.$$

Applying Lemma 2 on the above inequality, we get

$$\theta(t) \leq \theta(t_0)e^{-\frac{1}{2\|\mathcal{P}_2\|}(t-t_0)} + \frac{2\|\mathcal{P}_2\|M_\phi}{\sqrt{\nu}}.$$

Hence,

$$\|e(t)\| \leq \sqrt{\frac{\|\mathcal{P}_2\|}{\nu}}\|e_0\|e^{-\frac{1}{2\|\mathcal{P}_2\|}(t-t_0)} + \frac{2\|\mathcal{P}_2\|M_\phi}{\nu}.$$

We deduce that the system (7) is globally uniformly practically exponentially stable. Consequently, the system (6) is a global uniform practical exponential Luenberger observer for the system (1).

Theorem 2 is proved.

3.3. Compensator design. To obtain a compensator design to (1) just consider (1) controlled by the linear feedback control $u(t) = D\hat{x}(t)$ estimated by the Luenberger observer (6).

Theorem 3. *Consider the nonlinear system* (1) *and assume that assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *hold. If* $D \in L(H,U)$ *and* $L \in L(Y,H)$ *are such that* $A+BD$ *and* $A+LC$ *generate exponentially stable semigroups, then the controller* $u = D\hat{x}$, *where* $\hat{x}$ *is the Luenberger observer with output injection* L, *stabilizes the closed-loop system. The stabilizing compensator is given by*

$$\dot{\hat{x}} = (A+LC)\hat{x} + Bu(t) + G(t,\hat{x}) - Ly(t),$$
$$u(t) = D\hat{x}(t).$$

Proof. Under assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, there exist operators D and L such that $S_{BD}(t)$ and $S_{LC}(t)$ are exponentially stable. Combining the abstract differential equations, we see that the closed-loop system is given by the dynamics of the extended state $x^e = \begin{pmatrix} \hat{x} \\ e \end{pmatrix}$,

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{e} \end{pmatrix}(t) = \begin{pmatrix} A+BD & LC \\ 0 & A+LC \end{pmatrix} \times \begin{pmatrix} \hat{x} \\ e \end{pmatrix}(t) + \begin{pmatrix} G(t,\hat{x}) \\ G(t,\hat{x}) - G(t,x) \end{pmatrix}, \quad t \geq 0. \tag{8}$$

Let us define the Lyapunov function

$$Y(x^e) = \alpha W(\hat{x}) + Z(e),$$

where $W(\hat{x}) = \langle \mathcal{P}_1\hat{x}, \hat{x}\rangle$, $Z(e) = \langle \mathcal{P}_2 e, e\rangle$ and $\alpha > 0$ is a Lyapunov parameter to be determined. The Lie derivative of Y along the trajectories of system (8) is given as follows:

$$\dot{Y}(x^e) = \alpha\dot{W}(\hat{x}) + \dot{Z}(e) =$$

$$= \alpha(\langle \mathcal{P}_1\dot{\hat{x}}, \hat{x}\rangle + \langle \mathcal{P}_1\hat{x}, \dot{\hat{x}}\rangle) + \langle \mathcal{P}_2\dot{e}, e\rangle + \langle \mathcal{P}_2 e, \dot{e}\rangle =$$

$$= \alpha(\langle \mathcal{P}_1[A\hat{x} + BD\hat{x} + G(t,\hat{x}) + LCe], \hat{x}\rangle + \langle \mathcal{P}_1\hat{x}, A\hat{x} + BD\hat{x} + G(t,\hat{x}) + LCe\rangle +$$

$$+ \langle \mathcal{P}_2[(A+LC)e + G(t,\hat{x}) - G(t,x)], e\rangle + \langle \mathcal{P}_2 e, (A+LC)e + G(t,\hat{x}) - G(t,x)\rangle.$$

By using $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, with the help of Cauchy – Schwartz inequality, we obtain

$$\dot{Y}(x^e) \leq \alpha(-\langle \hat{x}, \hat{x}\rangle + 2\|\mathcal{P}_1\|\|G(t,\hat{x})\|\|\hat{x}\| +$$

$$+2\|\mathcal{P}_1\|\|LCe\|\|\hat{x}\|) - \langle e, e\rangle + 2\|\mathcal{P}_2\|\|G(t,\hat{x}) - G(t,x)\|\|e\|.$$

It follows that

$$\dot{Y}(x^e) \leq \alpha\left(-\frac{1}{\|\mathcal{P}_1\|}W(\hat{x}) + 2\|\mathcal{P}_1\|\phi(t)\|\hat{x}\| + 2\|\mathcal{P}_1\|\|LCe\|\|\hat{x}\|\right) -$$

$$-\frac{1}{\|\mathcal{P}_2\|}Z(e) + 4\|\mathcal{P}_2\|\phi(t)\|e(t)\|.$$

Let $\varepsilon > 0$. Using Young's inequality

$$2\|\hat{x}\|\|e\| \leq \frac{1}{\varepsilon}\|\hat{x}\|^2 + \varepsilon\|e\|^2,$$

we can continue the above estimates as

$$\dot{Y}(x^e) \leq \left(-\frac{1}{\|\mathcal{P}_1\|} + \frac{\|\mathcal{P}_1\|\|LC\|}{\mu\varepsilon}\right)\alpha W(\hat{x}) +$$

$$+\left(-\frac{1}{\|\mathcal{P}_2\|} + \frac{\alpha\varepsilon\|\mathcal{P}_1\|\|LC\|}{\nu}\right)Z(e) + \frac{2\alpha\|\mathcal{P}_1\|\phi(t)}{\sqrt{\mu}}\sqrt{W(\hat{x})} + \frac{4\|\mathcal{P}_2\|\phi(t)}{\sqrt{\nu}}\sqrt{Z(e)}.$$

Choose $\varepsilon > 0$ such that $\dfrac{1}{\|\mathcal{P}_1\|} - \dfrac{\|\mathcal{P}_1\|\|LC\|}{\mu\varepsilon} > 0$.

Let

$$\varepsilon = \frac{2\|\mathcal{P}_1\|^2\|LC\|}{\mu}.$$

Also, choose for this value of ε the scalar α such that $\dfrac{1}{\|\mathcal{P}_2\|} - \dfrac{\alpha\varepsilon\|\mathcal{P}_1\|\|LC\|}{\nu} > 0$. Then let

$$\alpha = \frac{\mu\nu}{4\|\mathcal{P}_1\|^3\|\mathcal{P}_2\|\|LC\|^2}.$$

We get

$$\dot{Y}(x^e) \le -\frac{\alpha}{2\|\mathcal{P}_1\|}W(\hat{x}) - \frac{1}{2\|\mathcal{P}_2\|}Z(e) + \frac{2\alpha\|\mathcal{P}_1\|\phi(t)}{\sqrt{\mu}}\sqrt{W(\hat{x})} + \frac{4\|\mathcal{P}_2\|\phi(t)}{\sqrt{\nu}}\sqrt{Z(e)}.$$

Thus,

$$\dot{Y}(x^e) \le -\lambda_1 Y(x^e) + \lambda_2\phi(t)\Big(\sqrt{\alpha W(\hat{x})} + \sqrt{Z(e)}\Big)$$

with

$$\lambda_1 = \min\left(\frac{\alpha}{2\|\mathcal{P}_1\|}, \frac{1}{2\|\mathcal{P}_2\|}\right)$$

and

$$\lambda_2 = \max\left(\frac{2\sqrt{\alpha}\|\mathcal{P}_1\|}{\sqrt{\mu}}, \frac{4\|\mathcal{P}_2\|}{\sqrt{\nu}}\right).$$

Since $\sqrt{a} + \sqrt{b} \le 2\sqrt{a+b}$, for all $a, b \ge 0$, one can get, for all $t \ge t_0$,

$$\Big(\sqrt{\alpha W(\hat{x})} + \sqrt{Z(e)}\Big) \le 2\sqrt{\alpha W(\hat{x}) + Z(e)},$$

which implies that

$$\dot{Y}(x^e) \le -\lambda_1 Y(x^e) + 2\lambda_2\phi(t)\sqrt{Y(x^e)}.$$

Let

$$\omega(t) = \sqrt{Y(x^e(t))}.$$

The derivative of ω is given by

$$\dot{\omega}(t) = \frac{\dot{Y}(x^e(t))}{2\sqrt{Y(x^e(t))}},$$

which implies that

$$\dot{\omega}(t) \le -\frac{\lambda_1}{2}\omega(t) + \lambda_2\phi(t).$$

It follows from Lemma 2 that

$$\omega(t) \le \omega(t_0)e^{-\frac{\lambda_1}{2}(t-t_0)} + \lambda_2 M_\phi.$$

By using the inequality, $(b_1+b_2)^2 \le 2b_1^2 + 2b_2^2$ for all $b_1, b_2 \ge 0$, we obtain

$$Y(x^e(t)) \le 2Y(x_0^e)e^{-\lambda_1(t-t_0)} + 2(\lambda_2 M_\phi)^2,$$

where $x_0^e = (\hat{x}_0, e_0)$.

Hence,

$$\|\hat{x}(t)\| \le \sqrt{\frac{2}{\alpha}}\Big[\max(\sqrt{\alpha\|\mathcal{P}_1\|}, \sqrt{\|\mathcal{P}_2\|})(\|\hat{x}_0\| + \|e_0\|)e^{-\frac{\lambda_1}{2}(t-t_0)} + \lambda_2 M_\phi\Big].$$

Consequently, the cascade system (8) is globally uniformly practically exponentially stable.

Theorem 3 is proved.

4. Illustrative example.

Example. We consider the controlled metal bar equation

$$\frac{\partial x(\zeta,t)}{\partial t} = \frac{\partial^2 x(\zeta,t)}{\partial \zeta^2} + b(\zeta)u(t) + \frac{(t+1)e^{-t}}{1+\|x(\zeta,t)\|},$$

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t), \quad x(\zeta,0) = x_0(\zeta), \quad t \geq 0,$$

$$y(t) = \int\limits_0^1 c(\zeta)x(\zeta,t)d\zeta,$$

where $x(\zeta,t)$ represents the temperature at position ζ at time $t \geq 0$ and x_0 represents the initial temperature profile, $u(t)$ the addition of heat along the bar and b, c represents the shaping functions around the control ζ_0 and the sensing point ζ_1, respectively,

$$b(\zeta) = \frac{1}{2\delta}\mathbf{1}_{[\zeta_0-\delta,\zeta_0+\delta]}$$

and

$$c(\zeta) = \frac{1}{2\kappa}\mathbf{1}_{[\zeta_1-\kappa,\zeta_1+\kappa]},$$

with $[\zeta_0-\delta,\zeta_0+\delta]\cap[\zeta_1-\kappa,\zeta_1+\kappa]=\varnothing$ and

$$\mathbf{1}_{[\vartheta,\upsilon]}(x) = \begin{cases} 1, & \text{if} \quad \vartheta \leq x \leq \upsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that b and c in this example are both elements in $L^2(0,1)$ for a fixed small, nonnegative constants δ and κ.

The partial differential equation is equivalent to system (1) where $H = L^2(0,1)$, $U = \mathbb{C}$, $Y = \mathbb{C}$, $A = \dfrac{\partial^2}{\partial^2\zeta}$, with $D(A) = \left\{h \in L^2(0,1),\ h,\ \dfrac{\partial h}{\partial \zeta} \text{ are absolutely continuous, } \dfrac{\partial^2 h}{\partial \zeta^2} \in L^2(0,1) \text{ and } \dfrac{dh}{d\zeta}(0) = 0 = \dfrac{dh}{d\zeta}(1)\right\}$, the input operator

$$Bu = b(\zeta)u,$$

$B \in L(\mathbb{C},H)$, and has norm $\dfrac{1}{\sqrt{2\delta}}$.

Furthermore, the measured output operator

$$Cx = \int\limits_0^1 c(\zeta)x(\zeta,t)d\zeta,$$

where $C \in L(H,\mathbb{C})$, and has norm $\dfrac{1}{\sqrt{2\kappa}}$, and

$$G(t,x) = \frac{(t+1)e^{-t}}{1 + \|x(\zeta,t)\|}.$$

A has the eigenvalues $0, -n^2\pi^2,\ n \geq 1$, and the corresponding orthogonal eigenvectors are

$$\upsilon_n = \begin{cases} 1, & \text{if} \quad n = 0, \\ \sqrt{2}\cos(n\pi\zeta), & \text{if} \quad n \geq 1. \end{cases}$$

It follows that A is the infinitesimal generator of the C_0-semigroup (see [11] for details).

We can take as a stabilizing feedback $u(t) = Dx$ with

$$Dx = -3\langle x, \upsilon_0\rangle = -3\langle x, 1\rangle,$$

where $\langle , \rangle$ is the inner product on $L^2(0,1)$.

It is easy to verify that $A + BD$ has the eigenvalues $-3,\ -(n\pi)^2,\ n \geq 1$. Then the pair $\{A, B\}$ is exponentially stabilizable.

In addition, the stabilizing output injection is given by

$$Ly = -3y\upsilon_0 = -3y.1.$$

The system $A + LC$ has the eigenvalues $-3,\ -(n\pi)^2,\ n \geq 1$. Then the pair $\{A, C\}$ is exponentially detectable.

Moreover, the assumption $(\mathcal{H}_1)$ is satisfied with $\phi(t) = (t+1)e^{-t}$ is a continuous nonnegative function with

$$\int_0^{+\infty} (t+1)e^{-t} = M_\phi = 2 < \infty.$$

From Theorem 3, we conclude that a stabilizing compensator is given by

$$\frac{\partial \hat{x}(\zeta,t)}{\partial t} = \frac{\partial^2 \hat{x}(\zeta,t)}{\partial \zeta^2} - \frac{3}{2\kappa}\int_{\zeta_1-\kappa}^{\zeta_1+\kappa} \hat{x}(\zeta,t)d\zeta + \frac{1}{2\delta}\mathbf{1}_{[\zeta_0-\delta,\zeta_0+\delta]}(\zeta)u(t) + \frac{(t+1)e^{-t}}{1+\|\hat{x}(\zeta,t)\|} + 3y(t),$$

$$\frac{\partial \hat{x}}{\partial \zeta}(0,t) = 0 = \frac{\partial \hat{x}}{\partial \zeta}(1,t), \quad \hat{x}(\zeta,0) = \hat{x}_0(\zeta), \quad t \geq 0,$$

$$u(t) = -3\int_0^1 \hat{x}(\zeta,t)d\zeta.$$

5. Conclusion. We have presented the problems of state observation and state trajectory control via output feedback for a class of nonlinear system. It is shown that the system can be practically stabilizable by means of an estimated state feedback given by a designated Luenberger observer. Furthermore, a compensator design of a class of nonlinear control systems has been considered. An example has been introduced to validate the developed methods.

References

1. M. Balas, *Towards a (more) practical control theory for distributed parameter systems*, Control and Dynamics Systems: Advances in Theory and Applications, Acad. Press, New York (1980).
2. A. Benabdallah, *A separation principle for the stabilization of a class of time-delay nonlinear systems*, Kybernetika, **51**, 99 – 111 (2015).
3. A. Benabdallah, I. Ellouze, M. A. Hammami, *Practical exponential stability of perturbed triangular systems and a separation principle*, Asian J. Control, **13**, 445 – 448 (2011).
4. D. S. Bernstein, D. C. Hyland, *The optimal projection equations for finite-dimensional fixed-order dynamic compensation of infinite-dimensional systems*, SIAM J. Control and Optim., **24**, 122 – 151 (1986).
5. P. D. Christofides, *Nonlinear and robust control of PDE systems: methods and applications to transport-reaction processes*, Birkhäuser, Boston (2001).
6. R. F. Curtain, *Compensators for infinite dimensional linear systems*, J. Franklin Inst., **315**, 331 – 346 (1983).
7. R. F. Curtain, *Finite-dimensional compensators for parabolic distributed systems with unbounded control and observation*, SIAM J. Control and Optim., **22**, 225 – 276 (1984).
8. R. F. Curtain, *A comparaison of finite-dimensional controller designs for distributed parameter systems*, Control Theory and Technol., **9**, 609 – 628 (1993).
9. R. F. Curtain, *Stabilization of boundary control distributed systems via integral dynamic output feedback of a finite-dimensional compensator*, Anal. and Optim. Systems, **44**, 761 – 776 (1982).
10. R. F. Curtain, D. Salamon, *Finite-dimensional compensators for infinite-dimensional systems with unbounded input operators*, SIAM J. Control and Optim., **24**, 797 – 816 (1986).
11. R. F. Curtain, H. J. Zwart, *An introduction to infinite dimensional linear systems theory*, Springer-Verlag, New York (1995).
12. H. Damak, M. A. Hammami, *Stabilization and practical asymptotic stability of abstract differential equations*, Numer. Funct. Anal. and Optim., **37**, 1235 – 1247 (2016).
13. H. Damak, I. Ellouze, M. A. Hammami, *A separation principle of a class of time-varying nonlinear systems*, Nonlinear Dyn. and Syst. Theory, **13**, 133 – 143 (2013).
14. R. Datko, *Extending a theorem of A. M. Liapunov to Hilbert spaces*, J. Math. Anal. and Appl., **32**, 610 – 616 (1970).
15. I. Ellouze, *On the practical separation principle of time-varying perturbed systems*, IMA J. Math. Control and Inform., **37**, № 1, 1 – 16 (2019).
16. I. Ellouze, M. A. Hammami, *A separation principle of time-varying dynamical systems: a practical stability approach*, Math. Model. and Anal., **12**, 297 – 308 (2007).
17. R. Gressang, G. Lamont, *Observers for systems characterized by semigroups*, IEEE Trans. Automat. Control, **AC-20**, 523 – 528 (1975).
18. W. Harmon Ray, *Advanced process control*, McGraw-Hill, New York (1981).
19. W. He, S. S. Ge, B. V. E How, Y. S. Choo, *Dynamics and control of mechanical systems in offshore engineering*, Springer-Verlag, London (2014).
20. S. Kitamura, H. Sakairi, M. Mishimura, *Observers for disturbuted parameter systems*, Electric. Eng. Jap., **92**, 142 – 149 (1972).
21. A. Loria, E. Panteley, *A separation principle for a class of Euler – Langrange systems*, New Directions in Nonlinear Observer Design, Lect. Notes Control and Inform. Sci., vol. 244, Springer, London (1999).
22. Y. Orlov, Y. Lou, P. D. Christofides, *Robust stabilization of infinite-dimensional systems using sliding-mode output feedback control*, Int. J. Control, **77**, 1115 – 1136 (2004).
23. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York (1983).
24. Y. Qiang, J. W. Wang, C. Y. Sun, *Observer-based output feedback compensator design for linear parabolic PDEs with local piecewise control and pointwise observation in space*, IET Control Theory Appl., **12**, 1812 – 1821 (2018).
25. Y. Sakawa, T. Matsushita, *Feedback stabilization for a class of distributed systems and construction of a state estimator*, IEEE Trans. Automat. Control, **AC-20**, 748 – 753 (1975).
26. G. Teschl, *Ordinary differential equations and dynamical systems, graduate studies in mathematics*, Amer. Math. Soc. (2012).
27. B. Zhou, *Stability analysis of nonlinear time-varying systems by Lyapunov functions with indefinite derivatives*, IET Control Theory Appl., **11**, 1434 – 1442 (2017).

Received 05.06.20