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SOME PROPERTIES OF A GENERALIZED MULTIPLIER TRANSFORM ON ANALYTIC p-VALENT FUNCTIONS

ДЕЯКІ ВЛАСТИВОСТІ УЗАГАЛЬНЕНОГО МУЛЬТИПЛІКАТИВНОГО ПЕРЕТВОРЕННЯ НА АНАЛІТИЧНИХ p-ВАЛЕНТНИХ ФУНКЦІЯХ

For a function

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$$

where $p \in \mathbb{N}$, the authors investigate some properties of a more general multiplier transform on analytic p-valent functions in an open unit disk. The applications of the obtained results to fractional calculus are pointed out, while several other corollaries follow as simple consequences.

Для функції

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

де $p \in \mathbb{N}$, досліджено деякі властивості більш загального мультиплікативного перетворення на аналітичних p-валентних функціях у відкритому одиничному колі. Розглянуто застосування отриманих результатів до дробового числення, а деякі інші результати отримано як прості наслідки.

1. Introduction and preliminaries. Let Γ denote the class of analytic functions $f(z)$, having the series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and normalized by $f'(0) - 1 = 0 = f(0)$ in the open unit disk $U = \{z \in \mathbb{C} \colon |z| < 1\}$.

Also let Γ_p denote the class of analytic p-valent functions $f(z)$ having the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad p \in \mathbb{N}. \tag{2}$$

A function $U = u(x, y)$ is said to be harmonic if it is a real-valued function having continuous partial derivatives of order one and two, and satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

However, a continuous complex-valued function $f(z) = u(x, y) + iv(x, y)$ is said to be harmonic in a complex domain D say, if both the real and imaginary parts $u(x, y)$ and $v(x, y)$, respectively, are harmonic in D. The geometric function theory is mostly interested in the survey of properties of analytic functions (see [2, 5]). Given any simply connected region $R \subset D$, we can say that

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$$f(z) = h(z) + \overline{g(z)},$$

where h and g are analytic in the connected region R. Conventionally, we refer to h and g as the analytic and co-analytic parts, respectively. Thus a necessary and sufficient condition for function f to be locally univalent and orientation preserving is that

$$\left|h'(z)\right| > |g'(z)| \in R,$$

see [1, 4] among others. Let $\mathcal{H}$ denote the family of p-valent harmonic function in U. Then h and g can be expressed as

$$h(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad g(z) = \sum_{k=0}^{\infty} b_{k+p} z^{k+p}$$

for $p \in \mathbb{N}$ and, in particular, $0 \leq |b_p| < 1$.

Therefore, we write that

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} + \overline{\sum_{k=0}^{\infty} b_{k+p} z^{k+p}}. \tag{3}$$

As a special case, if the co-analytic part of f is identically zero (i.e., $g = 0$), then the family of orientation preserving, normalized harmonic univalent functions reduces to the usual class of normalized analytic functions. For function $f(z)$ of the form (1), Swamy [10], in 2012 introduced and studied a multiplier differential operator $I^n_{\alpha,\beta} f(z)$ given by

$$I^n_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^n a_k z^k,$$

see also [8].

Furthermore, we define for function $f(z)$ of the form (2), a linear differential operator $L^{n,p}_{\alpha,\beta,\gamma} f(z)$ such that

$$L^{1,p}_{\alpha,\beta,\gamma} f(z) = \frac{\alpha f(z) + \beta z f'(z) + \gamma z (z f'(z))'}{\alpha + \beta p + \gamma p^2},$$

$$L^{2,p}_{\alpha,\beta,\gamma} f(z) = L^{1,p}_{\alpha,\beta,\gamma} f(z) \Big(L^{1,p}_{\alpha,\beta,\gamma} f(z) \Big),$$

$$L^{3,p}_{\alpha,\beta,\gamma} f(z) = L^{1,p}_{\alpha,\beta,\gamma} f(z) \Big(L^{2,p}_{\alpha,\beta,\gamma} f(z) \Big),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$L^{n,p}_{\alpha,\beta,\gamma} f(z) = L^{1,p}_{\alpha,\beta,\gamma} f(z) \Big(L^{n-1,p}_{\alpha,\beta,\gamma} f(z) \Big), \tag{4}$$

where $p \in \mathbb{N}$, $n, \alpha, \beta \geq 0$ and α is real such that $\alpha + \beta + \gamma > 0$. It follows from (4) that

$$L^{n,p}_{\alpha,\beta,\gamma} f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{\alpha + \beta(k+p) + \gamma(k+p)^2}{\alpha + \beta p + \gamma p^2} \right)^n a_{k+p} z^{k+p}. \tag{5}$$

Remark. Suppose that the function $f(z)$ has the form (2), it is easily verified from (5) that

$$L^{0,p}_{\alpha,0,0}f(z) = f(z) \in \Gamma_p \quad \text{and} \quad L^{0,1}_{\alpha,0,0}f(z) = f(z) \in \Gamma_p.$$

It is obvious that the operator $L^{n,p}_{\alpha,\beta,\gamma}f(z)$ generalizes many existing operators of this kind which were introduced and studied by different authors. For instance,

(i) $L^{n,1}_{\alpha,\beta,\gamma}f(z) = I^{n}_{\alpha,\beta,\gamma}f(z)$ studied by Makinde et al. [8];

(ii) $L^{n,1}_{\alpha,\beta,0}f(z) = I^{n}_{\alpha,\beta}f(z)$ studied by Swamy [10];

(iii) $L^{n,1}_{\alpha,1,\gamma}f(z) = I^{n}_{\alpha}f(z),\ \alpha > -1$ studied by Cho and Srivastava [3];

(iv) $L^{n,1}_{1,\beta,0}f(z) = N^{n}_{\beta}f(z)$ studied by Swamy [10].

With reference to (5), we can write that

$$L^{n,p}_{\alpha,\beta,\gamma}f(z) = H^{n,p}_{\alpha,\beta,\gamma}f(z) + \overline{G^{n,p}_{\alpha,\beta,\gamma}f(z)}. \tag{6}$$

Now using (6), we give the following definition.

Definition 1. *Let $f(z)$ be of the form* (3), *then $f(z) \in \mathcal{H}^{n,p}_{\mu}(\alpha,\beta,\gamma)$ if it satisfies the condition that*

$$\Re\left\{\frac{z\Big(H^{n,p}_{\alpha,\beta,\gamma}f(z)\Big)' - \overline{z\Big(G^{n,p}_{\alpha,\beta,\gamma}f(z)\Big)'}}{H^{n,p}_{\alpha,\beta,\gamma}f(z) + \overline{G^{n,p}_{\alpha,\beta,\gamma}f(z)}}\right\} \geq \mu \tag{7}$$

for $p \in \mathbb{N},\ 0 \leq \mu < p,\ n,\ \beta,\ \gamma \geq 0$ and α is real such that $\alpha + \beta + \gamma > 0$.

In addition, suppose that

$$\mathcal{V}^{n,p}_{\mathcal{H}}(\alpha,\beta,\gamma,\mu) = \mathcal{V}^{n,p}_{\mathcal{H}} \cap \mathcal{H}^{n,p}_{\mu}(\alpha,\beta,\gamma), \tag{8}$$

where $\mathcal{V}^{n,p}_{\mathcal{H}}$ is the harmonic functions with varying arguments consists of functions f of the form (3) in $\mathcal{H}^{n,p}_{\mu}$ for which there exists a real number σ such that

$$\psi_{k+p} + k\sigma \equiv \pi (\operatorname{mod} 2\pi), \quad \tau_{k+p} + k\sigma \equiv 0 (\operatorname{mod} 2\pi), \quad k \geq 1, \tag{9}$$

where

$$\psi_{k+p} = \arg(a_{k+p}) \quad \text{and} \quad \tau_{k+p} = \arg(b_{k+p}).$$

At this juncture, we shall obtain a sufficient coefficient condition for function f of the form (3) to be in the aforementioned class $\mathcal{H}^{n,p}_{\mu}(\alpha,\beta,\gamma)$. It is noted that this coefficient condition is also necessary for functions belonging to the class $\mathcal{V}^{n,p}_{\mathcal{H}}(\alpha,\beta,\gamma)$.

2. Necessary and sufficient coefficient for the class $\mathcal{H}^{n,p}_{\mu}(\alpha,\beta,\gamma)$.

Theorem 2.1. *Let $f(z)$ be of the form* (3). *Then, for $p \in \mathbb{N},\ 0 \leq \mu < p,\ \alpha > 0,\ \beta,\gamma \geq 0,$ $\alpha + \beta + \gamma > 0$, and $|b_p| < \dfrac{p-\mu}{p+\mu}$, $f \in \mathcal{H}^{n,p}_{\mu}(\alpha,\beta,\gamma)$ if*

$$\sum_{k=1}^{\infty}\left[\frac{k+p-\mu}{p-\mu}\,|a_{k+p}| + \frac{k+p+\mu}{p-\mu}\,|b_{k+p}|\right]\times$$

$$\times\left(\frac{\alpha + \beta(k+p) + \gamma(k+p)^2}{\alpha + \beta p + \gamma p^2}\right)^{n} \leq 1 - \frac{p+\mu}{p-\mu}|b_p|. \tag{10}$$

Proof. We begin the prove by showing that the condition (7) is satisfied if the inequality (10) holds true for the coefficient of f defined in (3). Using (6) and (7), we have

$$\omega(z)=\left\{\frac{z\Big(H^{n,p}_{\alpha,\beta,\gamma}f(z)\Big)'-\overline{z\Big(G^{n,p}_{\alpha,\beta,\gamma}f(z)\Big)'}}{p\Big(H^{n,p}_{\alpha,\beta,\gamma}f(z)+\overline{G^{n,p}_{\alpha,\beta,\gamma}f(z)}\Big)}\right\}=\frac{M(z)}{N(z)}, \tag{11}$$

where

$$M(z)=\frac{z}{p}\Big(H^{n,p}_{\alpha,\beta,\gamma}f(z)\Big)'-\overline{\frac{z}{p}\Big(G^{n,p}_{\alpha,\beta,\gamma}f(z)\Big)'}$$

and

$$N(z)=H^{n,p}_{\alpha,\beta,\gamma}f(z)+\overline{G^{n,p}_{\alpha,\beta,\gamma}f(z)}.$$

Here, we recall that $\Re(\omega)>\dfrac{\mu}{p}$ if and only if

$$|p-\mu+p\omega|\geq|p+\mu-p\omega|.$$

Then, from (11), it suffices to show that

$$|M(z)+(p-\mu)N(z)|\geq|M(z)-(p+\mu)N(z)|$$

and

$$|M(z)+(p-\mu)N(z)|-|M(z)-(p+\mu)N(z)|\geq 0. \tag{12}$$

Having substituted for the values of $M(z)$ and $N(z)$ in (12), we obtain

$$|M(z)+(p-\mu)N(z)|-|M(z)-(p+\mu)N(z)|\geq$$

$$\geq 2(p-\mu)|z|^p-\sum_{k=1}^{\infty}2(k+p)\left(\frac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n|a_{k+p}|\,|z|^{k+p}-$$

$$-\sum_{k=0}^{\infty}2(k+p)\left(\frac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n|b_{k+p}|\,|z|^{k+p}+$$

$$+2\mu\sum_{k=1}^{\infty}\left(\frac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n|a_{k+p}|\,|z|^{k+p}-$$

$$-2\mu\sum_{k=0}^{\infty}\left(\frac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n|b_{k+p}|\,|z|^{k+p},$$

that is,

$$|M(z)+(p-\mu)N(z)|-|M(z)-(p+\mu)N(z)|\geq$$

$$\geq 2(p-\mu)|z|^p-\sum_{k=1}^{\infty}2(k+p-\mu)Y^n|a_{k+p}|\,|z|^{k+p}-\sum_{k=0}^{\infty}2(k+p+\mu)Y^n|b_{k+p}|\,|z|^{k+p}\geq$$

$$\geq 2(p-\mu)|z|^p\left\{1-\sum_{k=1}^{\infty}\frac{k+p-\mu}{p-\mu}\,Y^n|a_{k+p}|-\sum_{k=0}^{\infty}\frac{k+p+\mu}{p-\mu}\,Y^n|b_{k+p}|\right\}\geq$$

$$\geq 2(p-\mu)|z|^p\left\{1-\frac{p+\mu}{p-\mu}|b_p|-\sum_{k=1}^{\infty}\left[\frac{k+p-\mu}{p-\mu}\,|a_{k+p}|-\frac{k+p+\mu}{p-\mu}\,|b_{k+p}|\right]Y^n\right\}\geq 0$$

by virtue of inequality (10) where $Y=\dfrac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}$.

This shows that $f\in\mathcal{H}_{\mu}^{n,p}(\alpha,\beta,\gamma)$.

Theorem 2.1 is proved.

***Corollary* 2.1.** *Let* $f(z)\in\mathcal{H}_{\mu}^{n,1}(\alpha,\beta,\gamma)$. *Then, for* $0\leq\mu<1,\ \alpha>0,\ \beta,\gamma\geq 0$ *and* $\alpha+\beta+\gamma>$ >0,

$$\sum_{k=2}^{\infty}\left[\frac{k-\mu}{1-\mu}\,|a_k|+\frac{k+\mu}{1-\mu}\,|b_k|\right]\left(\frac{\alpha+\beta k+\gamma k^2}{\alpha+\beta+\gamma}\right)^n\leq 1-\frac{1+\mu}{1-\mu}\,|b_1|. \tag{13}$$

***Corollary* 2.2.** *Let* $f(z)\in\mathcal{H}_{0}^{n,1}(\alpha,\beta,\gamma)$. *Then, for* $\alpha>0,\ \beta,\gamma\geq 0$ *and* $\alpha+\beta+\gamma>0$,

$$\sum_{k=2}^{\infty}[k|a_k|+k|b_k|]\left(\frac{\alpha+\beta k+\gamma k^2}{\alpha+\beta+\gamma}\right)^n\leq 1-|b_1|.$$

***Corollary* 2.3.** *Let* $f(z)\in\mathcal{H}_{0}^{0,1}(\alpha,\beta,\gamma)$. *Then, for* $\alpha>0,\ \beta,\gamma\geq 0$ *and* $\alpha+\beta+\gamma>0$,

$$\sum_{k=2}^{\infty}[k|a_k|+k|b_k|]\leq 1-|b_1|. \tag{14}$$

Next we obtain both the necessary and sufficient condition for function f of the form (3) given the condition (8).

Theorem 2.2. $f\in\mathcal{V}_{\mathcal{H}}^{n,p}(\alpha,\beta,\gamma,\mu)$ *if and only if*

$$\sum_{k=1}^{\infty}\left[\frac{k+p-\mu}{p-\mu}\,|a_{k+p}|+\frac{k+p-2\mu}{p-\mu}\,|b_{k+p}|\right]\left(\frac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n\leq 1-\frac{p+\mu}{p-\mu}\,|b_p| \tag{15}$$

for $p\in\mathbb{N},\ 0\leq\mu<p,\ \alpha>0,\ \beta,\ \gamma\geq 0$ *and* $\alpha+\beta+\gamma>0$.

Proof. Since $\mathcal{V}_{\mathcal{H}}^{n,p}(\alpha,\beta,\gamma,\mu)\subset\mathcal{H}_{\mu}^{n,p}(\alpha,\beta,\gamma)$. Then the necessary condition part of the theorem shall be established. Suppose that $f\in\mathcal{V}_{\mathcal{H}}^{n,p}(\alpha,\beta,\gamma,\mu)$, then appealing to (6) and (7), we have that

$$\Re\left\{\left[\frac{z\Big(H_{\alpha,\beta,\gamma}^{n,p}f(z)\Big)'-\overline{z\Big(G_{\alpha,\beta,\gamma}^{n,p}f(z)\Big)'}}{H_{\alpha,\beta,\gamma}^{n,p}f(z)+\overline{G_{\alpha,\beta,\gamma}^{n,p}f(z)}}\right]-\mu\right\}\geq 0.$$

Equivalently, we can write that

$$\Re\left\{\frac{pz^p+\sum_{k=1}^{\infty}(k+p)Y^na_{k+p}z^{k+p}-\sum_{k=0}^{\infty}(k+p)Y^nb_{k+p}\overline{z^{k+p}}}{z^p+\sum_{k=1}^{\infty}Y^na_{k+p}z^{k+p}+\sum_{k=0}^{\infty}Y^nb_{k+p}\overline{z^{k+p}}}-\mu\right\}\geq$$

$$\geq \left\{ \frac{[(p-\mu)-(p+\mu)|b_p|]}{(1+|b_p|)+\sum_{k=1}^{\infty} Y^n |a_{k+p}|\,|z^k| + \left|\dfrac{\overline{z}}{z}\right|^p \displaystyle\sum_{k=1}^{\infty} Y^n |b_{k+p}| |\overline{z^k}|} \right\} -$$

$$-\left\{ \frac{-\sum_{k=1}^{\infty}(k+p-\mu)Y^n |a_{k+p}||z^k|}{(1+|b_p|)+\sum_{k=1}^{\infty} Y^n |a_{k+p}||z^k| + \left|\dfrac{\overline{z}}{z}\right|^p \sum_{k=1}^{\infty} Y^n |b_{k+p}| |\overline{z^k}|} \right\} -$$

$$-\left\{ \frac{-\left|\dfrac{\overline{z}}{z}\right|^p \sum_{k=1}^{\infty}(k+p+\mu)Y^n |b_{k+p}||\overline{z^k}|}{(1+|b_p|)+\sum_{k=1}^{\infty} Y^n |a_{k+p}||z^k| + \left|\dfrac{\overline{z}}{z}\right|^p \sum_{k=1}^{\infty} Y^n |b_{k+p}| |\overline{z^k}|} \right\} \geq 0$$

and Y is as earlier defined.

The above condition must hold for all the values of z such that $|z| = r < 1$. With σ as in (9), we obtain

$$\frac{[(p-\mu)-(p+\mu)|b_p|] - \sum_{k=1}^{\infty}[(k+p-\mu)|a_{k+p}| - (k+p+\mu)|b_{k+p}|]}{(1+|b_p|)+\sum_{k=1}^{\infty}[|a_{k+p}|+|b_{k+p}|]\left(\dfrac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n r^k} \times$$

$$\times \frac{\left(\dfrac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n r^k}{(1+|b_p|)+\sum_{k=1}^{\infty}\left[|a_{k+p}|+|b_{k+p}|\right]\left(\dfrac{\alpha+\beta(k+p)+\gamma(k+p)^2}{\alpha+\beta p+\gamma p^2}\right)^n r^k} \geq 0. \tag{16}$$

Suppose that (15) does not hold, then the numerator in (16) is negative for r sufficiently close to 1. Thus there exists point $z_0 = r_0,\ 0 < r_0 < 1$ for which the quotient in (16) is negative and this negates our assumption that $f \in \mathcal{V}_{\mathcal{H}}^{n,p}(\alpha,\beta,\gamma,\mu)$. Therefore, we can conclude that it is necessary as well as sufficient that (15) holds true whenever $f \in \mathcal{V}_{\mathcal{H}}^{n,p}(\alpha,\beta,\gamma,\mu)$ and this ends the proof of Theorem 2.2.

Next we obtain both the growth and distortion results.

Theorem 2.3. *Let $f \in \mathcal{H}_{\mu}^{n,p}(\alpha,\beta,\gamma)$. Then*

$$\left|f(z)\right| \geq (1-|b_p|)r^p - \left[\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}|b_p|\right]\left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)^n r^{p+1}$$

or

$$|f(z)| \leq (1+|b_p|)r^p + \left[\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}|b_p|\right]\left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)^n r^{p+1}.$$

Proof. From (3), we have

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} + \overline{\sum_{k=0}^{\infty} b_{k+p} z^{k+p}}.$$

Now

$$\left|f(z)\right| \le (1+|b_p|)r^p + \frac{p-\mu}{1+p-\mu}\left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)^n \times$$

$$\times \sum_{k=1}^{\infty}\left[\frac{k+p-\mu}{p-\mu}|a_{p+k}| + \frac{k+p-\mu}{p-\mu}\left|b_{p+k}\right|\right] Y^n r^{p+1} \le$$

$$\le (1+|b_p|)r^p + \left[\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}\left|b_p\right|\right]\left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)^n r^{p+1},$$

where Y is as defined earlier.

Theorem 2.3 is proved.

Theorem 2.4. *Let* $f \in \mathcal{H}^{n,p}_{\mu}(\alpha,\beta,\gamma)$. *Then*

$$\left|f'(z)\right| \ge p(1-|b_p|)r^{p-1} - \left[\frac{(1+p)(p-\mu)}{1+p-\mu} - \frac{(1+p)(p+\mu)}{1+p-\mu}|b_p|\right]\times$$

$$\times\left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)^n r^p$$

or

$$\left|f'(z)\right| \le p(1+|b_p|)r^{p-1} + \left[\frac{(1+p)(p-\mu)}{1+p-\mu} - \frac{(1+p)(p+\mu)}{1+p-\mu}|b_p|\right]\times$$

$$\times\left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)^n r^p.$$

Proof is much similar to that of Theorem 2.2.

3. Application of fractional calculus. Given function $f(z)$ of the form (1). The fractional integral of order ϵ, $0<\epsilon\le 1$, is defined such that

$$D_z^{-\epsilon} f(z) = \frac{1}{\Gamma(\epsilon)}\int_0^z \frac{f(t)}{(z-t)^{1-\epsilon}}\,dt, \tag{17}$$

where $f(z)$ is analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\epsilon-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Similarly, the fractional derivative of order ϵ, $0\le\epsilon<1$, is given by

$$D_z^{\epsilon} f(z) = \frac{1}{\Gamma(1-\epsilon)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\epsilon}}\,dt, \tag{18}$$

where $f(z)$ is as defined above and the multiplicity of $(z-t)^{-\epsilon}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$. Interestingly both (17) and (18) have the series representations

$$D_z^{-\epsilon} f(z) = \frac{1}{\Gamma(2+\epsilon)} z^{\epsilon+1} + \sum_{k=2}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1+\epsilon)} a_k z^{k+\epsilon}$$

and

$$D_z^{\epsilon} f(z) = \frac{1}{\Gamma(2-\epsilon)} z^{1-\epsilon} + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\epsilon)}\, a_k z^{k+\epsilon},$$

respectively (see [6, 7, 9, 11]).

Theorem 3.1. *Let $f(z)$ be of the form* (3). *If $f \in \mathcal{H}_{\mu}^{n,p}(\alpha,\beta,\gamma)$, then*

$$\left|D_z^{-\epsilon} f(z)\right| \le \frac{\Gamma(p+1)|z|^{p+\epsilon}}{\Gamma(p+1+\epsilon)} \left\{ \left(1+|b_p|\right) + \frac{p+1}{p+1+\epsilon}\left(\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{p+1+\epsilon}|b_p|\right) X^n |z| \right\}$$

and

$$\left|D_z^{-\epsilon} f(z)\right| \ge \frac{\Gamma(p+1)|z|^{p+\epsilon}}{\Gamma(p+1+\epsilon)} \left\{ \left(1-|b_p|\right) - \frac{p+1}{p+1+\epsilon}\left(\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}|b_p|\right) X^n |z| \right\},$$

where

$$X = \left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right).$$

Proof. Following the representation of $D_z^{-\epsilon} f(z)$, we have

$$D_z^{-\epsilon} f(z) = \frac{1}{\Gamma(\epsilon)} \int_0^z (z-t)^{-(1-\epsilon)} [f(t) + \overline{g(z)}] dt =$$

$$= \frac{1}{\Gamma(\epsilon)} \left\{ \int_0^z (z-t)^{-(1-\epsilon)} \left(t^p + \sum_{k=1}^{\infty} a_{k+p} t^{k+p}\right) dt + \overline{\int_0^z (z-t)^{-(1-\epsilon)} \left(\sum_{k=0}^{\infty} a_{k+p} t^{k+p}\right) dt} \right\} =$$

$$= \frac{\Gamma(p+1)}{\Gamma(p+1+\epsilon)} z^{\epsilon+p} + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1+\epsilon)} a_{k+p} z^{k+p+\epsilon} + \sum_{k=0}^{\infty} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1+\epsilon)}\, b_{k+p} z^{k+p+\epsilon}.$$

Then

$$\frac{\Gamma(p+1+\epsilon)}{\Gamma(p+1)} z^{-\epsilon} D_z^{-\epsilon} f(z) =$$

$$= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(p+1+\epsilon)\Gamma(k+p+1)}{\Gamma(p+1)\Gamma(k+p+1+\epsilon)} a_{k+p} z^{k+p} + \sum_{k=0}^{\infty} \frac{\Gamma(p+1+\epsilon)\Gamma(k+p+1)}{\Gamma(p+1)\Gamma(k+p+1+\epsilon)} b_{k+p} z^{k+p}.$$

Simple computation of the above yields

$$\left|\frac{\Gamma(p+1+\epsilon)}{\Gamma(p+1)} z^{-\epsilon} D_z^{-\epsilon} f(z)\right| \le$$

$$\le |z|^p + |b_p||z|^p + \frac{p+1}{p+1+\epsilon}\left(\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}|b_p|\right) X^n |z|^{p+1},$$

where $X = \left(\dfrac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2}\right)$. Therefore,

$$\left|D_z^{-\epsilon}f(z)\right| \leq \frac{\Gamma(p+1)\left|z\right|^{p+\epsilon}}{\Gamma(p+1+\epsilon)}\left\{\left(1+\left|b_p\right|\right)+\frac{p+1}{p+1+\epsilon}\left(\frac{p-\mu}{1+p-\mu}-\frac{p+\mu}{1+p-\mu}|b_p|\right)X^n\left|z\right|\right\}$$

and

$$\left|D_z^{-\epsilon}f(z)\right| \geq \frac{\Gamma(p+1)\left|z\right|^{p+\epsilon}}{\Gamma(p+1+\epsilon)}\left\{\left(1-\left|b_p\right|\right)-\frac{p+1}{p+1+\epsilon}\left(\frac{p-\mu}{1+p-\mu}-\frac{p+\mu}{1+p-\mu}|b_p|\right)X^n\left|z\right|\right\}.$$

Theorem 3.1 is proved.

***Corollary* 3.1.** *Let $f(z)$ be of the form* (3). *If $f(z) \in \mathcal{H}_\mu^{n,1}(\alpha,\beta,\gamma)$, then*

$$\left|D_z^{-\epsilon}f(z)\right| \leq \frac{\left|z\right|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{\left(1+\left|b_1\right|\right)+\frac{2}{2+\epsilon}\left(\frac{1-\mu}{2-\mu}-\frac{1+\mu}{2-\mu}|b_1|\right)X^n\left|z\right|\right\}$$

and

$$\left|D_z^{-\epsilon}f(z)\right| \geq \frac{\left|z\right|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{\left(1-\left|b_1\right|\right)-\frac{2}{2+\epsilon}\left(\frac{1-\mu}{2-\mu}-\frac{1+\mu}{2-\mu}|b_1|\right)X^n\left|z\right|\right\},$$

where

$$X=\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right).$$

***Corollary* 3.2.** *Let $f(z)$ be of the form* (3). *If $f(z) \in \mathcal{H}_0^{n,1}(\alpha,\beta,\gamma)$, then*

$$\left|D_z^{-\epsilon}f(z)\right| \leq \frac{\left|z\right|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{\left(1+\left|b_1\right|\right)+\frac{1}{2+\epsilon}(1-|b_1|)X^n\left|z\right|\right\}$$

and

$$\left|D_z^{-\epsilon}f(z)\right| \geq \frac{\left|z\right|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{\left(1-\left|b_1\right|\right)-\frac{1}{2+\epsilon}(1-|b_1|)X^n\left|z\right|\right\},$$

where

$$X=\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right).$$

***Corollary* 3.3.** *Let $f(z)$ be of the form* (3). *If $f(z) \in \mathcal{H}_0^{0,1}(\alpha,\beta,\gamma)$, then*

$$\left|D_z^{-\epsilon}f(z)\right| \leq \frac{\left|z\right|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{\left(1+\left|b_1\right|\right)+\frac{1}{2+\epsilon}(1-|b_1|)z\right|\right\}$$

and

$$\left|D_z^{-\epsilon}f(z)\right| \geq \frac{\left|z\right|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{\left(1-\left|b_1\right|\right)-\frac{1}{2+\epsilon}(1-|b_1|)\left|z\right|\right\}.$$

Theorem 3.2. *Let* $f(z)$ *be of the form* (3). *If* $f \in \mathcal{H}_{\mu}^{n,p}(\alpha,\beta,\gamma)$, *then*

$$|D_z^{\epsilon} f(z)| \leq \frac{\Gamma(p+1)|z|^{p-\epsilon}}{\Gamma(p+1-\epsilon)} \left\{ (1+|b_p|) + \frac{p+1}{p+1-\epsilon} \left(\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}\, b_p \right) X^n |z| \right\}$$

and

$$\left|D_z^{\epsilon} f(z)\right| \geq \frac{\Gamma(p+1)\left|z\right|^{p-\epsilon}}{\Gamma(p+1-\epsilon)} \left\{ \left(1-\left|b_p\right|\right) - \frac{p+1}{p+1-\epsilon} \left(\frac{p-\mu}{1+p-\mu} - \frac{p+\mu}{1+p-\mu}|b_p| \right) X^n \left|z\right| \right\},$$

where

$$X = \left(\frac{\alpha+\beta p+\gamma p^2}{\alpha+\beta(1+p)+\gamma(1+p)^2} \right).$$

Proof is similar to that of Theorem 3.1.

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