DOI: 10.37863/umzh.v74i9.6193

UDC 519.688

G. R. P. Teruel[1] (Dep. Math., IES Carrús, Elche-Alicante, Spain)

NEW FAST METHODS TO COMPUTE THE NUMBER OF PRIMES LESS THAN A GIVEN VALUE

НОВІ ШВИДКІ МЕТОДИ ОБЧИСЛЕННЯ КІЛЬКОСТІ ПРОСТИХ ЧИСЕЛ, МЕНШИХ ЗА ДЕЯКУ ЗАДАНУ ВЕЛИЧИНУ

The paper describes new fast algorithms for evaluating $\pi(x)$ inspired by the harmonic and geometric mean integrals that can be used on any pocket calculator. In particular, the formula $h(x)$ based on the harmonic mean is within ≈ 15 of the actual value for $3 \leq x \leq 10000$. The approximation verifies the inequality, $h(x) \leq \mathrm{Li}(x)$ and, therefore, is better than $\mathrm{Li}(x)$ for small x. We show that $h(x)$ and their extensions are more accurate than other famous approximations, such as Locker – Ernst's or Legendre's also for large x. In addition, we derive another function $g(x)$ based on the geometric mean integral that employs $h(x)$ as an input, and allows one to significantly improve the quality of this method. We show that $g(x)$ is within ≈ 25 of the actual value for $x \leq 50000$ (to compare $\mathrm{Li}(x)$ lies within ≈ 40 for the same range) and asymptotically $g(x) \sim \dfrac{x}{\ln x} \exp\left(\dfrac{1}{\ln x - 1}\right)$.

Описано нові швидкі алгоритми для обчислення $\pi(x)$ на основі інтегралів гармонічних та геометричних середніх, які можна використовувати на будь-якому кишеньковому калькуляторі. Зокрема, формула $h(x)$, що отримана на основі середнього гармонічного, знаходиться в межах ≈ 15 від фактичного значення для $3 \leq x \leq 10000$. Це наближення задовольняє нерівність $h(x) \leq \mathrm{Li}(x)$ і тому є кращим за $\mathrm{Li}(x)$ для малих x. Показано, що $h(x)$ та їхні розширення є точнішими за інші відомі наближення, такі як Локера – Ернста або Лежандра, і для великих x. Крім того, отримано ще одну функцію $g(x)$ на основі середньогеометричного інтеграла, яка використовує $h(x)$ як вхідну величину і дозволяє суттєво покращити такий метод. Показано, що $g(x)$ знаходиться в межах ≈ 25 від фактичного значення для $x \leq 50000$ (для порівняння, $\mathrm{Li}(x)$ знаходиться в межах ≈ 40 в тому ж самому діапазоні) і має таку асимптотику: $g(x) \sim \dfrac{x}{\ln x} \exp\left(\dfrac{1}{\ln x - 1}\right)$.

1. Introduction. As is widely known, the prime counting function $\pi(x)$ computes the number of primes less than or equal to a given number x. Since there are no primes ≤ 1, then $\pi(1) = 0$, there are two primes ≤ 3, so $\pi(3) = 2$. And so on. In the last two centuries, there have been different attempts to approximate $\pi(x)$ by a smooth and easy computable function. Already in 1808, Legendre [1] noticed that $\pi(x) \approx \dfrac{x}{\ln x - B}$, where he originally proposed $B = 1.08366\ldots$. Some years before, Gauss[2] had already observed that the logarithmic integral, $\mathrm{li}(x) = \displaystyle\int_0^x \frac{dt}{\ln t}$ approximates $\pi(x)$ quite accurately, but for small x it overestimates the number of primes less or equal to x. To solve this problem, in 1859 Riemann [3] introduced

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{1/n}) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k! k \zeta(k+1)},$$

where $\mu(n)$ is the Mobius function [4, 5], and the last form above for $R(x)$ is the Gram series which is the better way to calculate this function. Riemann's approximation turns out to be 10 times better than $\mathrm{li}(x)$ for $x \leq 10^9$ but has been proven to be worse infinitely often by Littlewood [6, 7]. Other approximations to $\pi(x)$ are Lehmer's formula, Mapes' method, or Meissel's formula [8 – 12].

[1] e-mail:gines.landau@gmail.com.

Currently, the best known approximation to $\pi(x)$ is due to A. V. Kulsha [14]:

$$\pi(x) \approx R(x) - \frac{1}{\ln x} + \frac{1}{\pi} \arctan \frac{\pi}{\ln x}.$$

In this work, we present and derive several other approximations, the first one is denoted as $h(x)$ and is based on the harmonic integral (continuous harmonic mean) of the function $1/\ln x$. We show that it provides an accurate approximation of $\pi(x)$, specially for small x. Asymptotically, it behaves as $x/(\ln x - 1)$ which turns out to be superior than other famous approximations such as Legendre's or Locker – Ernst's formula. In the following, we refer to the first method derived here as "harmonic approximation", and it should not be confused with the approximate method due to Locker – Ernst [16 – 18]: $\pi(x) \approx \dfrac{x}{h_x}$, where $h_x = H_x - 3/2$, with $H(x)$ the harmonic number. The other algorithm that we present in this letter to evaluate $\pi(x)$ is a function denoted as $g(x)$, based on the geometric continuous mean. The function $g(x)$ depends on $h(x)$, it is also a fast an easy computable method and provides a close approximation to $\pi(x)$ improving the results of $h(x)$.

This paper is organized as follows. In Section 2, we begin by rewriting the logarithmic integral in a form that makes explicit their dependence on the average value from calculus. Then we explore what happens if we replace in this formula the standard average value by other means, such as the harmonic mean or the geometric mean, which unlike the standard average, can be analytically computed. The approximation that arises by the first possibility is studied in detail, including some natural generalizations. In the last part of the work, the second possibility that arises by replacing the standard average by the geometric mean integral in the definition of $\mathrm{Li}(x)$ is considered. This is the subject of Section 3.

We should mention that the new methods presented in this letter are not meant to replace the current approximation methods of $\pi(x)$; they merely add to the toolbox of available techniques and approximations. However, they have the advantage of being smooth and easy computable functions that may be used on any pocket calculator.

2. The harmonic mean integral approximation. Let us begin by writing the standard definition of the offset logarithmic integral or Eulerian logarithmic integral

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}.$$

As is well-known, the offset logarithmic integral appears in estimates of the number of prime numbers less than a given value. In particular, the prime number theorem states that $\pi(x) \sim \mathrm{Li}(x)$ for large x, where $\pi(x)$ is again the number of primes smaller than or equal to x. For small x, it has been always found that $\mathrm{Li}(x) > \pi(x)$, namely, $\mathrm{Li}(x)$ overestimates the number of primes less or equal than x.

For our purposes in this letter, it is more convenient to rewrite $\mathrm{Li}(x)$ in the following form:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t} = (x-2)\left\langle \frac{1}{\ln t} \right\rangle_2^x, \tag{1}$$

where we have defined

$$\left\langle \frac{1}{\ln t} \right\rangle_2^x = \frac{1}{x-2} \int\limits_2^x \frac{dt}{\ln t},$$

as the standard average of the function $1/\ln t$ among 2 and x. Let us assume now that $f(t) \neq 0$ for all t in $[a, b]$. The harmonic integral of $f(t)$ can be defined by

$$\mathcal{H}_a^b(f) = \frac{b-a}{\displaystyle\int_a^b \frac{1}{f(t)}\, dt}.$$

Then the continuous harmonic mean of $f(t) = 1/\ln t$ among 2 and x is equal to the reciprocal of the average of the reciprocal of $1/\ln t$, that is,

$$\mathcal{H}_2^x \left(\frac{1}{\ln t} \right) = \frac{1}{\langle \ln t \rangle_2^x} = (x-2) \left(\int\limits_2^x \ln t\, dt \right)^{-1}. \tag{2}$$

Contrary to the case of the standard average of $1/\ln t$ which does not admit an elementary primitive, the harmonic integral admits a completely elementary primitive, which can be found integrating by parts, and is given by

$$\mathcal{H}_2^x \left(\frac{1}{\ln t} \right) = \frac{x-2}{x(\ln x - 1) - 2(\ln 2 - 1)}.$$

Theorem 2.1.

$$\mathrm{Li}(x) \geq \frac{(x-2)^2}{x(\ln x - 1) - 2(\ln 2 - 1)}. \tag{3}$$

Proof. We begin by noting that

$$\left\langle \frac{1}{\ln t} \right\rangle_2^x \geq \mathcal{H}_2^x \left(\frac{1}{\ln t} \right).$$

This is nothing but the generalization to the continuum of the condition $AM \geq HM$ of the discrete case, i.e., the arithmetic mean AM is always at least as large as the harmonic mean HM. This implies the inequality

$$\frac{1}{x-2} \int\limits_2^x \frac{dt}{\ln t} \geq \frac{x-2}{\displaystyle\int_2^x \ln t\, dt}.$$

Reordering the last inequality

$$\left(\int\limits_2^x \frac{dt}{\ln t} \right) \left(\int\limits_2^x \ln t\, dt \right) \geq (x-2)^2.$$

The second integral of the left-hand side can be computed analytically

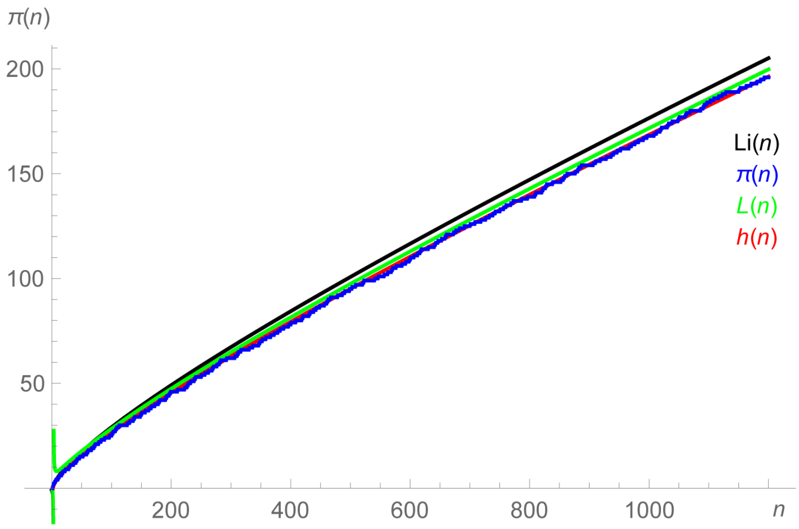

Fig. 1. Representation of $\pi(n)$ (blue) and several of their approximations, $\mathrm{Li}(n)$ (black), $L(n)$ (green), and the harmonic approximation $h(n)$ (red).

$$\left(\int_2^x \frac{dt}{\ln t}\right)\Big[t\ln t - t\Big]_2^x \geq (x-2)^2.$$

Finally we obtain

$$\int_2^x \frac{dt}{\ln t} \geq \frac{(x-2)^2}{x(\ln x - 1) - 2(\ln 2 - 1)}.$$

Define the function of the right-hand side of Eq. (3)

$$h(x) = \frac{(x-2)^2}{x(\ln x - 1) - 2(\ln 2 - 1)}. \tag{4}$$

Since $\mathrm{Li}(x) > \pi(x)$ for small x, and given that $\mathrm{Li}(x) \geq h(x)$, then it seems natural to wonder if $h(x)$ could be a good approximation of $\pi(x)$. In order to test $h(x)$ as a possible approximation of $\pi(x)$, we have presented here some plots. In Fig. 1, we have plotted the prime-counting function $\pi(x)$ together with $\mathrm{Li}(x)$, Legendre's $L(x)$ and $h(x)$ until $x \sim 1200$. Fig. 2 presents a graph of the compared behavior of the harmonic integral approximation to Locker – Ernst's until $x \sim 10000$.

Before concluding this section, let us mention that there are several possible generalizations of $h(x)$ that are quite natural. The first case are the functions of the type

$$\mathfrak{h}(x) = \frac{(x-2)^2}{x(\ln x - 1) + A}$$

with A an arbitrary constant. On the other hand, we have the family of functions

$$\mathfrak{p}_p(x) := (x-p)\,\mathcal{H}_p^x(f) = (x-p)^2\left(\int_p^x \ln t\, dt\right)^{-1} = \frac{(x-p)^2}{x(\ln x - 1) - p(\ln p - 1)},$$

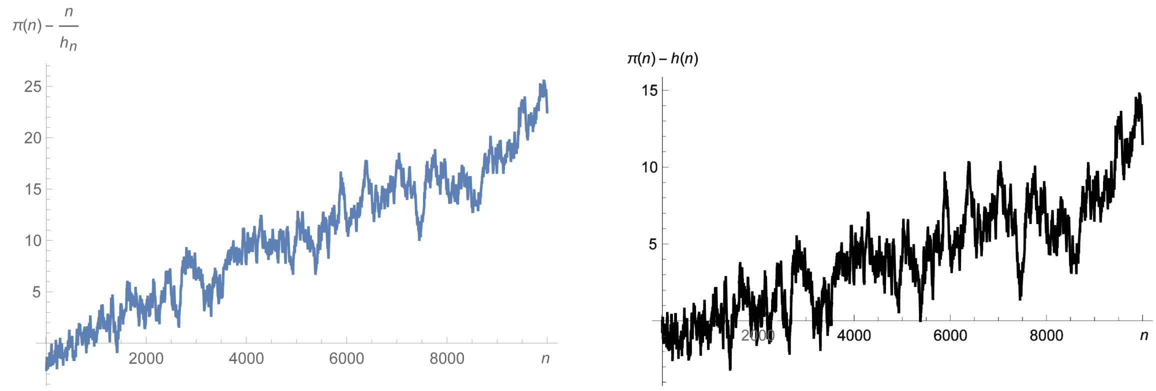

Fig. 2. Comparison among $\pi(n) - \dfrac{n}{h_n}$ (blue) and $\pi(n) - h(n)$ (black), where n/h_n is Locker – Ernst's approximation and $h(n)$ the harmonic mean integral approximation. $\pi(n) - h(n)$ has more zeros than $\pi(n) - \dfrac{n}{h_n}$. Moreover, $h(n)$ is within ≈ 15 of the actual value for $3 \leq x \leq 10000$, while Locker – Ernst's is within ≈ 25 for the same range.

where p is any prime that satisfies the condition $p < x$. All these functions present a discontinuity at $x = p$. The harmonic integral approximation $h(x)$ corresponds to the case $p = 2$, i.e., $h(x) = \mathfrak{p}_2(x)$. Regarding the case $p = 3$, we have

$$\mathfrak{p}_3(x) := (x-3)\, \mathcal{H}_3^x(f) = (x-3)^2 \left(\int\limits_3^x \ln t \, dt \right)^{-1} = \frac{(x-3)^2}{x(\ln x - 1) - 0.29583686601}.$$

On the other hand, the limit $x \to p$ of $\mathfrak{p}_p(x)$ can be computed with the aid of the L'Hopital – Bernoulli rule

$$\lim_{x \to p} \frac{(x-p)^2}{x(\ln x - 1) - p(\ln p - 1)} = \lim_{x \to p} \frac{2(x-p)}{\ln x} = 0.$$

It can be shown that $\pi(x) \sim \mathfrak{p}_3(x)$, i.e., this function is also a good approximation of $\pi(x)$, in fact the relative error is even lower that that provided by $h(x)$ for small x. We show now that asymptotically, $\mathfrak{p}_p(x) \sim x/(\ln x - 1)$, which is a better asymptotic behavior than Legendre's function (it is well-known that 1 turns out to be a better constant for large x than $1.08366\ldots$).

Lemma 2.1. *For $x >> p$,*

$$\mathfrak{p}_p(x) \sim \frac{x}{\ln x - 1}.$$

Proof. When $x >> p$, the factor $p(\ln p - 1)$ can be neglected with respect to $x(\ln x - 1)$

$$\frac{(x-p)^2}{x(\ln x - 1) - p(\ln p - 1)} \sim \frac{x}{\ln x - 1}.$$

Theorem 2.2.

$$\lim_{x \to \infty} \frac{\pi(x)}{\mathfrak{p}_p(x)} = 1.$$

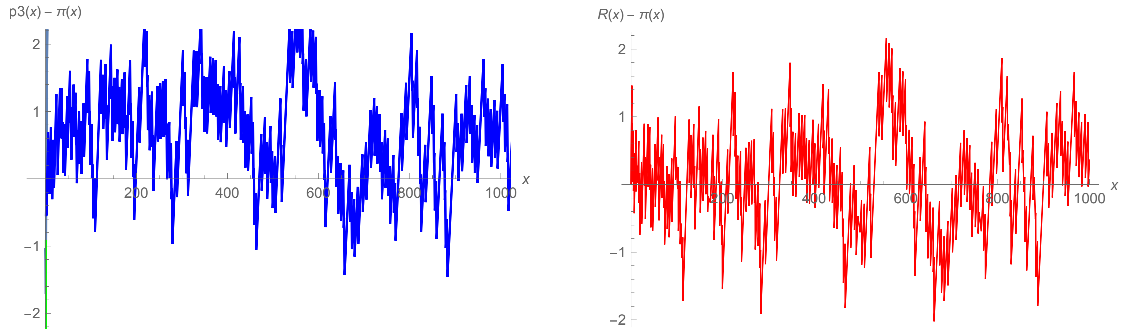

Fig. 3. Representation of $\mathfrak{p}_3(x) - \pi(x)$ and $R(x) - \pi(x)$, where $R(x)$ is the Riemann function.

Proof. First, we use the result of the previous lemma and the factorization

$$\frac{x}{\ln x - 1} = \frac{x}{\ln x}\left(1 - \frac{1}{\ln x}\right)^{-1}.$$

Now, we can employ the binomial theorem to expand $\left(1 - \dfrac{1}{\ln x}\right)^{-1}$:

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathfrak{p}_p(x)} \sim \lim_{x\to\infty} \frac{\pi(x)}{\left[\dfrac{x}{\ln x}\left(1 + \dfrac{1}{\ln x} + \dfrac{1}{(\ln x)^2} + \ldots\right)\right]} \sim \lim_{x\to\infty} \frac{\pi(x)}{\left[\dfrac{x}{\ln x}\right]}.$$

Finally, the prime number theorem (PNT) states that

$$\lim_{x\to\infty} \frac{\pi(x)}{\left[\dfrac{x}{\ln x}\right]} = 1.$$

In Fig. 3, we have represented the quantities $\mathfrak{p}_3(x) - \pi(x)$ and $R(x) - \pi(x)$, where $R(x)$ is the Riemann function. Although $R(x) - \pi(x)$ has more zeros for $10 \le x \le 1000$, $\mathfrak{p}_3(x)$ is not much worse, and it is faster and easier to use on any scientific calculator. Notice also that $\mathfrak{p}_2 > \mathfrak{p}_p$ for $p > 2$, namely, in a representation, the graph of all of these functions would be located below $\mathfrak{p}_2$. In the next section, we will see that it is possible to significantly improve the quality of the harmonic approximation $h(x)$ and their family $\mathfrak{p}_p(x)$.

3. Improving $h(x)$ by the geometric mean integral approximation. In Section 2 we showed that, by virtue of Eq. (1), the offset logarithmic integral $\mathrm{Li}(x)$ can be rewritten as $(x-2)\left\langle \dfrac{1}{\ln t} \right\rangle_2^x$ this form of expressing $\mathrm{Li}(x)$ makes explicit their dependence on the standard average. It is therefore natural to wonder what happens if one replaces in the latter formula the standard average by other possible means. Having studied the case of the harmonic mean in the previous section, in this last part of the work we investigate the replacement of the standard average by the geometric mean in the definition of $\mathrm{Li}(x)$. Indeed, if $f(t) > 0$ for all t in $[a, b]$ and f is integrable on $[a, b]$, then the geometric mean (product integral) of $f(t)$ exists, and its definition is given by

$$\mathcal{G}_a^b(f) = \exp\left(\frac{\displaystyle\int_a^b \ln f(t)\, dt}{b - a}\right). \tag{5}$$

Notice that $\mathcal{G}_a^b(f)$ satisfies the inequality

$$\langle f\rangle_a^b \geq \mathcal{G}_a^b(f) \geq \mathcal{H}_a^b(f). \tag{6}$$

This is again the continuous analog of the inequality $AM \geq GM \geq HM$ of the discrete case. A consequence is that

$$\mathrm{Li}(x) = \int\limits_2^x \frac{dt}{\ln t} = (x-2)\left\langle \frac{1}{\ln t}\right\rangle_2^x \geq (x-2)\,\mathcal{G}_2^x\left(\frac{1}{\ln t}\right), \tag{7}$$

where $\mathcal{G}_2^x\left(\frac{1}{\ln t}\right)$ is given by

$$\mathcal{G}_2^x\left(\frac{1}{\ln t}\right) = \exp\left(\frac{\int_2^x \ln\left(\frac{1}{\ln t}\right) dt}{x-2}\right).$$

Integrating by parts, the integral that appears in the argument of the exponential can be expressed in terms of a function, a constant, and the logarithmic integral

$$\int\limits_2^x \ln\left(\frac{1}{\ln t}\right) dt = x\ln\left(\frac{1}{\ln x}\right) - 2\ln\left(\frac{1}{\ln 2}\right) + \int\limits_2^x \frac{dt}{\ln t}.$$

By virtue of this relation, we have found that $\mathcal{G}_2^x\left(\frac{1}{\ln t}\right)$ depends on $\mathrm{Li}(x)$ as

$$\mathcal{G}_2^x\left(\frac{1}{\ln t}\right) = \exp\left(\frac{x\ln\left(\frac{1}{\ln x}\right) - 2\ln\left(\frac{1}{\ln 2}\right) + \mathrm{Li}(x)}{x-2}\right).$$

Then, combining all these results, the inequality of Eq. (7) turns out

$$\mathrm{Li}(x) \geq (x-2)\exp\left(\frac{2\ln(\ln 2) - x\ln(\ln x) + \mathrm{Li}(x)}{x-2}\right). \tag{8}$$

Notice that, since $\mathrm{Li}(x) \geq h(x)$, where $h(x)$ is the harmonic mean integral approximation given by Eq. (4), it is also true that

$$\mathrm{Li}(x) \geq (x-2)\exp\left(\frac{2\ln(\ln 2) - x\ln(\ln x) + h(x)}{x-2}\right). \tag{9}$$

Therefore, the functions of the right-hand sides of Eqs. (8), (9) are both smooth functions with a closed and compact form that can be studied as two other possible algorithms for evaluating $\pi(x)$.

Define the couple of functions $g_{\mathrm{Li}}(x)$ and $g_h(x)$, respectively, as

$$g_{\mathrm{Li}}(x) = (x-2)\exp\left(\frac{2\ln(\ln 2) - x\ln(\ln x) + \mathrm{Li}(x)}{x-2}\right), \tag{10}$$

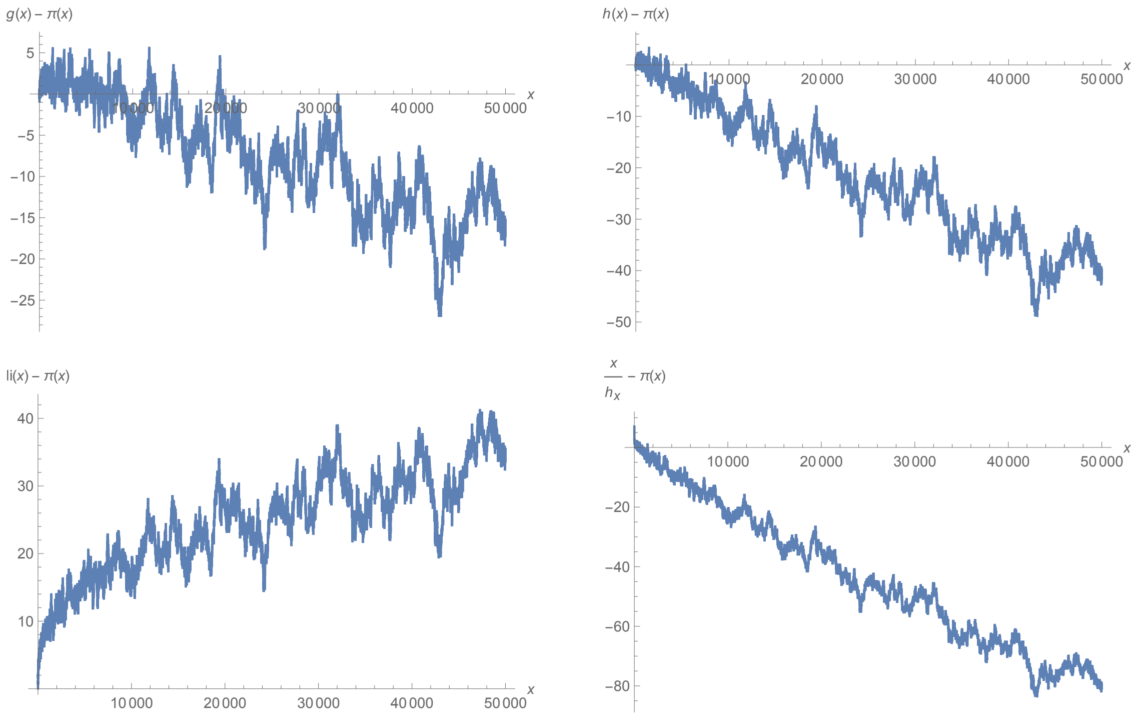

Fig. 4. Representation of the differences to $\pi(x)$ for $2 < x \leq 50000$, of four different approximations, where $g(x)$, $h(x)$ and x/h_x represent the geometric mean, the harmonic mean, and Locker – Ernst's approximations, respectively.

$$g_h(x) = (x-2)\exp\left(\frac{2\ln(\ln 2) - x\ln(\ln x) + h(x)}{x-2}\right). \tag{11}$$

Given that $\mathrm{Li}(x) \geq h(x)$, it automatically follows an inequality among $g_h(x)$, $g_{\mathrm{Li}}(x)$,

$$g_{\mathrm{Li}}(x) \geq g_h(x).$$

The pair of functions $g_{\mathrm{Li}}(x)$, $g_h(x)$ employ the couple $\mathrm{Li}(x)$, $h(x)$ as inputs, but improve their results for a wide range of values. For instance, in order to test the quality of $g_h(x)$ as approximation of $\pi(x)$, in Fig. 4 we have plotted the differences $g_h(x) - \pi(x)$, $h(x) - \pi(x)$, $\mathrm{Li}(x) - \pi(x)$ and $x/h_x - \pi(x)$, for $2 < x \leq 50000$. The superiority of $g_h(x)$ over the rest of approximations for such domain turns out to be manifest.

Before concluding, let us investigate the asymptotic behavior of $g_h(x)$. For this purpose, it is convenient to factorize $g_h(x)$ in the form

$$g_h(x) = (x-2)\exp\left(\frac{2\ln(\ln 2) - x\ln(\ln x)}{x-2}\right)\exp\left(\mathcal{H}_2^x\left(\frac{1}{\ln t}\right)\right), \tag{12}$$

where we have used Eq. (2). Now we provide a proof that $g_h(x)$ approaches $\pi(x)$ for $x \to \infty$.

Theorem 3.1. *Let* $\pi(x)$, $g_h(x)$ *be the prime counting function and the function defined by* Eq. (11), *respectively. Then*

$$\lim_{x\to\infty} \frac{\pi(x)}{g_h(x)} = 1.$$

Proof. With the aid of the factorization given by Eq. (12), it is easy to see that asymptotically,

$$g_h(x) \sim x\, e^{\frac{-x\ln(\ln x)}{x}}\, e^{\frac{1}{\ln x - 1}} = \frac{x}{\ln x}\, e^{\frac{1}{\ln x - 1}}.$$

Then, given that for sufficiently large x, $e^{\frac{1}{\ln x-1}} \sim 1$, we have

$$\lim_{x\to\infty} \frac{\pi(x)}{g_h(x)} \sim \lim_{x\to\infty} \frac{\pi(x)}{\left[\dfrac{x}{\ln x}\, e^{\frac{1}{\ln x-1}}\right]} \sim \lim_{x\to\infty} \frac{\pi(x)}{\left[\dfrac{x}{\ln x}\right]}.$$

Finally, according to the PNT

$$\lim_{x\to\infty} \frac{\pi(x)}{\left[\dfrac{x}{\ln x}\right]} = 1.$$

On the other hand, by virtue of Eq. (6) and the results obtained in this section, it is clear that, $\mathrm{Li}(x) \geq g_{\mathrm{Li}}(x) \geq g_h(x) \geq h(x)$.

Let us briefly mention that a natural generalization of $g_{\mathrm{Li}}(x)$ can also be written down using (5)

$$(x-p)\,\mathcal{G}_p^x\left(\frac{1}{\ln t}\right) = (x-p)\exp\left(\frac{\displaystyle\int_p^x \ln\left(\frac{1}{\ln t}\right) dt}{x-p}\right) =$$

$$= (x-p)\exp\left(\frac{p\ln(\ln p) - x\ln(\ln x) + \displaystyle\int_p^x \frac{dt}{\ln t}}{x-p}\right),$$

where p is an arbitrary prime subjected to the constraint $x > p$. For $p = 2$, we naturally recover the case of the function $g_{\mathrm{Li}}(x)$.

It can be shown that $g_h(x) \sim \dfrac{x}{\ln x}\, e^{\frac{1}{\ln x-1}}$, is a more accurate approximation than $h(x)$, Legendre's $L(x)$ and any function that asymptotically converges to $f(x) = x/(\ln x - 1)$. The same also applies to $g_{\mathrm{Li}}(x)$, which is even better, although to compute $g_{\mathrm{Li}}(x)$ we need to know first the value of $\mathrm{Li}(x)$, and then insert it into Eq. (10).

As a quick example, for $x = 10^{27}$ (the current record is $x = 10^{29}$) we have $g_h(10^{27}) = 1.63500982221 \times 10^{25}$, $L(10^{27}) = 1.63703262434 \times 10^{25}$, $h(10^{27}) = 1.63479370652 \times 10^{25}$, being the actual value $\pi(10^{27}) = 1.6352460426841680446427399 \times 10^{25}$. The prediction of $g_h(x)$ is not as good as that provided by $\mathrm{Li}(x)$, but it gives a surprisingly accurate result for such an algebraically simple function. In fact, the function $g_h(x)$ does have some advantages over $\mathrm{Li}(x)$; unlike $\mathrm{Li}(x)$, $g_h(x)$ can be expressed in closed form, which renders it fairly easy to evaluate $g_h(x)$ on any scientific calculator. Moreover, since the arithmetic mean is always at least as large as the geometric mean, we always have that $\mathrm{Li}(x) \geq g_h(x)$; it is well-known that $\mathrm{Li}(x)$ overestimates $\pi(x)$ for small x, therefore $g_h(x)$ can serve as a much more accurate approximation to $\pi(x)$ for such values of x.

4. Summary and conclusions. In this paper, we have added some new fast methods to the toolbox of available approximations of $\pi(x)$, which are very easy to use on any pocket calculator. A natural question arises: which method is better for approximating $\pi(x)$? It depends on the domain of values of interest. Our results show that $g_h(x)$, for instance, is a much better formula than $\mathrm{Li}(x)$ until $x \sim 10^6$, and is much faster to use. However, if one requires high asymptotic precision, then $\mathrm{Li}(x)$ is a better formula, although $g_h(x)$, $h(x)$ are also good, and superior than other fast methods based on simple functions like Legendre's or Locker – Ernst's.

De la Vallée-Poussin proved that $\mathrm{Li}(x)$ is better approximation of $\pi(x)$ than any rational function of x and $\ln x$ for large x. $h(x)$ is a rational function of x and $\ln x$, and therefore it will be inferior than $\mathrm{Li}(x)$ in the long run, but $g_h(x)$ is not rational (contains an exponential). Unfortunately, we do not know if $g_h(x)$ can improve the results of $\mathrm{Li}(x)$ for sufficiently large x, because there is no record of the number of primes beyond $x = 10^{29}$, the current available record announced in 2022 by D. Baugh and K. Walisch.

References

1. A. M. Legendre, *Essai sur la théorie des nombres*, Courcier, Paris (1808).
2. C. F. Gauss, *Werke*, vol. II. Königliche Gesellschaft der Wissenschaften zu Göttingen, 444 – 447 (1863).
3. G. F. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Königl. Preuss. Akad. Wiss. Berlin, 671 – 680 (1859).
4. Hardy, G. H. Ramanujan, *Twelve lectures on subjects suggested by his life and work*, 3rd ed., Chelsea, New York (1999).
5. J. M. Borwein, D. M. Bradley, R. E. Crandall, *Computational strategies for the Riemann Zeta function*, J. Comput. and Appl. Math., **121**, 247 – 296 (2000).
6. E. W. Weisstein, *Gram series*; http://mathworld.wolfram.com/GramSeries.html.
7. A. E. Ingham, *Ch. 5 in the distribution of prime numbers*, Cambridge Univ. Press, New York (1990).
8. H. Riesel, *Lehmer's formula*, Prime Numbers and Computer Methods for Factorization, 2nd ed., Birkhäuser, Boston, MA (1994), p. 13 – 14.
9. D. C. Mapes, *Fast method for computing the number of primes less than a given limit*, Math. Comput., **17**, 179 – 185 (1963).
10. H. Riesel, *Mapes' method*, Prime Numbers and Computer Methods for Factorization, 2nd ed., Birkhäuser, Boston, MA (1994), p. 23.
11. E. D. F. Meissel, *Berechnung der Menge von Primzahlen, welche innerhalb der ersten Milliarde naturlicher Zahlen vorkommen*, Math. Ann., **25**, 251 – 257 (1885).
12. H. Riesel, *Meissel's formula*, Prime Numbers and Computer Methods for Factorization, 2nd ed., Birkhäuser, Boston, MA (1994), p. 12 – 13.
13. R. Séroul, *Meissel's formula*, § 8.7.3 in Programming for Mathematicians, Springer-Verlag, Berlin (2000), p. 179 – 181.
14. A. V. Kulsha, *Values of $\pi(x)$ and $\Delta(x)$ for various values of x*, Retrieved 2008-09-14.
15. C.-J. de la Vallée Poussin, *Recherches analytiques la théorie des nombres premiers*, Ann. Soc. Sci. Bruxelles, **20**, 183 – 256 (1896).
16. L. Locker-Ernst, *Bemerkung über die Verteilung der Primzahlen*, Elem. Math. (Basel), **14**, 1 – 5 (1959).
17. L. Panaitopol, *Several approximations of $\pi(x)$*, Math. Inequal. Appl., **2**, 317 – 324 (1999).
18. J. Havil, *Gamma: exploring Euler's constant*, Princeton Univ. Press, Princeton, NJ (2003).
19. C. K. Caldwell, *How many primes are there*?; https://primes.utm.edu/howmany.htmlbetter.
20. https://oeis.org/A006880

Received 28.06.20