DOI: 10.37863/umzh.v74i9.6222

UDC 511.3

S. K. Sekatskii[1] (Laboratory of Biological Electron Microscopy, IPHYS, Ecole Polytechnique Fédérale de Lausanne, Switzerland),
S. Beltraminelli (CERFIM, Research Center for Math. and Phys., Locarno, Switzerland)

SOME SIMPLEST INTEGRAL EQUALITIES EQUIVALENT TO THE RIEMANN HYPOTHESIS

ДЕЯКІ НАЙПРОСТІШІ ІНТЕГРАЛЬНІ РІВНОСТІ, ЩО ЕКВІВАЛЕНТНІ ГІПОТЕЗІ РІМАНА

We show that the following integral equalities are equivalent to the Riemann hypothesis for any real $a > 0$ and any real $0 < \epsilon < 1$, $\epsilon \neq 1$:

$$\int\limits_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{a+it}\,dt = -2\pi \ln \frac{a+\frac{1}{2}}{a},$$

$$\int\limits_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}\,dt = -\frac{2\pi}{1-\epsilon}\left(\left(a+\frac{1}{2}\right)^{1-\epsilon} - a^{1-\epsilon}\right).$$

Показано, що інтегральні рівності

$$\int\limits_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{a+it}\,dt = -2\pi \ln \frac{a+\frac{1}{2}}{a},$$

$$\int\limits_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}\,dt = -\frac{2\pi}{1-\epsilon}\left(\left(a+\frac{1}{2}\right)^{1-\epsilon} - a^{1-\epsilon}\right)$$

еквівалентні гіпотезі Рімана для довільного дійсного $a > 0$ і довільного дійсного $0 < \epsilon < 1$, $\epsilon \neq 1$.

1. Introduction. In recent papers [1, 2] (see also reprints [3]), we, together with Merlini, have established and proved an infinite number of integral equalities equivalent to the Riemann hypothesis (RH; see, e.g., [4] for the general discussion of the Riemann ζ-function). In particular, we have shown that Balazard – Saias – Yor equality [5] and Volchkov equality [6] are certain particular cases of our general approach.

The aim of this short note is to establish and briefly discuss what we believe possibly are the *simplest integral equalities* of this type.

2. Integral equalities. Let us introduce the function $g(z) = \dfrac{i}{\left(a+\left(z-\frac{1}{2}\right)\right)^{\epsilon}}$, where a is real positive, $i = \sqrt{-1}$ and $1 \geq \epsilon > 0$. We use our "standard" [1 – 3] contour C com-

[1] Corresponding author, e-mail: serguei.sekatski@epfl.ch.

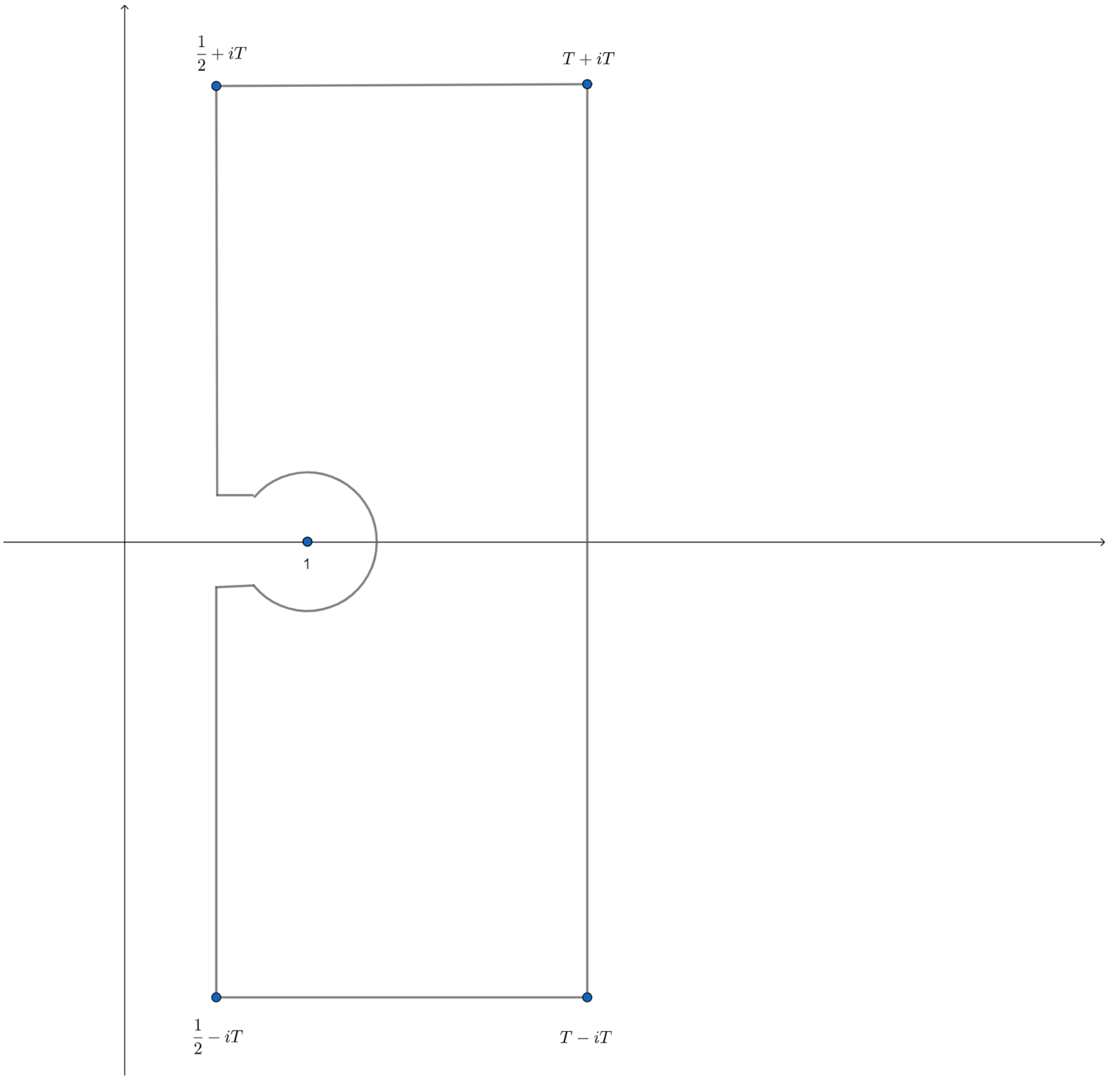

Fig. 1. The contour C used for the calculation of the integrals. The cut made to exclude the simple pole of the Riemann ζ-function is shown, see [1–3] for details.

posed by the segments of straight lines $\left[\frac{1}{2}-iT, \frac{1}{2}+iT\right]$, $\left[\frac{1}{2}+iT, T+iT\right]$, $[T+iT, T-iT]$ and $\left[T-iT, \frac{1}{2}-iT\right]$, where T is a real positive number larger than 1 (see Fig. 1), and consider a contour integral $\int_C g(z)\ln(\zeta(z))dz$. Consideration of similar integrals with $\epsilon > 1$ is given in [1].

Lemma. *Assume RH. Then, for $T \to +\infty$,*

$$\int\limits_C g(z)\ln(\zeta(z))dz = \int\limits_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}\,dt = -2\pi \int\limits_{\frac{1}{2}}^{1} \frac{dx}{\left(a+\left(x-\frac{1}{2}\right)\right)^{\epsilon}}.$$

Proof. To calculate the contour integral at question we use the generalized Littlewood theorem concerning contour integrals of the logarithm of an analytical function [1 – 3]. This theorem expresses the contour integral value via the sum over residues given by the poles of the function $g(z)$ lying inside the contour, and the sum of the integrals $\int_{\frac{1}{2}+it}^{\sigma+it} g(z)dz$, where $\sigma+it$, $\sigma>\frac{1}{2}$, is a zero or the pole (these contributions have different signs) of the Riemann ζ-function; see [1 – 3] for details.

Inside the contour we have no poles of the function $g(z)$ and, due to the fact that we assume RH, also no zeros of the Riemann ζ-function. Thus for any T we obtain

$$\int_C g(z)\ln(\zeta(z))dz = -2\pi\int_{\frac{1}{2}}^{1}\frac{dx}{\left(a+\left(x-\frac{1}{2}\right)\right)^{\epsilon}}, \tag{1}$$

where, of course, the integral in the right-hand side is the contribution of the simple pole of the ζ-function at $z=1$.

From the other side, the contour integral at question is equal to $\int_{-T}^{T}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}dt$ plus corresponding integrals taken over three remaining sides of the quadratic contour C. To finish the proof of the lemma we need to show that the latter tend to zero. First we consider an integral $\int_{\frac{1}{2}+iT}^{T+iT} g(z)\ln(\zeta(z))dz$ (consideration for $\int_{\frac{1}{2}-iT}^{T-iT} g(z)\ln(\zeta(z))dz$ is quite similar) and divide the integration area into the segments $\left[\frac{1}{2}+iT, N+iT\right]$ and $[N+iT, T+iT]$, where $N\ll T$ is some positive number depending on T (we can take it as large as we please; we will select its value later on).

We use a "standard" definition of $\arg\left(\zeta\left(\frac{1}{2}+it\right)\right)$, that is, that where the argument starts from zero value at $z=b>1$, $t=0$ ($\ln(\zeta(b))>0$ here) and then is obtained by the continued variation along the segments of straight lines $[b, b+it]$, $\left[b+it, \frac{1}{2}+it\right]$. With this definition, for the segment $\left[\frac{1}{2}+iT, N+iT\right]$, we have $|\arg(\zeta(z))|=\mathcal{O}(\ln T)$ ([4], this is Backlund theorem [7]) and $|\ln(\zeta(z))|=\mathcal{O}(\ln T)$ [4]. Hence

$$\left|\int_{\frac{1}{2}+iT}^{N+iT} g(z)\ln(\zeta(z))\,dz\right| = \mathcal{O}\big(NT^{-\epsilon}\ln T\big).$$

For the point $x+iT$ with sufficiently large x belonging to the segment $[N+iT, T+iT]$, we have $|\ln(\zeta(z))|=\mathcal{O}\big(2^{-x}\big)$, which follows trivially from the definition of the Riemann ζ-function, hence

$$\int_{x+iT}^{T+iT} g(z)\ln(\zeta(z))dz = \mathcal{O}\big(2^{-N}T^{-\epsilon}\ln T\big).$$

Thus it is enough to select, say, $N = T^{\frac{\epsilon}{2}}$ to see the disappearance of these two integrals in the limit of large T. Due to the same $|\ln(\zeta(z))| = \mathcal{O}\big(2^{-\Re(z)}\big)$, the disappearance of the integral taken along the line $[T + iT, T - iT]$ in the limit of large T is evident, which finishes the proof of the lemma.

Thus on RH the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}\,dt$ for $1 \geq \epsilon > 0$ do exists. This should not be surprising because "similar" integrals are known, such as, e.g., the following examples from Gradshtein and Ryzhik book [8]. Example 4.421.1 gives $\displaystyle\int_0^{\infty} \frac{\ln x \sin(ax)}{x}\,dx = -\frac{\pi}{2}(\gamma + \ln a)$; more generally we have Example 4.422.1: for $|\Re(\mu)| < 1,\ a > 0$,

$$\int_0^{\infty} \frac{\ln x \sin(ax)}{x^{1-\mu}}\,dx = -\frac{\Gamma(\mu)}{a^{\mu}} \sin\frac{\mu\pi}{2}\left(\psi(\mu) - \ln a + \frac{\pi}{2}\cos\frac{\mu\pi}{2}\right),$$

and Example 4.422.2: for $0 < \Re(\mu) < 1,\ a > 0$,

$$\int_0^{\infty} \frac{\ln x \cos(ax)}{x^{1-\mu}}\,dx = -\frac{\Gamma(\mu)}{a^{\mu}} \cos\frac{\mu\pi}{2}\left(\psi(\mu) - \ln a - \frac{\pi}{2}\tan\frac{\mu\pi}{2}\right).$$

Here γ is Euler–Mascheroni constant and Γ, ψ are gamma- and digamma-functions, respectively.

Our next aim is the proof of the following theorem.

Theorem 1. *For any real $a > 0$ the integral equality*

$$\int_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{a+it}\,dt = -2\pi \ln\frac{a+\frac{1}{2}}{a} \tag{2}$$

is equivalent to the Riemann hypothesis.

Proof. Applying Eq. (1) for $\epsilon = 1$, we see that on RH Eq. (2) takes place. If RH fails and a non-trivial Riemann zero $\sigma_k + it_k$ with $\sigma_k > \frac{1}{2}$ exists, it contributes

$$-2\pi i \int_0^{\delta_k} \frac{i}{a + x + it_k}\,dx = 2\pi \ln\frac{a + \delta_k + it_k}{a + it_k},$$

where $\delta_k = \sigma_k - \frac{1}{2} > 0$, to the integral value. Pairing the contributions of zeros with $\sigma_k \pm it_k$, we have the total contribution of $2\pi \ln\dfrac{(a+\delta_k)^2 + t_k^2}{a^2 + t_k^2}$, always a purely real and strictly positive number.

Theorem 1 is proved.

For large t_k we see that the contribution of non-trivial zeros $\sigma_k \pm it_k$ is equal to

$$2\pi \ln\frac{(a+\delta_k)^2 + t_k^2}{a^2 + t_k^2} = \mathcal{O}\left(\frac{1}{t_k^2}\right).$$

The known counting function of the Riemann zeta-function zeros $\mathcal{O}(T\ln T)$ [4] thus guaranties that this sum converges, so that the integral $\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{a+it}dt$ exists irrespective on the validity of RH.

Similarly, we have the following theorem.

Theorem 2. *For any real* $a>0$ *and any* $0<\epsilon<1$, *the integral equality*

$$\int\limits_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}dt=-\frac{2\pi}{1-\epsilon}\left(\left(a+\frac{1}{2}\right)^{1-\epsilon}-a^{1-\epsilon}\right) \tag{3}$$

is equivalent to the Riemann hypothesis.

Proof. On RH, Eq. (3) is an immediate consequence of Eq. (1). If there is a non-trivial Riemann zero $\frac{1}{2}+\delta_k+it_k$, we take $t_k>0$ here, it contributes $-2\pi i\int_0^{\delta_k}\frac{i}{(a+x+it_k)^{\epsilon}}dx$ to the integral value. Two complex conjugate zeros contribute

$$2\pi\int\limits_0^{\delta_k}\frac{(a+x+it_k)^{\epsilon}+(a+x-it_k)^{\epsilon}}{\left((a+x)^2+t_k^2\right)^{\epsilon}}dx=2\pi\int\limits_0^{\delta_k}\frac{2\cos(\epsilon\phi_k)}{\left((a+x)^2+t_k^2\right)^{\frac{\epsilon}{2}}}dx, \tag{4}$$

where $\phi_k=\arctan\frac{t_k}{a+x}$. The expression under the integral sign in the right-hand side here is certainly positive if $|\epsilon\phi_k|<\frac{\pi}{2}$ so that if $\left|\arctan\frac{t_k}{a+x}\right|<\frac{\pi}{2\epsilon}$. This is always true because $0<$ $<\arctan\frac{t_k}{a+x}<\frac{\pi}{2}$ and $0<\epsilon<1$.

Using $\arctan\frac{t_k}{a+x}=\frac{\pi}{2}-\arctan\frac{a+x}{t_k}$ we see that for large t_k asymptotically $\cos(\epsilon\phi_k)\cong$ $\cong\cos\frac{\epsilon\pi}{2}=\mathcal{O}(1)$. Hence, as it follows from Eq. (4), asymptotically each pair of complex conjugate non-trivial Riemann zeros contributes $\mathcal{O}\left(\delta_k t_k^{-\epsilon}\right)$. Thus the existence of the integral

$$\int\limits_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}dt$$

depends on the unknown density of the non-trivial Riemann zeros. If, for example, the density of non-trivial Riemann zeros with $\delta\geq\delta_0>\frac{1}{2}$ in a limit of large T is constant, the sum over such zeros diverges and the integral does not exist. Of course, this makes the (not easy due to the slow convergence) numerical study of the integral $\int_{-N}^{N}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}dt$ with $0<\epsilon<1$ for large N especially interesting from the viewpoint of the RH testing.

3. Discussion and numerical testing. For the rest of this section, we assume RH. Then, for $a = \frac{1}{2}$, we have

$$\int_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{\frac{1}{2}+it}\,dt = -2\pi\ln 2 = -4.35517\ldots. \tag{5}$$

This equation has been tested numerically with Mathematica. We used the built-in function *NIntegrate*. After some testing we found out that a good trade-off between precision and computational time was to choose *GlobalAdaptive* as the integration strategy for *NIntegrate* (in general it has better performance than local adaptive strategy) and to set the value of the parameter *MaxRecursion* to 100 and that of *MaxErrorIncrease* to $10'000$. In addition, we explicitly specified the singularities (i.e., where $\zeta(s) = 0$) in the range of integration. The convergence of an integral is quite slow (as it should be following the proof of Lemma 1), it takes about 4 and a half hours to calculate the integral on a standard desktop PC but for $\int_{-N}^{N} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{\frac{1}{2}+it}\,dt$ with $N = 9878$ (i.e., integrating over the region including the first $10'000$ non-trivial Riemann zeros), we succeed to obtain the "reasonable" value $-4.35518\ldots.$

The value of the integral (3) for $\epsilon = \frac{1}{2}$, $a = \frac{1}{2}$, that is,

$$\int_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{\sqrt{\frac{1}{2}+it}}\,dt = -4\pi\left(1-\frac{\sqrt{2}}{2}\right) = -3.6806\ldots \tag{6}$$

also has been tested numerically with the same limits obtaining $-3.683\ldots.$

Now we want to make the following remarks. First, nothing prevents us to take $a = 0$ in Eq. (3) obtaining

$$\int_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(it)^{\epsilon}}\,dt = -\frac{2^{\epsilon}\pi}{1-\epsilon}. \tag{7}$$

In particular,

$$\int_{-\infty}^{\infty} \frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{\sqrt{it}}\,dt = -2\sqrt{2}\pi. \tag{8}$$

Of course, integrals (7), (8) are to be understood as Cauchy principal values.

Next, if in the same contour integral $\int_C g(z)\ln(\zeta(z))dz$ as above we use $g(z) = \dfrac{i}{-a+\left(z-\frac{1}{2}\right)}$, we get a simple pole of the function $g(z)$ with the residue i at $z = a + \frac{1}{2}$ inside the contour. This gives the contribution $-2\pi\ln\left(\zeta\left(a+\frac{1}{2}\right)\right)$ which should be summed up with the contribution of the simple pole of $\zeta(z)$, $-2\pi\ln\frac{a-\frac{1}{2}}{a}$, hence

$$\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{-a+it}\,dt=-2\pi\ln\left(\zeta\left(\frac{1}{2}+a\right)\frac{a-\frac{1}{2}}{a}\right).$$

Taking $a=\frac{1}{2}+\delta$, $\delta\to 0$, we have $-2\pi\ln\delta-2\pi\ln 2$ for the contribution of a simple pole of $g(z)$ and $2\pi\ln\delta+\mathcal{O}(\delta)$ for the contribution of a simple pole of $\zeta(z)$, that is, a total of $-2\pi\ln 2+\mathcal{O}(\delta)$. Thus we get

$$\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{-\frac{1}{2}+it}\,dt=-2\pi\ln 2. \tag{9}$$

Certainly, $\dfrac{1}{\frac{1}{4}+t^2}=\dfrac{1}{\frac{1}{2}+it}-\dfrac{1}{-\frac{1}{2}+it}$, thus combining (5) and (9) we obtain again the Balazard – Saias – Yor theorem stating that RH is equivalent to the integral equality [5]

$$\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{\frac{1}{4}+t^2}\,dt=0.$$

Similarly, combining the values of the integrals

$$\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{a+it}\,dt\quad\text{and}\quad\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{-a+it}\,dt$$

given above for $a\neq\frac{1}{2}$, we restore the Theorem 3 from [1]: the equality

$$\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{a^2+t^2}dt=\ln\left|\frac{\zeta\left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right)}{a+\frac{1}{2}}\right|$$

is equivalent to the Riemann hypothesis. Numerous examples of this equality were tested numerically.

This is worthwhile to note that if $\epsilon\to 0$, expression (3) has a *finite* limit (of course) independent on a, which we formulate as the following proposition.

Proposition. *Assume RH. Then for any real positive a*

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)}{(a+it)^{\epsilon}}\,dt=-\pi.$$

Thus, in a certain sense, on RH the function $\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)$ is indeed “almost periodic”.

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Received 10.07.20