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EXISTENCE OF A WEAK SOLUTION FOR A CLASS OF NONLINEAR ELLIPTIC EQUATIONS ON THE SIERPIŃSKI GASKET

ІСНУВАННЯ СЛАБКОГО РОЗВ'ЯЗКУ ДЛЯ КЛАСУ НЕЛІНІЙНИХ ЕЛІПТИЧНИХ РІВНЯНЬ НА ПРОКЛАДЦІ СЕРПІНСЬКОГО

We study the existence of a weak (strong) solution of the nonlinear elliptic problem

$$-\Delta u - \lambda u g_1 + h(u) g_2 = f \quad \text{in} \quad V \setminus V_0,$$

$$u = 0 \quad \text{on} \quad V_0,$$

where V is a Sierpiński gasket in $\mathbb{R}^{N-1}$, $N \geq 2$, V_0 is its boundary (consisting of N its corners), and λ is a real parameter. Here, $f, g_1, g_2 : V \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are functions satisfying suitable hypotheses.

Досліджується існування слабкого (сильного) розв'язку нелінійної еліптичної задачі

$$-\Delta u - \lambda u g_1 + h(u) g_2 = f \quad \text{в} \quad V \setminus V_0,$$

$$u = 0 \quad \text{на} \quad V_0,$$

де V — прокладка Серпінського в $\mathbb{R}^{N-1}$, $N \geq 2$, V_0 — її межа (що складається з її N кутів) і λ — дійсний параметр. Тут $f, g_1, g_2 : V \to \mathbb{R}$ і $h : \mathbb{R} \to \mathbb{R}$ — функції, що задовольняють відповідні гіпотези.

1. Introduction. We study the existence of weak solutions for the following class of elliptic problem:

$$\begin{gathered} -\Delta u - \lambda u g_1 + h(u) g_2 = f \quad \text{in } V \setminus V_0 \\ u = 0 \quad \text{on } V_0, \end{gathered} \tag{1.1}$$

where V denotes the Sierpiński gasket in $\mathbb{R}^{N-1}$ $N \geq 2$, V_0 is its boundary (consisting of its N corners). Δ denotes the Laplacian operator on V, $\lambda \in \mathbb{R}$ and $f, g_1, g_2 : V \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are functions satisfying the following hypotheses:

(H_1) $h : \mathbb{R} \to \mathbb{R}$ is bounded (i.e., $|h(t)| \leq A, t \in \mathbb{R},$ for a fixed $A > 0$) and continuous;

(H_2) $g_1 \in L^\infty(V)$, $g_2 \in L^2(V)$ and $f \in L^2(V)$.

Recently, there has been a considerable interest in the study of nonlinear partial differential equations on fractal domains and in particular on the Sierpiński gasket. Many physical problems on fractal regions such as reaction-diffusion problems, elastic properties of fractal media and flow through fractal regions are modeled by nonlinear equations. Now, a natural question is whether the classical existence results (we refer to [1, 24, 29]) in the standard framework of the Laplacian also hold in the corresponding fractal framework. To answer this we have to overcome several difficulties that arise due to the geometrical structure of fractal domains. One main difficulty is how to define differential operators, like the Laplacian operator, on the fractal domains for there is no concept of a generalized derivative of functions defined on such domains. However, a Laplacian is defined on

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a few special fractals, we refer to [2, 3, 20, 21] and a Hilbert space structure is introduced in [20]. This enables us to investigate the existence of solutions for equations of type (1.1) defined on fractal domains.

The study of nonlinear elliptic equations on the Sierpiński gasket was essentially initiated by Falconer and Hu in the paper [15]. Since then many authors have contributed to the literature in this direction. In [15], Falconer and Hu, considered the problem

$$\begin{aligned} \Delta u + a(x)u = f(x,u), &\quad x \in V \setminus V_0, \\ u|_{V_0} = 0, & \end{aligned} \tag{1.2}$$

where V denotes the Sierpiński gasket with boundary V_0 and $a \in L^1(V)$ satisfies suitable condition. The authors formulated the problem in a suitable function space over Sierpiński gasket and used the Mountain Pass theorem [1] to prove the existence of a solution. In [5], Molica Bisci et al. considered a similar problem

$$\Delta u + \alpha(x)u = \lambda f(x,u), \quad x \in V \setminus V_0,$$

$$u|_{V_0} = 0,$$

where λ is a positive real parameter and proved the existence of at least two solutions for small values of λ. For this the authors used a variational result due to Ricceri [26]. In [8], Breckner et al. studied the existence of infinitely many solutions of the problem

$$\Delta u(x) + \alpha(x)u(x) = g(x)f\big(u(x)\big), \quad x \in V \setminus V_0,$$

$$u|_{V_0} = 0.$$

The authors proved the existence of infinitely many solutions by extending a method introduced by Faraci and Kristály [16] in the framework of Sobolev spaces to the case of function spaces on fractal domains. For more results on the existence and multiplicity of solutions of nonlinear elliptic equations on the Sierpiński gasket and on other fractals we refer to the papers [4, 6 – 11, 13, 14] and [18, 19, 27, 28] as well as the references therein. The main tools used in these papers to prove the existence of nontrivial solutions are basically Mountain Pass theorems, saddle-point theorems or certain minimization procedures.

In [5, 15] on of the assumptions on the nonlinearity $f(x,u)$ is

(f) there exist constants $\nu > 2$ and $r \geq 0$ such that, for $|t| \geq r$,

$$tf(x,t) < \nu F(x,t) < 0,$$

where $F(x,t) = \displaystyle\int_0^t f(x,s)ds$.

If the condition (f) does not hold then the energy functional associated to the problem (1.2) in general does not satisfy certain conditions needed to apply the Mountain Pass theorem in order to prove the existence of solutions. In this paper, we show an application of demicontinuous operators to nonlinear elliptic problems in the fractal setting. In particular, the main tool we used to establish the existence of at least one solution is a result due to P. Hess [12] on linear demicontinuous operators.

Our paper was inspired by a problem on bounded domains given in Section 29.9 of the book by Zeidler [31, p. 661], we refer also to [25] where the existence of a weak solution is established for the fractional counterpart of (1.1).

This paper is organized as follows. Section 2 deals with preliminaries and the weak formulation of the problem. Section 3 concerns with the main result concerning the existence of a weak solution of (1.1). Finally, Section 4 deals with an extension to a class of continuous functions h that are not necessarily bounded.

2. Preliminaries. Let $C(V)$ denotes the space of real-valued continuous functions on V and $C_0(V) = \{u \in C(V) : u|_{V_0} = 0\}$ both equipped with the usual supremum norm $\|\cdot\|_\infty$. Let $H_0^1(V)$ be the Hilbert space structure defined on V with inner product denoted by $\mathcal{W}(u, v)$. We refer to [15] (see also [3–10, 13]), for more details. The space $H_0^1(V)$ with the inner product $\mathcal{W}(u, v)$ is a separable Hilbert space (we refer to [9]). The weak Laplacian Δ of u on V is defined as

$$\langle \Delta u, v\rangle = -\mathcal{W}(u, v) \quad \text{for all} \quad v \in H_0^1(V).$$

Now, we can define the weak solution for the problem (1.1).

Definition 2.1. *We say that a function $u \in H_0^1(V)$ is a weak solution of* (1.1) *if it satisfies*

$$\mathcal{W}(u, v) - \lambda \int\limits_V g_1(x)u(x)v(x)\, d\mu + \int\limits_V h(u(x))g_2(x)v(x)\, d\mu = \int\limits_V f(x)v(x)\, d\mu$$

for all $v \in H_0^1(V)$.

For further details on the Laplacian operator on certain fractals, we refer to the paper [20].

We note that if the functions f, g_1, g_2 and h are continuous, then the weak solutions of the equation (1.1) are also strong solutions of it; as shown by the following result.

Lemma 2.1. *Assume that $u \in H_0^1(V)$ is a weak solution to the problem* (1.1). *If $f, g_1, g_2 \in C(V)$ and $h \in C(\mathbb{R})$, then u is a strong solution of* (1.1).

Proof is similar to [15] (Lemma 2.16), hence omitted.

At each step, a generic constant is denoted by C or c to avoid too many suffixes. We recall the embedding properties of $H_0^1(V)$ into the spaces $C_0(V)$ and $L^2(V, \mu)$ (we refer to [15]), for the sake of completeness.

Lemma 2.2. *The embedding $j : H_0^1(V) \hookrightarrow C_0(V)$ is compact and, for every $u \in H_0^1(V)$,*

$$|u(x)| \le (2N + 3)\|u\|_{H_0^1(V)} \quad \textit{for any} \quad x \in V.$$

Also, the embedding $j : H_0^1(V) \hookrightarrow L^2(V)$ is compact and

$$\|u\|_2 \le C\|u\|_{H_0^1(V)}, \tag{2.1}$$

where $\|u\|_2 = \left(\int_V |u(x)|^2 d\mu\right)^{\frac{1}{2}}$.

Let Y^* denote the dual of the real Banach space Y. Let $\|.\|$ and $\|.\|_{Y^*}$ be the norms on Y and Y^*, respectively. For $x \in Y$ and $f \in Y^*$, we write $(f|x)$ for $f(x)$.

Definition 2.2. *Let B, $N : Y \to Y^*$ be operators on the real separable reflexive Banach space Y. Then*:

(i) *$B + N$ is asymptotically linear if B is linear and*

$$\frac{\|Nu\|}{\|u\|} \to 0 \quad as \quad \|u\| \to \infty;$$

(ii) *B satisfies condition (S) if*

$$u_n \rightharpoonup u \quad and \quad \lim_{n\to\infty}(Bu_n - Bu|u_n - u) = 0, \quad implies \quad u_n \to u.$$

We say that B is an (S)-operator if B satisfies condition (S).

The following definition deals with real Gårding forms (compare with [30, p. 364]).

Definition 2.3. *Let X and Z be Hilbert spaces over $\mathbb{R}$ with the continuous embedding $X \subseteq Z$. Then $G\colon X \times X \to \mathbb{R}$ is called a Gårding form if and only if G is bilinear and bounded, and there exist a constant $c > 0$ and a real constant C such that*

$$G(u,u) \geq c\|u\|_X^2 - C\|u\|_Z^2 \quad for\ all \quad u \in X. \tag{2.2}$$

The relation (2.2) *is called a Gårding inequality. If $C = 0$, then G is called a strict Gårding form. The Gårding form G is called regular if and only if the embedding $X \subseteq Z$ is compact.*

In Section 3, we need the following result.

Proposition 2.1. *Let $B, N\colon Y \to Y^*$ be operators on the real separable reflexive Banach space Y. Assume that*:

(i) *the operator $B\colon Y \to Y^*$ is linear and continuous*;

(ii) *the operator $N\colon Y \to Y^*$ is demicontinuous and bounded*;

(iii) *$B + N$ is asymptotically linear*;

(iv) *for each $T \in Y^*$ and for each $t \in [0,1]$, the operator $A_t\colon Y \to Y^*$ defined by $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S).*

If $Bu = 0$ implies $u = 0$, then, for each $T \in Y^*$, the equation $Bu + Nu = T$ has a solution in Y.

For a detailed proof of the above theorem, we refer to [12] or [31] (Theorem 29.C).

We define the functionals $B_1, B_2\colon H_0^1(V) \times H_0^1(V) \to \mathbb{R}$ by

$$B_1(u,\varphi) = \mathcal{W}(u,\varphi) - \lambda \int_V u(x)g_1(x)\varphi(x)d\mu,$$

$$B_2(u,\varphi) = \int_V h(u(x))g_2(x)\varphi(x)d\mu.$$

Also define $T\colon H_0^1(V) \to \mathbb{R}$ by

$$T(\varphi) = \int_V f(x)\varphi(x)d\mu.$$

A function $u \in H_0^1(V)$ is a weak solution of (1.1) if

$$B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi) \quad \forall \varphi \in H_0^1(V). \tag{2.3}$$

By applying the Cauchy – Schwarz inequality and the inequality (2.1), we note that, for every $(u,\varphi) \in H_0^1(V) \times H_0^1(V)$,

$$\big|B_1(u,\varphi)\big| \le |\mathcal{W}(u,\varphi)| + |\lambda| \int\limits_V |g_1(x)||u(x)||\varphi(x)|d\mu \le$$

$$\le \|u\|_{H_0^1(V)}\|\varphi\|_{H_0^1(V)} + |\lambda|\|g_1\|_\infty\|u\|_2\|\varphi\|_2 \le$$

$$\le (1 + C|\lambda|\|g_1\|_\infty)\|u\|_{H_0^1(V)}\|\varphi\|_{H_0^1(V)}.$$

By the hypotheses (H_1), (H_2), Hölder's inequality and (2.1), we have, for every $(u,\varphi) \in \in H_0^1(V) \times H_0^1(V)$,

$$|B_2(u,\varphi)| \le \int\limits_V \big|h(u(x))\big||\varphi(x)||g_2(x)|d\mu \le$$

$$\le A \int\limits_V |\varphi(x)||g_2(x)|d\mu \le$$

$$\le A\|\varphi\|_2\|g_2\|_2 \le AC\|\varphi\|_{H_0^1(V)}\|g_2\|_2. \tag{2.4}$$

Also, we have

$$\big|T(\varphi)\big| \le \int\limits_V |f(x)||\varphi(x)|d\mu \le \big\|f\big\|_2\|\varphi\|_2 \le C\big\|f\big\|_2\|\varphi\|_{H_0^1(V)},$$

where C is a constant arising out of the inequality (2.1). Now, $B_1(u,\cdot)$ and $B_2(u,\cdot)$ are linear and bounded for every $u \in H_0^1(V)$. We define the operators

$$B, N : H_0^1(V) \to H^{-1}(V)$$

as

$$(Bu|\varphi) = B_1(u,\varphi), \qquad (Nu|\varphi) = B_2(u,\varphi) \quad \text{for} \quad u,\varphi \in H_0^1(V).$$

Then, (2.3) is equivalent to the operator equation $Bu + Nu = T,\ u \in H_0^1(V)$.

3. Main results. In this section, we study the existence of a weak solution for (1.1).

Theorem 3.1. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Let*

$$1 > \lambda C^2\big\|g_1\big\|_\infty, \tag{3.1}$$

where C *is the constant in inequality* (2.1). *Then the BVP* (1.1) *has at least one weak solution* $u \in H_0^1(V)$. *Moreover, every* (*weak*) *solution* u *of* (1.1) *satisfies*

$$\|u\|_{H_0^1(V)} \le \frac{C\big\{A\|g_2\|_2 + \|f\|_2\big\}}{(1 - \lambda C^2\|g_1\|_\infty)},$$

where A *is the constant from hypothesis* (H_1).

Proof. First we write the BVP (1.1) as the following operator equation in $H^{-1}(V)$:

$$u \in H_0^1(V) : Bu + Nu = T.$$

We prove that $T \in H^{-1}(V),\ B, N : H_0^1(V) \to H^{-1}(V)$ satisfy all the conditions given in Proposition 2.1. For convenience, we divide the proof into five steps.

***Step* 1.** From the previous section we know that the operator B is linear and continuous. By Lemma 2.2 the embedding of $H_0^1(V) \hookrightarrow L^2(V)$ is compact which shows that $B_1(\cdot,\cdot)$ is a regular Gårding form. Furthermore, the inequality (3.1) and

$$B_1(u,u) = \mathcal{W}(u,u) - \lambda \int_V u^2(x) g_1(x) d\mu \geq$$

$$\geq \|u\|^2_{H_0^1(V)} - \lambda \|g_1\|_\infty \|u\|_2^2 \tag{3.2}$$

shows that $B_1(\cdot,\cdot)$ is a strict Gårding form. Let $\{u_k\}$ be any sequence in $H_0^1(V)$ and

$$\lim_{k\to\infty} (Bu_k|u_k) = 0. \tag{3.3}$$

Claim: B satisfies condition (S). Since B is linear, as in (3.2), we have

$$(Bu_k|u_k) = (B(u_k)|u_k) = B_1(u_k,u_k) \geq$$

$$\geq \|u_k\|^2_{H_0^1(V)} - \lambda \|g_1\|_\infty \|u_k\|_2^2 \geq$$

$$\geq \ (1 - \lambda C^2 \|g_1\|_\infty) \|u_k\|^2_{H_0^1(V)}. \tag{3.4}$$

From (3.3) and (3.4), we note that

$$0 \leq \left(1 - \lambda C^2 \|g_1\|_\infty\right) \lim_{k\to\infty} \|u_k\|^2_{H_0^1(V)} \leq \lim_{k\to\infty} \left(Bu_k|u_k\right) = 0.$$

Since $(1-\lambda C^2\|g_1\|_\infty) > 0$, we have $\|u_k\|^2_{H_0^1(V)} \to 0$ as $k \to \infty$, which implies $u_k \to 0$ as $k \to \infty$. Hence, B satisfies condition (S).

***Step* 2.** *Claim*: $B + N$ is asymptotically linear. By (2.4), we have

$$\|Nu\|_{H^{-1}(V)} \leq C',$$

where $C' = AC\|g_2\|_2$ is a constant depending on V. Consequently,

$$\frac{\|Nu\|_{H^{-1}(V)}}{\|u\|_{H_0^1(V)}} \to 0 \quad \text{as} \quad \|u\|_{H_0^1(V)} \to \infty,$$

which shows that $B + N$ is asymptotically linear. Next we show that N is strongly continuous. Since h is continuous and is a function of u only, the hypotheses $(H1)$ and $(H6*)$ of [31] (Corollary 26.14) are satisfied. The hypothesis $(H2)$ follows from the fact that h is bounded. Hence, by [31] (Corollary 26.14), N is a strongly continuous operator.

***Step* 3.** From Step 2, we note that the operator B satisfies condition (S). Since, N is strongly continuous, we note that the operator $u \in H_0^1(V) \mapsto t(Nu - T) \in \left(H_0^1(V)\right)^*$ is strongly continuous for $t \in [0,1]$. For each $t \in [0,1]$, the operator $A_t(u) = Bu + t(Nu - T)$ is thus a strongly continuous perturbation of the (S)-operator B. So, the operator $A_t(u)$ satisfies condition (S) (we refer to [31], Problem 27.1).

***Step* 4.** Now, $Bu = 0$, with $u \in H_0^1(V)$, implies

$$\mathcal{W}(u,u) - \lambda \int_V u^2(x) g_1(x) d\mu = 0.$$

Consequently, we have

$$\big(1 - \lambda C^2\|g_1\|_\infty\big)\|u\|^2_{H^1_0(V)} \le 0,$$

which shows that $u = 0$, since $1 - \lambda C^2\|g_1\|_\infty > 0$.

By Proposition 2.1, $Bu + Nu = T$ has a solution $u \in H^1_0(V)$ which equivalently shows, the BVP (1.1) has a solution $u \in H^1_0(V)$.

***Step* 5.** Let $u \in H^1_0(V)$ be a weak solution of (1.1). As in (3.4) (with the help of the embedding in Lemma 2.2), we obtain

$$B_1(u,u) \ge \big(1 - \lambda C^2\|g_1\|_\infty\big)\|u\|^2_{H^1_0(V)}.$$

Since, $1 > \lambda C^2\|g_1\|_\infty$, we have

$$\|u\|^2_{H^1_0(V)} \le \frac{1}{1 - \lambda C^2\|g_1\|_\infty} B_1(u,u). \tag{3.5}$$

Also, we note that

$$|B_1(u,u)| \le C\big\{A\|g_2\|_2 + \|f\|_2\big\}\|u\|_{H^1_0(V)}. \tag{3.6}$$

By (3.5) and (3.6), we get

$$\|u\|_{H^1_0(V)} \le \frac{C\big\{A\|g_2\|_2 + \|f\|_2\big\}}{\big(1 - \lambda C^2\|g_1\|_\infty\big)}.$$

Theorem 3.1 is proved.

Next, we dispense with the condition (3.1) when g_1 does not change sign. The two results are related to the cases when $g_1 \ge 0$ with $\lambda \le 0$ and $g_1 \le 0$ with $\lambda > 0$. These results are similar to that of Theorem 3.1 but with suitable changes.

Theorem 3.2. *Suppose that the hypotheses* (H_1) *and* (H_2) *hold. Let* $g_1 \ge 0$ *and* $\lambda \le 0$, *then the BVP* (1.1) *has at least one weak solution. For every weak solution* $u \in H^1_0(V)$ *of* (1.1) *the inequality*

$$\|u\|_{H^1_0(V)} \le C\big\{A\|g_2\|_2 + \|f\|_2\big\}$$

holds, *where* C *is the constant in inequality* (2.1).

Proof. As in Theorem 3.1, the basic idea is to reduce the problem (1.1) to the operator equation $Bu+Nu = T$ and then to apply Proposition 2.1. For this, we define B, N and T, as in Theorem 3.1. The compact embedding of $H^1_0(V) \hookrightarrow L^2(V)$ and (3.1) shows that $B_1(\cdot,\cdot)$ is a strict regular Gårding form. Also, $\lambda \le 0$ and $g_1 \ge 0$ yields

$$B_1(u,u) = \mathcal{W}(u,u) - \lambda \int\limits_V u^2(x) g_1(x) d\mu \ge \|u\|^2_{H^1_0(V)}. \tag{3.7}$$

Let $\{u_k\}$ be any sequence in $H^1_0(V)$ and

$$\lim_{k\to\infty} (Bu_k|u_k) = 0. \tag{3.8}$$

We claim that B satisfies condition (S). Since B is linear, as in (3.7), we have

$$(Bu_k|u_k) = (B(u_k)|u_k) = B_1(u_k,u_k) \ge \|u_k\|^2_{H^1_0(V)}. \tag{3.9}$$

From (3.8) and (3.9), we note that

$$0 \le \lim_{k\to\infty} \|u_k\|^2_{H^1_0(V)} \le \lim_{k\to\infty} (Bu_k|u_k) = 0$$

which implies $\|u_k\|_{H^1_0(V)} \to 0$ as $k \to \infty$ and, consequently, B satisfies condition (S). Next, we show that $B+N$ is asymptotically linear and, N is strongly continuous. The proof is similar to that of Theorem 3.1 and we omit it for brevity. Since $\lambda \le 0$, we get from (3.7) that $Bu = 0$ implies $u = 0$. By Proposition 2.1, $Bu + Nu = T$ has a solution $u \in H^1_0(V)$ which equivalently shows that the BVP (1.1) has at least one weak solution. Let $u \in H^1_0(V)$ be such a solution. Then, by (3.7) and a similar argument as in Theorem 3.1, we have

$$\|u\|_{H^1_0(V)} \le C\big\{A\|g_2\|_2 + \|f\|_2\big\},$$

where C is the constant in inequality (2.1).

Theorem 3.2 is proved.

With suitable modifications in the proof of Theorem 3.2, we have the following result.

Theorem 3.3. *Suppose that the hypotheses* (H_1) *and* (H_2) *hold. Let* $g_1 \le 0$ *and* $\lambda > 0$, *then* (1.1) *has a weak solution* $u \in H^1_0(V)$ *and there is a constant* k_0 *such that* $\|u\|_{H^1_0(V)} \le k_0$ *for every* (*weak*) *solution* u.

4. Extensions. In Section 3, the nonlinearity h is assumed to be continuous and bounded. In this section, we extend these results for a class of functions h which are continuous only. The generalized Hölder inequality comes handy for getting suitable estimates. We establish the existence of a weak solution for (1.1), where $h\colon \mathbb{R} \to \mathbb{R}$ is required to be continuous and to satisfy $|h(t)| \le |t|^\epsilon$, $0 < \epsilon < 1$, for all $t \in \mathbb{R}$. Again, we consider the cases $\lambda \le 0$ and $\lambda > 0$ separately. Since the proofs are similar to the ones in Section 3, we restrict ourselves to sketch only the differences wherever needed. The Corollary 26.14 in [31] is not applicable here since h is not bounded. We collect the common hypotheses for convenience:

(H'_1) the function $h\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $|h(t)| \le |t|^\epsilon, t \in \mathbb{R}$, for a fixed $0 < \epsilon < 1$;

(H'_2) $g_1 \in L^\infty(V)$, $g_2 \in L^{\frac{2}{1-\epsilon}}(V)$, $0 < \epsilon < 1$ and $f \in L^2(V)$.

Theorem 4.1. *Assume that the hypotheses* (H'_1), (H'_2) *hold. If* $g_1 \ge 0$ *and* $\lambda \le 0$, *then* (1.1) *has at least one weak solution and there is a constant* k_0 *such that* $\|u\|_{H^1_0(V)} \le k_0$ *for every (weak) solution* $u \in H^1_0(V)$.

Proof. We give only a sketch of the proof since it is similar to the proof of Theorem 3.2. By Lemma 2.2 and generalized Hölder's inequality [23, p. 67], we have

$$|B_2(u,\varphi)| \le \int_V |h(u(x))||\varphi(x)||g_2|d\mu \le \|u\|_2^\epsilon \|\varphi\|_2 \|g_2\|_{\frac{2}{1-\epsilon}} \quad \text{for every} \quad \varphi \in H^1_0(V).$$

By a similar argument as in Theorem 3.2 we observe that the operator B_1 satisfies condition (S). We also observe that

$$|(Nu|\varphi)| = |B_2(u,\varphi)| \le C^{\epsilon+1} \|u\|^\epsilon_{H^1_0(V)} \|\varphi\|_{H^1_0(V)} \|g_2\|_{\frac{2}{1-\epsilon}},$$

which implies

$$\|Nu\|_{H^{-1}(V)} \le C^{\epsilon+1} \|u\|^\epsilon_{H^1_0(V)} \|g_2\|_{\frac{2}{1-\epsilon}} = c\|u\|^\epsilon_{H^1_0(V)},$$

where the constant $c = C^{\epsilon+1}\|g_2\|_{\frac{2}{1-\epsilon}}$. So

$$\frac{\|Nu\|_{H^{-1}(V)}}{\|u\|_{H_0^1(V)}} \leq \frac{c\|u\|^{\epsilon}_{H_0^1(V)}}{\|u\|_{H_0^1(V)}} \to 0 \quad \text{as} \quad \|u\|_{H_0^1(V)} \to \infty. \tag{4.1}$$

This shows that $B + N$ is asymptotically linear. Also, $u \in L^2(V)$ implies that $h(u) \in L^{\frac{2}{\epsilon}}(V)$ and define the Nemytskii operator

$$F : L^2(V) \to L^{\frac{2}{\epsilon}}(V)$$

by $(Fu)(x) = h(u(x))$; we have that F is continuous (by [22], Theorem 2.1). Now, the hypotheses (H_1'), (H_2') and the generalized Hölder inequality imply that

$$\big|(Nu_n|\varphi) - (Nu|\varphi)\big| \leq \int\limits_V |h(u_n) - h(u)\|g_2\|\varphi| d\mu \leq$$

$$\leq C\big\|h(u_n) - h(u)\big\|_{\frac{2}{\epsilon}}\|g_2\|_{\frac{2}{1-\epsilon}}\|\varphi\|_{H_0^1(V)}.$$

Let $u_n \rightharpoonup u$ weakly in $H_0^1(V)$. Then, by the continuity of F in $L^{\frac{2}{\epsilon}}(V)$ and by the compact embedding $H_0^1(V) \hookrightarrow L^2(V)$, we have

$$\|Nu_n - Nu\|_{H^{-1}(V)} \to 0 \quad \text{as} \quad n \to \infty.$$

By a similar argument as in Theorem 3.1, we can show that the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S). If $\lambda \leq 0$, then $Bu = 0$ implies as in the proof of Theorem 3.2 that $u = 0$. By Proposition 2.1, problem (1.1) has at least one weak solution $u \in H_0^1(V)$, which completes the proof of existence result.

Let $u \in H_0^1(V)$ be a weak solution of (1.1). Then

$$\big|B_1(u,u)\big| \leq C\big\{C^{\epsilon}\|u\|^{\epsilon}_{H_0^1(V)}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big\}\|u\|_{H_0^1(V)}. \tag{4.2}$$

By (3.7) and (4.2), we have

$$\|u\|_{H_0^1(V)} \leq C\big\{C^{\epsilon}\|u\|^{\epsilon}_{H_0^1(V)}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big\}. \tag{4.3}$$

If $\|u\|_{H_0^1(V)} \geq 1$, then, from (4.3), we have

$$\|u\|_{H_0^1(V)} \leq C\big(C^{\epsilon}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)\|u\|^{\epsilon}_{H_0^1(V)},$$

which implies that

$$\|u\|^{1-\epsilon}_{H_0^1(V)} \leq c \quad \text{or} \quad \|u\|_{H_0^1(V)} \leq c^{\frac{1}{1-\epsilon}},$$

where $c = C\big(C^{\epsilon}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)$. If $\|u\|_{H_0^1(V)} \leq 1$, we have nothing to prove. Let $k_0 = \max\{1, c^{\frac{1}{1-\epsilon}}\}$. Hence, we have

$$\|u\|_{H_0^1(V)} \leq k_0.$$

Theorem 4.1 is proved.

Remark. Theorem 4.1 also holds if $g_1 \leq 0$ and $\lambda > 0$. But when $\lambda > 0$ and g_1 changes sign, we need additional conditions on λ and g_1 (stated below) as in Theorem 3.1. We state these results below in Theorem 4.2 for which we give only a sketch of the proof. We note that in (4.1) the asymptotic linearity of $B + N$ is a consequence of ϵ lying between 0 and 1.

Theorem 4.2. *Let the hypotheses* (H_1'), (H_2') *hold. Also, let* $1 > \lambda C^2 \|g_1\|_\infty$. *Then the BVP* (1.1) *has at least one weak solution and there is a constant* k_0 *such that* $\|u\|_{H_0^1(V)} \leq k_0$ *for every (weak) solution* $u \in H_0^1(V)$.

Proof. The proof for the existence of at least one weak solution $u \in H_0^1(V)$ for (1.1) is similar to the arguments in the proof of Theorem 4.1 and Theorem 3.1 and hence it is omitted. As in Theorem 3.1, we note that

$$(1 - \lambda C^2 \|g_1\|_\infty)\|u\|^2_{H_0^1(V)} \leq C\{C^\epsilon \|u\|^\epsilon_{H_0^1(V)} \|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\}\|u\|_{H_0^1(V)},$$

where C is a constant. Since $1 > \lambda C^2 \|g_1\|_\infty$, we obtain

$$\|u\|_{H_0^1(V)} \leq \frac{C\big(C^\epsilon \|u\|^\epsilon_{H_0^1(V)} \|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)}{(1 - \lambda C^2 \|g_1\|_\infty)}. \tag{4.4}$$

If $\|u\|_{H_0^1(V)} \geq 1$, from (4.4), we have

$$\|u\|_{H_0^1(V)} \leq \frac{C\big(C^\epsilon \|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)\|u\|^\epsilon_{H_0^1(V)}}{\big(1 - \lambda C^2 \|g_1\|_\infty\big)},$$

which implies that

$$\|u\|^{1-\epsilon}_{H_0^1(V)} \leq c \quad \text{or} \quad \|u\|_{H_0^1(V)} \leq c^{\frac{1}{1-\epsilon}},$$

where $c = \dfrac{C\big(C^\epsilon \|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)}{(1 - \lambda C^2 \|g_1\|_\infty)}$ and $0 < \epsilon < 1$.

If $\|u\|_{H_0^1(V)} \leq 1$, we have nothing to prove. Let $k_0 = \max\{1, c^{\frac{1}{1-\epsilon}}\}$. Then we have

$$\|u\|_{H_0^1(V)} \leq k_0.$$

Theorem 4.2 is proved.

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