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ADMISSIBLE INTEGRAL MANIFOLDS FOR PARTIAL NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATIONS

ДОПУСТИМІ ІНТЕГРАЛЬНІ МНОГОВИДИ ДЛЯ НЕЙТРАЛЬНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We prove the existence and attraction property for admissible invariant unstable and center-unstable manifolds of admissible classes of solutions to the partial neutral functional-differential equation in Banach space X of the form

$$\frac{\partial}{\partial t}Fu_t = A(t)Fu_t + f(t,u_t), \quad t \geq s, \quad t,s \in \mathbb{R},$$

$$u_s = \phi \in \mathcal{C} := C([-r,0],X)$$

under the conditions that the family of linear partial differential operators $(A(t))_{t\in\mathbb{R}}$ generates the evolution family $(U(t,s))_{t\geq s}$ with an exponential dichotomy on the whole line $\mathbb{R}$; the difference operator $F:\mathcal{C}\to X$ is bounded and linear, and the nonlinear delay operator f satisfies the φ-Lipschitz condition, i.e., $\|f(t,\phi)-f(t,\psi)\| \leq \varphi(t)\|\phi-\psi\|_{\mathcal{C}}$ for $\phi,\psi\in\mathcal{C}$, where $\varphi(\cdot)$ belongs to an admissible function space defined on $\mathbb{R}$. We also prove that an unstable manifold of the admissible class attracts all other solutions with exponential rates. Our main method is based on the Lyapunov – Perron equation combined with the admissibility of function spaces. We apply our results to the finite-delayed heat equation for a material with memory.

Доведено існування та властивість притягання для допустимих інваріантних нестійких та центрально-нестійких многовидів допустимих класів розв'язків нейтрального функціонально-диференціального рівняння з частинними похідними в банаховому просторі X вигляду

$$\frac{\partial}{\partial t}Fu_t = A(t)Fu_t + f(t,u_t), \quad t \geq s, \quad t,s \in \mathbb{R},$$

$$u_s = \phi \in \mathcal{C} := C([-r,0],X)$$

за умови, що множина лінійних операторів частинного диференціювання $(A(t))_{t\in\mathbb{R}}$ породжує еволюційну множину $(U(t,s))_{t\geq s}$, що має експоненціальну дихотомію на всій прямій $\mathbb{R}$; різницевий оператор $F:\mathcal{C}\to X$ є обмеженим і лінійним, а нелінійний оператор затримки f задовольняє умову φ-Ліпшиця, тобто $\|f(t,\phi)-f(t,\psi)\| \leq \varphi(t)\|\phi-\psi\|_{\mathcal{C}}$ для $\phi,\psi\in\mathcal{C}$, де $\varphi(\cdot)$ належить допустимому функціональному простору, визначеному на $\mathbb{R}$. Ми також доводимо, що нестійкий многовид з допустимого класу притягує всі інші розв'язки з експоненціальною швидкістю. Наш основний метод базується на рівнянні Ляпунова – Перрона в поєднанні з допустимістю функціональних просторів. Отримані результати застосовано до рівняння теплопровідності зі скінченною затримкою для матеріалу з пам'яттю.

1. Introduction and preliminaries. The main concern of this paper is the existence and attraction property of an unstable manifold of $\mathcal{E}$-class for solutions to the partial neutral functional-differential equation (PNFDE)

$$\frac{\partial}{\partial t}Fu_t = A(t)Fu_t + f(t,u_t), \quad t \in \mathbb{R}, \tag{1.1}$$

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where $(A(t))_{t\in\mathbb{R}}$ is a family of (possibly unbounded) linear operators on a Banach X; $F:\mathcal{C}\to X$ is a bounded linear operator called a difference operator, $f:\mathbb{R}\times\mathcal{C}\to X$ is a continuous nonlinear operator called a delay operator, where $\mathcal{C}=C([-r,0],X)$, and u_t is the history function defined by $u_t(\theta):=u(t+\theta)$ for $\theta\in[-r,0]$.

The investigation for existence of an invariant manifold for solutions to (1.1) is of great importance since, on the one hand, it describes the behavior of solutions around a steady state or near some specific solution, and on the other hand, it attracts all other solutions of the equation so that the research of properties of all solutions can be deduced to studying the solutions on that manifold using the reduction principle. The classical conditions for the presence of such a manifold are two folds, firstly, the exponential dichotomy of the solution operators of corresponding linear homogeneous equations, and secondly the uniform Lipschitz continuity of the nonlinear term $f(t,u_t)$ with a sufficiently small Lipschitz constant, i.e., $\|f(t,\phi)-f(t,\psi)\|\le q\|\phi-\psi\|_{\mathcal{C}}$ for sufficiently small q (see, e.g., [9, 12, 13] and the references therein).

Huy [2] showed such results for general semilinear evolution equations with nonlinear terms being φ-Lipschitz and suitable for complicated diffusion processes. Moreover, in [1], Huy has proved the existence of a new type of invariant manifolds, called the invariant stable manifolds of admissible classes. Such manifolds have been constituted by trajectories belonging to the admissible Banach space E which can be L_p-space, Lorentz spaces $L_{p,q}$ or some interpolation space.

The purpose of the present paper is to prove the existence of unstable manifolds of admissible classes and their attraction property. We prove the existence of such manifolds for Eq. (1.1), when its linear part $(B(t))_{t\ge0}$ generates the evolution family having an exponential dichotomy on $\mathbb{R}$, and its nonlinear term is φ-Lipschitz, i.e., $\|f(t,\phi)-f(t,\psi)\|\le q\|\phi-\psi\|_{\mathcal{C}}$, where $\phi,\psi\in\mathcal{C}$ and $\varphi(t)$ is a real and positive function which belong admissible function space.

As mentioned in [4], when handling with PNFDE we face a difficult fact that the differential operators do not apply directly to $u(t)$ but to Fu_t, and hence the variation-of-constant formula is available only for Fu_t. Therefore, we write F in the form $F=\delta_0-(\delta_0-F)$, with Dirac distribution δ_0 concentrated at 0. Then we need certain "smallness" of $\Psi:=\delta_0-F$. It can be proved that, using a renorming procedure, the smallness of Ψ can be substituted by the fact that Ψ has "no mass in 0", and, in case that Ψ is written as an operator integral with a kernel η of bounded variation, the condition "having no mass in 0" of Φ is equivalent to the fact that η is non-atomic at 0 (see the details in [3]). Furthermore, another difficulty is lying in the fact that the admissibly inertial manifold is constituted by trajectories of the solutions belonging to (rescaledly) general admissible function spaces which are not necessary L_∞-spaces. Therefore, the techniques and methodology used in the paper [4] cannot directly be applied here. Instead, we use the duality arguments together with generalized Hölder inequalities to obtain necessary estimates corresponding to the dichotomy of the evolution family. Then we apply our techniques and results in [1] (see also [5]) of using admissibility of function spaces to construct the solutions of Lyapunov – Perron equation which will be used to derive the existence of invariant unstable manifolds of $\mathcal{E}$-class and center-invariant unstable of $\mathcal{E}$-class. Our main results are contained in Theorems 2.2, 2.3 and 3.1.

Next, we recall notions and concepts for latter use.

For a Banach spaces X (with norm $\|\cdot\|$) and a given $r > 0$, we denote by $\mathcal{C} := C([-r, 0], X)$ the Banach space of all continuous functions from $[-r, 0]$ into X, equipped the norm $\|\phi\|_{\mathcal{C}} = = \sup_{\theta\in[-r,0]} \|\phi(\theta)\|$ for $\phi \in \mathcal{C}$. For a continuous function $v : \mathbb{R} \longrightarrow X$ and each $t \in \mathbb{R}$, the history function $v_t \in \mathcal{C}$ is defined by $v_t(\theta) = v(t + \theta)$ for all $\theta \in [-r, 0]$.

Definition 1.1. *A family of bounded linear operators $\mathcal{U} = (U(t, s))_{t\geq s}$ on a Banach space X is a* (*strongly continuous, exponentially bounded*) *evolution family on the line if*

(i) $U(t, t) = Id$ *and* $U(t, r)U(r, s) = U(t, s)$ *for* $t \geq r \geq s$,

(ii) *the map* $(t, s) \mapsto U(t, s)x$ *is continuous for every* $x \in X$,

(iii) *there are constants* $K \geq 1$ *and* $\alpha \in \mathbb{R}$ *such that* $\|U(t, s)\| \leq Ke^{\alpha(t-s)}$ *for* $t \geq s$.

This notion has been invented to represent the solutions to Cauchy problem

$$\begin{aligned} \frac{du(t)}{dt} &= A(t)u(t), \quad t \geq s, \\ u(s) &= x_s \in X, \end{aligned} \tag{1.2}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of (unbounded) linear operators on X, which generates the evolution family $\mathcal{U} = (U(t, s))_{t\geq s}$. That is to say, under some appropriate conditions, the solutions to Cauchy problem (1.2) can be represented by that evolution family as $u(t) := U(t, s)u(s)$. We refer the reader to Pazy [11] (see also [10]) for a detailed treatment of the matter.

We then briefly recall some notions on function spaces taken from Massera and Schäffer [7] and Huy et al. [1, 5, 6].

Let E be admissible function spaces and E' be its associate space defined as in [5, 6]. Then we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}, \mathcal{C}) := \{g : \mathbb{R} \to \mathcal{C} : g \text{ is strongly measurable and } \|g(\cdot)\|_{\mathcal{C}} \in E\}$$

endowed with the norm

$$\|g\|_{\mathcal{E}} := \big\| \|g(\cdot)\|_{\mathcal{C}} \big\|_E.$$

Then clearly $\mathcal{E}$ is a Banach space, called *the Banach space corresponding to the admissible function space E*. Moreover, the following hypothesis is needed in our strategy.

Standing Hypothesis 1.1. *We will consider the Banach function space E and its associate space E' such that both are admissible spaces. Furthermore, we suppose that E' contains an exponentially E-invariant function $\varphi \geq 0$ satisfying that, for any fixed $\nu > 0$, the function $h_\nu(\cdot)$, defined by*

$$h_\nu(t) := \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'} \quad \textit{for} \quad t \in \mathbb{R},$$

belongs to E.

We refer the readers to [5] for various examples of admissible spaces and their applications to invariant manifolds of admissible classes. Typical examples of admissible spaces satisfying the above hypothesis are L_p-spaces with one type of exponentially L_p-invariant functions of the form $\beta e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed $\beta, \alpha > 0$.

2. Existence and attractiveness of admissible unstable manifolds. In this section, we prove the existence of an admissible unstable manifold of $\mathcal{E}$-class for the mild solutions of Eq. (1.1). Throughout this section we assume that the evolution family $\{U(t,s)\}_{t\geq s}$ has an exponential dichotomy on $\mathbb{R}$. We recall now the notion of exponential dichotomies on the whole line.

Definition 2.1. *An evolution family* $(U(t,s))_{t\geq s}$ *on the Banach space* X *is said to have an exponential dichotomy on* $\mathbb{R}$ *if there exist bounded linear projections* $P(t)$, $t\in\mathbb{R}$, *on* X *and positive constants* N, ν *such that*:

(a) $U(t,s)P(s)=P(t)U(t,s),\ t\geq s,$

(b) *the restriction* $U(t,s)_|:\ \operatorname{Ker}P(s)\to\operatorname{Ker}P(t),\ t\geq s,$ *is an isomorphism* $\big($*and we denote its inverse by* $(U(t,s)_|)^{-1}=U(s,t)_|$ *for* $t\geq s\big)$,

(c) $\|U(t,s)x\|\leq Ne^{-\nu(t-s)}\|x\|$ *for* $x\in P(s)X,\ t\geq s,$

(d) $\|U(s,t)_|x\|\leq Ne^{-\nu(t-s)}\|x\|$ *for* $x\in\operatorname{Ker}P(t),\ t\geq s.$

The projections $P(t)$, $t\in\mathbb{R}$, *are called the dichotomy projections*, *and the constants* N, ν *are the dichotomy constants*.

For an evolution family $(U(t,s))_{t\geq s}$ having an exponential dichotomy on the whole line, we can define the Green function on $\mathbb{R}$ as follows:

$$\mathcal{G}(t,\tau)=\begin{cases}P(t)U(t,\tau) & \text{for}\quad t\geq\tau,\\ -U(t,\tau)_|(I-P(\tau)) & \text{for}\quad t<\tau.\end{cases}\tag{2.1}$$

Thus, we have

$$\|\mathcal{G}(t,\tau)\|\leq N(1+H)e^{-\nu|t-\tau|}\quad\text{for all}\quad t\neq\tau,$$

where $H:=\sup_{t\in\mathbb{R}}\|P(t)\|<\infty$. Note that the exponential dichotomy of $(U(t,s))_{t\geq s}$ implies that $H:=\sup_{t\in\mathbb{R}}\|P(t)\|<\infty$ and the map $t\mapsto P(t)$ is strongly continuous (see [8], Lemma 4.2, for the same discussion).

We give next the notion of the φ-Lipschitz of the nonlinear term f.

Definition 2.2. *Let* E *be an admissible Banach function space and* φ *be a positive function belonging to* E. *A function* $f:\mathbb{R}\times\mathcal{C}\to X$ *is said to be* φ*-Lipschitz if* f *satisfies*:

(i) $\|f(t,0)\|=0$ *for all* $t\in\mathbb{R}$,

(ii) $\|f(t,\phi_1)-f(t,\phi_2)\|\leq\varphi(t)\|\phi_1-\phi_2\|_{\mathcal{C}}$ *for all* $t\in\mathbb{R}$ *and all* $\phi_1,\phi_2\in\mathcal{C}$.

Note that if $f(t,\phi)$ is φ-Lipschitz, then $\|f(t,\phi)\|\leq\varphi(t)\|\phi\|_{\mathcal{C}}$ for all $\phi\in\mathcal{C}$ and $t\in\mathbb{R}$. Note also that φ is locally integrable (because it belongs to an admissible space), it follows that $f(t,u_t)$ is locally integrable.

To prove the existence of an unstable manifold, instead of (1.1), we consider the following integral equations:

$$\begin{aligned}&Fu_t=U(t,s)F\phi+\int\limits_s^t U(t,\xi)f(\xi,u_\xi)d\xi\quad\text{for}\quad t\geq s,\\ &u_s=\phi\in\mathcal{C}.\end{aligned}\tag{2.2}$$

We note that if the evolution family $(U(t,s))_{t\geq s}$ arising from the well-posed Cauchy problem (1.2), then the function $u:\mathbb{R}\longrightarrow X$, which satisfies (2.2) for some given function f, is called a mild solution of the semilinear problems

$$\frac{\partial}{\partial t}Fu_t = A(t)Fu_t + f(t,u_t), \quad t \ge s,$$

$$u_s = \phi \in \mathcal{C}.$$

The following lemma gives the form of bounded solutions to Eq. (2.2).

Lemma 2.1. *Let the evolution family $(U(t,s))_{t\ge s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\nu > 0$. Assume Standing Hypothesis 1.1 and let φ be an exponentially E-invariant function defined as in that Standing Hypothesis 1.1. Let $F : \mathcal{C} \to X$ and $f : \mathbb{R}\times\mathcal{C} \to X$ be respectively the difference and delay operators. Suppose that the difference operator F is of the form $F = \delta_0 - \Psi$ for $\Psi \in \mathcal{L}(\mathcal{C}, X)$ with $\|\Psi\| \le 1$, and δ_0 being the Dirac function concentrated at 0. Suppose that f is φ-Lipschitz and $u(t)$ is a solution to Eq. (2.2) on $(-\infty, t_0]$ such that the function $x(t) = \begin{cases} u_t & \text{for } t \le t_0, \\ 0 & \text{for } t > t_0, \end{cases}$ $t \in \mathbb{R}$, belongs to $\mathcal{E}$.*

Then, for $t \le t_0$, the function $u(t)$ satisfies

$$Fu_t = U(t,t_0)_{|}\nu_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)f(\tau,u_\tau)d\tau \tag{2.3}$$

for some $\nu_1 \in X_1(t_0) = (I - P(t_0))X$, where $\mathcal{G}(t,\tau)$ is the Green function defined as in (2.1).

Proof. Put $z(t) = \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)f(\tau,u_\tau)d\tau$ for all $t \le t_0$. We have

$$\|z(t)\| \le \int_{-\infty}^{t_0} N(1+H)e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}d\tau \le$$

$$\le N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}d\tau.$$

Since $t+\theta \in [-r+t, t]$ for fixed $t \in (-\infty, t_0]$ and $\theta \in [-r, 0]$, we have

$$\|z_t\|_{\mathcal{C}} = \sup_{-r\le\theta\le 0}\|y(t+\theta)\| \le N(1+H)e^{\nu r}\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}\,d\tau \quad \text{for} \quad t \le t_0.$$

Since $e^{-\nu|t-\cdot|}\varphi(\cdot) \in E'$, $\|u_\cdot\|_{\mathcal{C}} \in E$ using the “Hölder inequality” [6] (inequality (15)), we obtain

$$\|z_t\|_{\mathcal{C}} \le N(1+H)e^{\nu r}\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}\|\|u_\cdot\|_{\mathcal{C}}\|_E = N(1+H)e^{\nu r}h_\nu(t)\|u(\cdot)\|_{\mathcal{E}} \quad \text{for} \quad t \le t_0.$$

Therefore, by Banach lattice properties we have that $z(\cdot) \in \mathcal{E}$ and

$$\|z(\cdot)\|_{\mathcal{E}} \le N(1+H)e^{\nu r}\|h_\nu(\cdot)\|_E\|u(\cdot)\|_{\mathcal{E}}.$$

By straightforward calculations, we get

$$z(t_0) = U(t_0,t)z(t) + \int_t^{t_0} U(t_0,\tau)f(\tau,u_\tau)d\tau \quad \text{for} \quad t \le t_0.$$

Indeed,

$$U(t_0,t)z(t)+\int\limits_t^{t_0}U(t_0,\tau)f(\tau,u_\tau)d\tau=$$

$$=\int\limits_t^{t_0}U(t_0,\tau)f(\tau,u_\tau)d\tau+U(t_0,t)\int\limits_{-\infty}^{t_0}\mathcal{G}(t,\tau)f(\tau,u_\tau)d\tau=$$

$$=\int\limits_t^{t_0}U(t_0,\tau)f(\tau,u_\tau)d\tau+\int\limits_{-\infty}^{t}U(t_0,\tau)P(\tau)f(\tau,u_\tau)d\tau-\int\limits_t^{t_0}U(t_0,\tau)(I-P(\tau))f(\tau,u_\tau)d\tau=$$

$$=\int\limits_{-\infty}^{t_0}U(t_0,\tau)P(\tau)f(\tau,u_\tau)d\tau=\int\limits_{-\infty}^{t_0}\mathcal{G}(t_0,\tau)f(\tau,u_\tau)d\tau=z(t_0).$$

On the other hand,

$$Fu_{t_0}=U(t_0,t)Fu_t+\int\limits_t^{t_0}U(t_0,\tau)f(\tau,u_\tau)d\tau \quad \text{for} \quad t\le t_0.$$

Hence, $Fu_{t_0}-z(t_0)=U(t_0,t)(Fu_t-z(t))$. For $\xi\le t$, we have

$$P(t)Fu_t=P(t)U(t,\xi)Fu_\xi+P(t)\int\limits_\xi^{t}U(t,\tau)f(\tau,u_\tau)d\tau=$$

$$=U(t,\xi)P(\xi)Fu_\xi+\int\limits_\xi^{t}U(t,\tau)P(\tau)f(\tau,u_\tau)d\tau.$$

Therefore, letting $\xi\to-\infty$, we obtain

$$P(t)[Fu_t-z(t)]=P(t)Fu_t-\int\limits_{-\infty}^{t}U(t,\tau)P(\tau)f(\tau,u_\tau)d\tau=\lim_{\xi\to-\infty}U(t,\xi)P(\xi)Fu_\xi.$$

We assume that $\lim_{\xi\to-\infty}U(t,\xi)P(\xi)Fu_\xi=m\ne0$. On the other hand,

$$\big\|U(t,\xi)P(\xi)Fu_\xi\big\|\le Ne^{-\nu(t-\xi)}\|P(\xi)Fu_\xi\|\le Ne^{-\nu(t-\xi)}H(1+\|\Psi\|)\|u_\xi\|_{\mathcal{C}}.$$

So, $e^{-\nu\xi}\|U(t,\xi)P(\xi)Fu_\xi\|\le Ne^{-\nu t}H(1+\|\Psi\|)\|u_\xi\|_{\mathcal{C}}$ for all $\xi\le t$. By Banach lattice property, we have $e^{-\nu\xi}\|U(t,\xi)P(\xi)Fu_\xi\|\in E$. Moreover, we also obtain $\lim_{\xi\to-\infty}e^{-\nu\xi}\|U(t,\xi)P(\xi)Fu_\xi\|=$ $=\infty$. Therefore,

$$\sup_{\xi\le t}\int\limits_{\xi-1}^{\xi}e^{-\nu\tau}\|U(t,\tau)P(\tau)Fu_\tau\|d\tau=\infty.$$

This contradict to $E \hookrightarrow \mathbf{M}(\mathbb{R})$ (see [6], Remark 1.5). So, $\lim_{\xi\to-\infty} U(t,\xi)P(\xi)Fu_\xi = 0$.

Thus, $Fu_t - z(t) \in \operatorname{Ker}(P(t))$. This leads to $Fu_{t_0} - z(t_0) = U(t_0,t)(Fu_t - z(t)) \in \operatorname{Ker}(P(t_0))$. Putting $\nu_1 = Fu_{t_0} - z(t_0)$, we have that $Fu_t = U(t,t_0)_|\nu_1 + z(t)$ for all $t \le t_0$.

Lemma 2.1 is proved.

***Remark* 2.1.** We call Eq. (2.3) the *Lyapunov–Perron equation*. By computing directly, we can see that the converse of Lemma 2.1 is also true in the sense that all solutions of Eq. (2.3) on $(-\infty, t_0]$ satisfy Eq. (2.2) for all $s \le t \le t_0$.

In case the evolution $(U(t,s))_{t\ge s}$ have an exponential dichotomy, using the projections $(P(t))_{t\in\mathbb{R}}$ on X, we can define the operators $\big(\widetilde{P}(t)\big)_{t\in\mathbb{R}}$ on $\mathcal{C}$ as follows.

For each $t \in \mathbb{R}$, we set that

$$\widetilde{P}(t)\colon \mathcal{C} \longrightarrow \mathcal{C},$$

$$(\widetilde{P}(t)\phi)(\theta) = U(t+\theta,t)_|(I - P(t))\phi(0) \quad \text{for all} \quad \theta \in [-r,0]. \tag{2.4}$$

We easily see that $\big(\widetilde{P}(t)\big)^2 = \widetilde{P}(t)$, so the operators $\big(\widetilde{P}(t)\big)_{t\in\mathbb{R}}$ are projections on $\mathcal{C}$. Moreover,

$$\operatorname{Im}\widetilde{P}(t) = \Big\{\phi \in \mathcal{C}\colon \phi(\theta) = U(t+\theta,t)_|\nu_1 \ \text{ for all } \ \theta\in[-r,0] \ \text{ for some } \ \nu_1 \in \operatorname{Ker}P(t)\Big\}. \tag{2.5}$$

We then come to our first result on the existence, uniqueness and exponential stability of solution to (2.3) with initial function belonging to $\operatorname{Im}\widetilde{P}(t)$. To do this, we first recall the notion of the integral translation operators Λ_1 (see [6], Definition 1.3, Proposition 1.6) as follows: for $\varphi \in E$, $\Lambda_1\varphi$ is defined by $\Lambda_1\varphi(t) := \int_t^{t+1}\varphi(\tau)d\tau$ belong to E for all $t \in \mathbb{R}$.

Theorem 2.1. *Let the evolution family $\{U(t,s)\}_{t\ge s}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t\in\mathbb{R}$, and constants N, $\nu > 0$. Consider the projections $\widetilde{P}(t)$ defined as in* (2.4), *and function h_ν defined as in Standing Hypothesis* 1.1. *Let the difference operator $F\colon \mathcal{C}\to X$ be of the form $F = \delta_0 - \Psi$ for $\Psi \in \mathcal{L}(\mathcal{C},X)$ with $\|\Psi\| \le 1$, and δ_0 being the Dirac function concentrated at* 0. *Suppose that the delay operator $f\colon \mathbb{R}\times\mathcal{C}\to X$ is φ-Lipschitz for $\varphi \in E'$ being an exponentially E-invariant function as in Standing Hypothesis* 1.1, *and set $k = N(1+H)e^{\nu r}\|h_\nu\|_E$. Then, if $\dfrac{k}{1-\|\Psi\|} < 1$, there corresponding to each $\phi \in \operatorname{Im}\widetilde{P}(t_0)$ one and only one solution $u(t)$ of* (2.3) *on $(-\infty,t_0]$ satisfying the conditions that $\widetilde{P}(t_0)\widetilde{u}_{t_0} = \phi$ and $x(t) = \begin{cases} u_t & \text{for } t\le t_0, \\ 0 & \text{for } t > t_0, \end{cases}$ $t\in\mathbb{R}$, belongs to $\mathcal{E}$, where the function $\widetilde{u}_{t_0}$ is defined by $\widetilde{u}_{t_0}(\theta) = Fu_{t_0+\theta}$ for all $-r\le\theta\le 0$. Moreover, if $\dfrac{N(1+H)e^{\nu r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-\|\Psi\|} < 1$, then the following estimate is valid for any two solutions $u(\cdot)$, $v(\cdot)$ corresponding to different initial function $\phi,\psi \in \operatorname{Im}\widetilde{P}(t_0)$:*

$$\|u_t - v_t\|_{\mathcal{C}} \le C_\mu e^{-\mu(t_0-t)}\|\phi(0)-\psi(0)\| \quad \textit{for all} \quad t\le t_0, \tag{2.6}$$

where μ is a positive number satisfying

$$0 < \mu < \nu + \ln\!\left(1 - \frac{N(1+H)e^{\nu r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-\|\Psi\|}\right)$$

and

$$C_\mu = \frac{Ne^{\nu r}}{1 - \|\Psi\| - \dfrac{N(1+H)e^{\nu r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{(1-\|\Psi\|)(1-e^{-(\nu-\mu)})}}.$$

Proof. Firstly, we prove that there corresponding to each $\phi \in \operatorname{Im}\widetilde{P}(t_0)$ one and only one solution $u(t)$ in $\mathcal{E}$ of Eq. (2.3) on $(-\infty, t_0]$. Since $\phi \in \operatorname{Im}\widetilde{P}(t_0)$, by (2.5), there exists $\nu_1 \in \operatorname{Ker} P(t_0)$ such that $\phi(\theta) = U(t_0+\theta, t_0)_{|}\nu_1$ for all $-r \le \theta \le 0$. Clearly, $\nu_1 = \phi(0)$. Denote by $C_b((-\infty, t_0], X)$ the Banach space of bounded, continuous and X-valued functions defined on $(-\infty, t_0]$. For $\nu_1 = \phi(0) \in \operatorname{Ker} P(t_0)$ as above, we define a mapping

$$\widetilde{F}_\phi \colon C_b((-\infty, t_0], X) \to C_b((-\infty, t_0], X)$$

by

$$(\widetilde{F}_\phi u)(t) = U(t, t_0)_{|}\nu_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau) f(\tau, u_\tau)\,d\tau.$$

We define the operator $\widetilde{\Psi} \colon C_b((-\infty, t_0], X) \to C_b((-\infty, t_0], X)$ by

$$(\widetilde{\Psi} u)(t) = \Psi u_t \quad \text{for} \quad t \le t_0.$$

Since $\|\Psi\| < 1$, we have $\|\widetilde{\Psi}\| \le \|\Psi\| < 1$. Therefore, the operator $(I - \widetilde{\Psi})$ is invertible.

We now put $T := (I-\widetilde{\Psi})^{-1}\widetilde{F}_\phi$. Then we have

$$\|(\widetilde{F}_\phi u)(t)\| \le Ne^{-\nu(t_0-t)}\|\nu_1\| + N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}\,d\tau =$$

$$= NT^+_{t_0}e_\nu(t)\|\nu_1\| + N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}\,d\tau.$$

Since $t+\theta \in [-r+t, t]$ for fixed $t \in (-\infty, t_0]$ and all $\theta \in [-r, 0]$, we obtain

$$\|(Tu)(t)\|_{\mathcal{C}} \le \sum_{n=0}^{\infty}\|\Psi\|^n \left(NT^+_{t_0}e_\nu(t)e^{\nu r}\|\nu_1\| + N(1+H)e^{\nu r}\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau\|_{\mathcal{C}}\,d\tau \right).$$

According to the “Hölder inequality”, we get

$$\|(Tu)(t)\|_{\mathcal{C}} \le \frac{1}{1-\|\Psi\|}\Big(NT^+_{t_0}e_\nu(t)e^{\nu r}\|\nu_1\| + N(1+H)e^{\nu r}h_\nu(t)\|u(\cdot)\|_{\mathcal{E}} \Big).$$

Therefore, by Banach lattice properties we have $(Tu)(\cdot) \in \mathcal{E}$ and

$$\|(Tu)(\cdot)\|_{\mathcal{E}} \le \frac{1}{1-\|\Psi\|}\Big(NN_1\|e_\nu\|_E e^{\nu r}\|\nu_1\| + N(1+H)e^{\nu r}\|h_\nu(\cdot)\|_E\|u(\cdot)\|_{\mathcal{E}} \Big).$$

Hence, the transformation T acts from $\mathcal{E}$ into $\mathcal{E}$. Next, we will prove T is a contraction mapping. Using the Neumann series, we obtain

$$(Tu)(t) - (Tv)(t) = \left[\left(\sum_{n=0}^{\infty} \widetilde{\Psi}^n\right)\widetilde{F}_\phi u\right](t) - \left[\left(\sum_{n=0}^{\infty} \widetilde{\Psi}^n\right)\widetilde{F}_\phi v\right](t) =$$

$$= \left[\left(\widetilde{F}_\phi u\right)(t) - \left(\widetilde{F}_\phi v\right)(t)\right] + \left[\left(\widetilde{\Psi}\widetilde{F}_\phi u\right)(t) - \left(\widetilde{\Psi}\widetilde{F}_\phi v\right)(t)\right] + \ldots .$$

We then estimate

$$\left\|\left(\widetilde{F}_\phi u\right)(t) - \left(\widetilde{F}_\phi v\right)(t)\right\| \le \int_{-\infty}^{t_0} \|\mathcal{G}(t,\tau)(f(\tau,u_\tau) - f(\tau,v_\tau))\| d\tau \le$$

$$\le N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \quad \text{for} \quad t \le t_0.$$

Next, by induction we can easily see that

$$\left\|\left(\widetilde{\Psi}^n\widetilde{F}_\phi u\right)(t) - \left(\widetilde{\Psi}^n\widetilde{F}_\phi v\right)(t)\right\| \le \|\Psi\|^n N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \quad \text{for} \quad t \le t_0.$$

From the above claim it follow that

$$\|(Tu)(t) - (Tv)(t)\| \le \sum_{n=0}^{\infty} \|\Psi\|^n N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau =$$

$$= \frac{1}{1-\|\Psi\|} N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \quad \text{for} \quad t \le t_0.$$

Since $t+\theta \in [-r+t, t]$ for fixed $t \in (-\infty, t_0]$ and all $\theta \in [-r, 0]$, we have

$$\|(Tu)(t) - (Tv)(t)\|_{\mathcal{C}} = \sup_{-r\le\theta\le 0} \|(Tu)(t+\theta) - (Tv)(t+\theta)\| \le$$

$$\le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r}\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau.$$

Since $e^{-\nu|t-\cdot|}\varphi(\cdot) \in E'$, $\|u_\tau - v_\tau\|_{\mathcal{C}} \in E$, and using the “Hölder inequality” [6] (inequality (15)) it follows from the above inequality that

$$\|(Tu)(t) - (Tv)(t)\|_{\mathcal{C}} \le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r}\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'}\|\|u(\cdot) - v(\cdot)\|_{\mathcal{C}}\|_E \le$$

$$\le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r}h_\nu(t)\|u(\cdot) - v(\cdot)\|_{\mathcal{E}} \quad \text{for} \quad t \le t_0.$$

By the Banach lattice property of E and the fact that $h_\nu(\cdot) \in E$ it follows that $\|Tu(\cdot)\|_{\mathcal{C}} \in E$. Thus, $Tu \in \mathcal{E}$, and we have

$$\|Tu - Tv\|_{\mathcal{E}} \leq \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r}\|h_\nu\|_E.\|u-v\|_{\mathcal{E}} = \frac{k}{1-\|\Psi\|}\|u-v\|_{\mathcal{E}}.$$

Next, if $\frac{k}{1-\|\Psi\|} < 1$, the transformation T is a contraction mapping from $\mathcal{E}$ to it self. Hence, there exists a unique $u(\cdot) \in \mathcal{E}$ such that $Tu = u$. This yield that $u(t),\ t \leq t_0$, is the unique solution of (2.3) with

$$(\widetilde{F}_\phi u_{t_0})(\theta) = U(t_0+\theta, t_0)_{|}\nu_1 + \int\limits_{-\infty}^{t_0} \mathcal{G}(t_0+\theta, \tau) f(\tau, u_\tau) d\tau \quad \text{for all} \quad \theta \in [-r, 0],$$

and $(I - P(t_0))Fu_{t_0} = \nu_1 = \phi(0)$. Therefore, $\widetilde{P}(t_0)\widetilde{u}_{t_0} = \phi$ by the definition of $\widetilde{P}(t_0)$ (see (2.4)).

Secondly, by using the similar arguments as in [4] (Theorem 3.5) combined with the "Hölder inequality" in the admissible function spaces E and E', we obtain (2.6).

Theorem 2.1 is proved.

Definition 2.3. *A set $\boldsymbol{U} \subset \mathbb{R} \times \mathcal{C}$ is said to be unstable manifold of $\mathcal{E}$-class for the solution to Eq.* (2.2) *if for every $t \in \mathbb{R}$ the phase space splits into a direct sum $\mathcal{C} = \widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$ with corresponding projections $\widetilde{P}(t),\ t \in \mathbb{R}$ (i.e., $\widetilde{X}_0(t) = \operatorname{Im}\widetilde{P}(t),\ \widetilde{X}_1(t) = \operatorname{Ker}\widetilde{P}(t)$) such that $\sup_{t\in\mathbb{R}} \|\widetilde{P}(t)\| < \infty$, and there exists a family of Lipschitz continuous mapping*

$$\widetilde{y}_t : \widetilde{X}_0(t) \to \widetilde{X}_1(t), \quad t \in \mathbb{R},$$

with the Lipschitz constants being independent of t such that

(i) $\boldsymbol{U} = \Big\{(t, \psi + \widetilde{y}_t(\psi)) \in \mathbb{R} \times (\widetilde{X}_0(t) \oplus \widetilde{X}_1(t)) | t \in \mathbb{R}, \psi \in \widetilde{X}_0(t)\Big\}$, *and we denote by*

$$\boldsymbol{U}_t = \{\psi + \widetilde{y}_t(\psi) : (t, \psi + \widetilde{y}_t(\psi)) \in \boldsymbol{U},\ t \in \mathbb{R}\};$$

(ii) *$\boldsymbol{U}_t$ is homeomorphic to $\widetilde{X}_0(t)$ for all $t \in \mathbb{R}$;*

(iii) *to each $t_0 \in \mathbb{R},\ \phi \in \boldsymbol{U}_{t_0}$ there corresponds one and only one solution $u(\cdot)$ of Eq.* (2.2) *on $(-\infty, t_0]$ satisfying the conditions that $\widetilde{u}_{t_0} = \phi$ and*

$$x(t) = \begin{cases} u_t & \textit{for} \quad t \leq t_0, \\ 0 & \textit{for} \quad t > t_0, \end{cases} \quad t \in \mathbb{R}, \quad \textit{belongs to} \quad \mathcal{E},$$

where the functions $\widetilde{u}_{t_0}$ is defined as in Theorem 2.1. *Moreover, any two solutions $u(\cdot)$ and $v(\cdot)$ of* (2.2) *corresponding to different initial functions $\phi_1, \phi_2 \in \boldsymbol{U}_{t_0}$ backwardly and exponentially attract each other in the sense that there exist positive constants μ and C_μ independent of t_0 such that*

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{-\mu(t_0-t)} \Big\| \Big(\widetilde{P}(t_0)\phi_1\Big)(0) - \Big(\widetilde{P}(t_0)\phi_2\Big)(0) \Big\| \quad \textit{for} \quad t \leq t_0;$$

(iv) *$\boldsymbol{U}$ is positively F-invariant under* (2.2), *i.e., if $u(t),\ t \in \mathbb{R}$, is a solution to* (2.2) *satisfying conditions that $\widetilde{u}_{t_0} \in \boldsymbol{U}_{t_0}$ and function $x(t) = \begin{cases} u_t & \textit{for } t \leq t_0, \\ 0 & \textit{for } t > t_0, \end{cases}$ $t \in \mathbb{R}$, belongs to $\mathcal{E}$ for some $t_0 \in \mathbb{R}$, then we have $\widetilde{u}_t \in U_t$ for all $t \in \mathbb{R}$, where the function $\widetilde{u}_t$ is defined as in Theorem* 2.1 *with t_0 being replaced by t, i.e., $\widetilde{u}_t(\theta) = Fu_{t+\theta}$ for all $-r \leq \theta \leq 0$ and $t \in \mathbb{R}$.*

We now prove the existence of an unstable manifold of $\mathcal{E}$-class.

Theorem 2.2. *Under the hypotheses of Theorem* 2.1 *and function* $e_\nu(t) = e^{-\nu|t|}$ *for all* $t \in \mathbb{R}$. *Then if the* f *is* φ*-Lipschitz with* φ *satisfying*

$$\max\left\{\frac{N(1+H)e^{\nu r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-\|\Psi\|}, \frac{N^2N_1(1+H)e^{\nu r}\|e_\nu\|_E\|\varphi\|_{E'}}{1-k-\|\Psi\|}\right\} < 1,$$

then there exists an invariant unstable manifold $\boldsymbol{U}$ *of* $\mathcal{E}$*-class for the solutions of Eq.* (2.2), *where* k *is defined as in Theorem* 2.1.

Proof. The proof of this theorem can be done by the similar way as in [4] (Theorem 3.7) and using the structures of bounded solution as in Lemma 2.1. We just note that the family of Lipschitz mapping $(\widetilde{y}_t)_{t\in\mathbb{R}}$ determining the unstable manifold of $\mathcal{E}$-class in Definition 2.3 by

$$\widetilde{y}_t : \widetilde{X}_0(t) \to \widetilde{X}_1(t), \quad t \in \mathbb{R},$$

$$\widetilde{y}_t(\phi)(\theta) = \int_{-\infty}^{t} \mathcal{G}(t+\theta,\tau)f(\tau,u_\tau)d\tau \quad \text{for all} \quad \theta \in [-r,0].$$

Here, $u(\cdot)$ is the unique solution of Eq. (2.2) on $(-\infty, t]$ satisfying $\widetilde{P}(t)\widetilde{u}_t = \phi$ and $x(t) = \begin{cases} u_t & \text{for } t \le t_0, \\ 0 & \text{for } t > t_0, \end{cases}$ $t \in \mathbb{R}$, belongs to $\mathcal{E}$ (note that the existence and uniqueness of $u(\cdot)$ is guaranteed by Theorem 2.1). Using the "Hölder inequality", we obtain $\widetilde{y}_t$ is Lipschitz continuous with the Lipschitz constant

$$k_1 = \frac{N^2N_1(1+H)e^{\nu r}\|e_\nu\|_E\|\varphi\|_{E'}}{1-k-\|\Psi\|} \tag{2.7}$$

independent of t.

Theorem 2.2 is proved.

The next we will prove the attraction property of an invariant unstable manifold of $\mathcal{E}$-class for solutions of Eq. (2.2). Concretely, we will show that the unstable manifold of $\mathcal{E}$-class $\mathbf{U} = \{(t, \mathbf{U}_t)\}_{t\in\mathbb{R}}$ F-exponentially attracts all solutions to Eq. (2.2) in the sense that any solution $u(\cdot)$ to (2.2) is exponentially attracted to some F-induced trajectory $u^*(\cdot)$ lying in the unstable manifold of $\mathcal{E}$-class (i.e., $\widetilde{u}^*_t \in \mathbf{U}_t$ for all $t \in \mathbb{R}$). Precisely, we will prove the following theorem.

Theorem 2.3. *Assume that conditions of Theorem* 2.2 *are satisfied. For each fixed* $0 < \alpha < \nu$, *we define the functions* $e_{\nu-\alpha}(t) = e^{-(\nu-\alpha)|t|}$ *and* $h_{\nu-\alpha}(t) = \|e^{-(\nu-\alpha)|t-\cdot|}\varphi(\cdot)\|_{E'}$ *for* $t \in \mathbb{R}$. *Suppose that* $\dfrac{l}{1-\|\Psi\|} < 1$, *where*

$$l = N(1+H)e^{2\nu r}\max\left\{Nk_1 + \frac{(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-e^{-(\nu-\alpha)}},\ NN_1k_1\|e_{\nu-\alpha}\|_E + \|h_{\nu-\alpha}\|_E\right\},$$

k_1 *is defined in* (2.7). *Then the unstable manifold of* $\mathcal{E}$*-class* $\boldsymbol{U} = \{(t, \boldsymbol{U}_t)\}_{t\in\mathbb{R}}$ F*-exponentially attracts all solutions to Eq.* (2.2) *in the sense that for any solution* $u(\cdot)$ *to* (2.2) *with initial function* u_ξ *there exists a solution* $u^*(\cdot)$ *such that* $\widetilde{u}^*_t \in \boldsymbol{U}_t$ *for all* $t \in \mathbb{R}$ *such that*

$$\|u_t - u^*_t\|_{\mathcal{C}} \le Ce^{-\alpha(t-\xi)}\|u_\xi - u^*_\xi\|_{\mathcal{C}} \quad \text{for} \quad t \ge \xi,$$

where $\widetilde{u}^*_t(\theta) = Fu^*_{t+\theta}$ *for all* $\theta \in [-r,0]$, $t \in \mathbb{R}$.

Proof. For any fixed $\xi \in \mathbb{R}$, we introduce the space

$$C_{\xi,\alpha} = \Big\{x(\cdot) \in \mathcal{E} \text{ such that } x(t) = 0 \text{ for } t < \xi \text{ and } e^{\alpha(\cdot - \xi)}\|x(\cdot)\|_{\mathcal{C}} \in E \cap L_\infty(\mathbb{R})\Big\},$$

which is a Banach space endowed with the norm

$$\|x(\cdot)\|_\alpha = \max\Big\{\|e^{\alpha(\cdot-\xi)}\|x(\cdot)\|_{\mathcal{C}}\|_E,\ \|e^{\alpha(\cdot-\xi)}\|x(\cdot)\|_{\mathcal{C}}\|_\infty\Big\}.$$

We will find $u^*(\cdot)$ in the form $u^*(t) = u(t) + \omega(t)$ such that $z(t) = \begin{cases} \omega_t & \text{for } t \geq \xi, \\ 0 & \text{for } t < \xi, \end{cases}$ belongs to $C_{\xi,\alpha}$.

We see that $u^*(\cdot)$ is a solution to (2.2) if and only if $\omega(\cdot)$ is a solution of the equation

$$F\omega_t = U(t,\xi)F\omega_\xi + \int_\xi^t U(t,\tau)\big[f(\tau, u_\tau + \omega_\tau) - f(\tau, u_\tau)\big]d\tau.$$

To simplify the representation, we put $g(t,\omega_t) = f(t, u_t + \omega_t) - f(t, u_t)$. Then $g\colon \mathbb{R} \times \mathcal{C} \to X$ is also φ-Lipschitz and $g(t,0) = 0$. The equation for $\omega(t)$ can be rewritten as

$$F\omega_t = U(t,\xi)F\omega_\xi + \int_\xi^t U(t,\tau)g(\tau,\omega_\tau)d\tau. \tag{2.8}$$

In the same way as in the proof of Lemma 2.1 and Remark 2.1, we observe that the solution $\omega(t)$ of (2.8) defines on $[\xi - r, \infty)$ (here $\omega(t) = 0$ for $t < \xi - r$) such that $z(t)$ belongs to $\mathcal{E}$ if and only if satisfies

$$F\omega_t = U(t,\xi)\nu_0 + \int_\xi^\infty \mathcal{G}(t,\tau)g(\tau,\omega_\tau)d\tau \quad \text{for some} \quad \nu_0 \in \operatorname{Im} P(\xi) \quad \text{and} \quad t \geq \xi \tag{2.9}$$

and

$$F\omega_t = U(2\xi - t,\xi)\nu_0 + \int_\xi^\infty \mathcal{G}(2\xi - t,\tau)g(\tau,\omega_\tau)d\tau \quad \text{for some} \quad \nu_0 \in \operatorname{Im} P(\xi) \quad \text{and} \quad t \in [\xi - r, \xi]. \tag{2.10}$$

We will choose $\nu_0 \in \operatorname{Im} P(\xi)$ such that $u^*_\xi = u_\xi + \omega_\xi \in \mathbf{U}_\xi$. This means

$$\Big(I - \widetilde{P}(\xi)\Big)(u_\xi + \omega_\xi)(0) = \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(\theta) \quad \text{for} \quad \theta \in [-r, 0].$$

Hence,

$$\nu_0 = \Big(\omega_\xi - \widetilde{P}(\xi)\omega_\xi\Big)(0) = -\Big(u_\xi - \widetilde{P}(\xi)u_\xi\Big)(0) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0) =$$

$$= -P(\xi)u(\xi) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0). \tag{2.11}$$

Substituting (2.11) into (2.9) and (2.10), we obtain

$$F\omega_t = \begin{cases} U(t,\xi)\Big[-P(\xi)u(\xi) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0)\Big] + \\ \quad + \int_\xi^\infty \mathcal{G}(t,\tau)g(\tau,\omega_\tau)d\tau & \text{for} \quad t \ge \xi, \\ U(2\xi - t,\xi)\Big[-P(\xi)u(\xi) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0)\Big] + \\ \quad + \int_\xi^\infty \mathcal{G}(2\xi - t,\tau)g(\tau,\omega_\tau)d\tau & \text{for} \quad t \in [\xi - r, \xi]. \end{cases} \tag{2.12}$$

Thus, $u^*(t)$ is a solution to (2.2) and satisfies $u^*_\xi \in \mathbf{U}_\xi$ if and only if $\omega(t)$ satisfies (2.12).

Next, in order to prove the existence of $u^*(t)$ satisfying assertions of the theorem, we will find solution $\omega(t)$ of Eq. (2.12) in the Banach space $C_{\xi,\alpha}$. To do this, we define a mapping

$$\widetilde{F}_\phi \colon C([\xi - r,\infty), X) \to C([\xi - r,\infty), X)$$

by

$$\Big(\widetilde{F}_\phi\omega\Big)(t) = \begin{cases} U(t,\xi)\Big[-P(\xi)u(\xi) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0)\Big] + \\ \quad + \int_\xi^\infty \mathcal{G}(t,\tau)g(\tau,\omega_\tau)d\tau & \text{for} \quad t \ge \xi, \\ U(2\xi - t,\xi)\Big[-P(\xi)u(\xi) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0)\Big] + \\ \quad + \int_\xi^\infty \mathcal{G}(2\xi - t,\tau)g(\tau,\omega_\tau)d\tau & \text{for} \quad t \in [\xi - r, \xi]. \end{cases}$$

We also define the operator $\widetilde{\Psi} \colon C([\xi - r,\infty), X) \to C([\xi - r,\infty), X)$ by

$$\Big(\widetilde{\Psi}u\Big)(t) = \begin{cases} \Psi(u_t) & \text{for} \quad t \ge \xi, \\ \Psi(u_\xi) & \text{for} \quad \xi - r \le t < \xi. \end{cases}$$

Since $\|\Psi\| < 1$, we have $\|\widetilde{\Psi}\| \le \|\Psi\| < 1$. Therefore, the operator $I - \widetilde{\Psi}$ is invertible and we now put $T = (I - \widetilde{\Psi})^{-1}\widetilde{F}_\phi$. We will prove that transformation T as above acts from $C_{\xi,\alpha}$ into $C_{\xi,\alpha}$ is a contraction mapping. Firstly, we show that $T\omega \in C_{\xi,\alpha}$. Indeed, for $t \ge \xi - r$, using the Neumann series, we have

$$(T\omega)(t) = \left[\left(\sum_{n=0}^\infty \widetilde{\Psi}^n\right)\widetilde{F}_\phi\omega\right](t).$$

Then we estimate

$$\left\|\Big(\widetilde{F}_\phi\omega\Big)(t)\right\| \le Ne^{-\nu(t-\xi)}\|\nu_0\| + N(1+H)\int\limits_\xi^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|\omega_\tau\|_{\mathcal{C}}d\tau \quad \text{for} \quad t \ge \xi$$

and similarly

$$\left\|\left(\widetilde{F}_{\phi}\omega\right)(t)\right\| \leq Ne^{-\nu(\xi-t)}\|\nu_0\| + N(1+H)\int\limits_{\xi}^{\infty} e^{-\nu|2\xi-t-\tau|}\varphi(\tau)\|\omega_\tau\|_{\mathcal{C}}d\tau \quad \text{for} \quad t\in[\xi-r,\xi].$$

From the inequality $\|\widetilde{\Psi}\| \leq \|\Psi\|$, it follows that

$$\|(T\omega)(t)\| \leq \sum_{n=0}^{\infty}\|\Psi\|^n\left[Ne^{-\nu(t-\xi)}\|\nu_0\| + N(1+H)\int\limits_{\xi}^{\infty} e^{-\nu|t-\tau|}\varphi(\tau)\|\omega_\tau\|_{\mathcal{C}}d\tau\right] \quad \text{for} \quad t\geq\xi$$

and

$$\left\|(T\omega)(t)\right\| \leq \sum_{n=0}^{\infty}\|\Psi\|^n\left[Ne^{-\nu(\xi-t)}\|\nu_0\| + \right.$$

$$\left. + N(1+H)\int\limits_{\xi}^{\infty} e^{-\nu|2\xi-t-\tau|}\varphi(\tau)\|\omega_\tau\|_{\mathcal{C}}d\tau\right] \quad \text{for} \quad t\in[\xi-r,\xi].$$

Therefore, for $t\geq\xi$,

$$\left\|(T\omega)(t)\right\|_{\mathcal{C}} \leq \frac{1}{1-\|\Psi\|}\left[Ne^{\nu r}e^{-\nu(t-\xi)}\|\nu_0\| + N(1+H)e^{\nu r}\int\limits_{\xi}^{\infty} e^{-\nu|t-\tau|}\varphi(\tau)\|\omega_\tau\|_{\mathcal{C}}d\tau\right].$$

Thus,

$$e^{\alpha(t-\xi)}\|(T\omega)(t)\|_{\mathcal{C}} \leq \frac{1}{1-\|\Psi\|}\left[Ne^{\nu r}\|\nu_0\| + N(1+H)e^{\nu r}\int\limits_{\xi}^{\infty} e^{-(\nu-\alpha)|t-\tau|}\varphi(\tau)e^{\alpha(\tau-\xi)}\|\omega_\tau\|_{\mathcal{C}}d\tau\right] \leq$$

$$\leq \frac{1}{1-\|\Psi\|}\left[Ne^{\nu r}\|\nu_0\| + \frac{N(1+H)e^{\nu r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-e^{-(\nu-\alpha)}}\|e^{\alpha(t-\xi)}\|\omega_t\|_{\mathcal{C}}\|_\infty\right].$$

So, $e^{\alpha(t-\xi)}\|(T\omega)(t)\|_{\mathcal{C}} \in L_\infty(\mathbb{R})$.

On the other hand, we also have

$$e^{\alpha(t-\xi)}\|(T\omega)(t)\|_{\mathcal{C}} \leq$$

$$\leq \frac{1}{1-\|\Psi\|}\left[Ne^{\nu r}e^{-(\nu-\alpha)(t-\xi)}\|\nu_0\| + N(1+H)e^{\nu r}\int\limits_{\xi}^{\infty} e^{-(\nu-\alpha)|t-\tau|}\varphi(\tau)e^{\alpha(\tau-\xi)}\|\omega_\tau\|_{\mathcal{C}}d\tau\right] \leq$$

$$\leq \frac{1}{1-\|\Psi\|}\left[Ne^{\nu r}\left(T_\xi^+ e_{\nu-\alpha}\right)(t)\|\nu_0\| + N(1+H)e^{\nu r}h_{\nu-\alpha}(t)\|e^{\alpha(\tau-\xi)}\|\omega_\tau\|_{\mathcal{C}}\|_E\right].$$

By the Banach lattice property of E the $e^{\alpha(t-\xi)}\|(T\omega)(t)\|_{\mathcal{C}} \in E \cap L_\infty(\mathbb{R})$. This leads to $T\omega \in C_{\xi,\alpha}$. Next, by the Lipschitz continuity of $\widetilde{y}_\xi$, we obtain

$$\|\nu_0\| = \left\| -P(\xi)u(\xi) + \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0)\right\| \leq$$

$$\leq \left\|\widetilde{y}_\xi\Big(\widetilde{P}(\xi)u(\xi)\Big)(0) - P(\xi)u(\xi)\right\| + \left\|\widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0) - \widetilde{y}_\xi\Big(\widetilde{P}(\xi)u(\xi)\Big)(0)\right\| \leq$$

$$\leq \left\|\widetilde{y}_\xi\Big(\widetilde{P}(\xi)u_\xi\Big) - (I - \widetilde{P}(\xi))u_\xi\right\|_{\mathcal{C}} + k_1\left\|\widetilde{P}(\xi)\omega_\xi\right\|_{\mathcal{C}} \leq$$

$$\leq m(\xi) + k_1 N(1+H)e^{\nu r}\|\omega_\xi\|_{\mathcal{C}}$$

for

$$m(\xi) = \left\|\widetilde{y}_\xi\Big(\widetilde{P}(\xi)u_\xi\Big) - (I - \widetilde{P}(\xi))u_\xi\right\|_{\mathcal{C}} \leq m(\xi) + k_1 N(1+H)e^{\nu r}\|\omega\|_\alpha.$$

So,

$$\|T\omega\|_\alpha \leq \max\{1, N_1\|e_{\nu-\alpha}\|_E\}\frac{Ne^{\nu r}m(\xi)}{1-\|\Psi\|} + \frac{l}{1-\|\Psi\|}\|\omega\|_\alpha. \tag{2.13}$$

We then prove that T is a contraction mapping. Indeed, for ω, v belongs to $C_{\xi,\alpha}$. Then, for $\nu_0 = \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + \omega_\xi)\Big)(0)$, $\mu_0 = \widetilde{y}_\xi\Big(\widetilde{P}(\xi)(u_\xi + v_\xi)\Big)(0)$, we have

$$e^{\alpha(t-\xi)}\|(T\omega)(t) - (Tv)(t)\|_{\mathcal{C}} \leq \frac{1}{1-\|\Psi\|}\Bigg[Ne^{\nu r}e^{-(\nu-\alpha)(t-\xi)}\|\nu_0 - \mu_0\| +$$

$$+ N(1+H)e^{\nu r}\int_\xi^\infty e^{-(\nu-\alpha)|t-\tau|}\varphi(\tau)e^{\alpha(\tau-\xi)}\|\omega_\tau - v_\tau\|_{\mathcal{C}}d\tau\Bigg].$$

On the other hand, $\|\nu_0 - \mu_0\| \leq k_1 N(1+H)e^{\nu r}\|\omega - v\|_\alpha$.

Thus,

$$\|e^{\alpha(t-\xi)}\|T\omega - Tv\|_{\mathcal{C}}\|_\infty \leq$$

$$\leq \frac{1}{1-\|\Psi\|}\left[k_1 N^2(1+H)e^{2\nu r}\|\omega - v\|_\alpha + \frac{N(1+H)e^{2\nu r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-e^{-(\nu-\alpha)}}\|\omega - v\|_\alpha\right]$$

and

$$\|e^{\alpha(t-\xi)}\|T\omega - Tv\|_{\mathcal{C}}\|_E \leq$$

$$\leq \frac{1}{1-\|\Psi\|}\left[k_1 N^2 N_1(1+H)e^{2\nu r}\|e_{\nu-\alpha}\|_E\|\omega - v\|_\alpha + N(1+H)e^{\nu r}\|h_{\nu-\alpha}\|_E\|\omega - v\|_\alpha\right].$$

Therefore,

$$\|T\omega - Tv\|_\alpha \leq \frac{l}{1-\|\Psi\|}\|\omega - v\|_\alpha.$$

Since $\dfrac{l}{1-\|\Psi\|} < 1$, we obtain that T is a contraction on the Banach space $C_{\xi,\alpha}$. Hence, the equation $T\omega = \omega$ has a unique solution $\omega \in C_{\xi,\alpha}$. From (2.13) we get

$$\|\omega\|_\alpha \leq \frac{\max\{1, N_1\|e_{\nu-\alpha}\|_E\} N e^{\nu r} m(\xi)}{1 - \|\Psi\| - l}.$$

We have therefore completed the proof of the existence of the solution $u^* = u + \omega$ of Eq. (2.2) satisfying $\widetilde{u}_t^* \in \mathbf{U}_t$ for $t \geq \xi$ and

$$\|u_t - u_t^*\|_{\mathcal{C}} = \|\omega_t\|_{\mathcal{C}} \leq e^{\nu r} e^{-\alpha(t-\xi)} \|\omega\|_\alpha \leq$$

$$\leq \frac{\max\{1, N_1\|e_{\nu-\alpha}\|_E\} N e^{\nu r} m(\xi) e^{-\alpha(t-\xi)}}{1 - \|\Psi\| - l} =$$

$$= \frac{\max\{1, N_1\|e_{\nu-\alpha}\|_E\} N e^{\nu r} e^{-\alpha(t-\xi)}}{1 - \|\Psi\| - l} \left\| \widetilde{y}_\xi \left(\widetilde{P}(\xi) u_\xi \right) - (I - \widetilde{P}(\xi)) u_\xi \right\|_{\mathcal{C}} =$$

$$= C\|u_\xi - u_\xi^*\|_{\mathcal{C}} \quad \text{for all} \quad t \geq \xi.$$

Theorem 2.3 is proved.

3. Exponential trichotomy and center-invariant unstable manifolds on $\mathbb{R}$. In this section, we will generalize Theorem 2.2 to the case that the evolution family $(U(t,s))_{t\geq s}$ has an exponential trichotomy on $\mathbb{R}$ and the nonlinear forcing term f is φ-Lipschitz. In this case, under similar conditions as in above section we will prove that there exists a center-invariant unstable manifold of $\mathcal{E}$-class for the solutions to Eq. (2.2). We now recall the definition of an exponential trichotomy and a center-invariant unstable manifold of $\mathcal{E}$-class.

Definition 3.1. *A given evolution family $(U(t,s))_{t\geq s}$ is said to have an exponential trichotomy on $\mathbb{R}$ if there are three families of projections $\{P_j(t)\}$, $t \in \mathbb{R}$, $j = 1,2,3$, and positive constants N, α, β with $\alpha < \beta$ such that the following conditions are fulfilled*:

(i) $\sup_{t\in\mathbb{R}} \|P_j(t)\| < \infty$, $j = 1,2,3$;

(ii) $P_1(t) + P_2(t) + P_3(t) = Id$ *for* $t \in \mathbb{R}$ *and* $P_j(t)P_i(t) = 0$ *for all* $j \neq i$;

(iii) $P_j(t)U(t,s) = U(t,s)P_j(s)$ *for* $t \geq s$ *and* $j = 1,2,3$;

(iv) $U(t,s)|_{\operatorname{Im} P_j(s)}$ *are homeomorphisms from* $\operatorname{Im} P_j(s)$ *onto* $\operatorname{Im} P_j(t)$ *for all* $t \geq s$ *and* $j = 2,3$, *respectively*; *we also denote the inverse of* $U(t,s)|_{\operatorname{Im} P_2(s)}$ *by* $U(s,t)_|$, $s \leq t$;

(v) *for all* $t \geq s$ *and* $x \in X$, *the following estimates hold:*

$$\|U(t,s)P_1(s)x\| \leq N e^{-\beta(t-s)} \|P_1(s)x\|,$$

$$\|U(s,t)_| P_2(t)x\| \leq N e^{-\beta(t-s)} \|P_2(t)x\|,$$

$$\|U(t,s)P_3(s)x\| \leq N e^{\alpha(t-s)} \|P_3(s)x\|.$$

The projections $\{P_j(t)\}$, $t \in \mathbb{R}$, $j = 1,2,3$, are called the trichotomy projections, and the constants N, α, β are the trichotomy constants.

Using the trichotomy projections we can now construct three families of projections $\{\widetilde{P}_j(t)\}$, $t \in \mathbb{R}$, $j = 1,2,3$, on $\mathcal{C}$ as follows:

$$(\widetilde{P}_j(t)\phi)(\theta) = U(t+\theta, t)_|(I - P_j(t))\phi(0) \quad \text{for all} \quad \theta \in [-r, 0] \quad \text{and} \quad \phi \in \mathcal{C}. \tag{3.1}$$

Definition 3.2. *Let the evolution family* $(U(t,s))_{t\geq s}$ *have an exponential trichotomy with the trichotomy projections* $\{P_j(t)\}$, $t\in\mathbb{R}$, $j=1,2,3$, *and constants* N, α, β *given as in Definition* 3.1.

A set $\boldsymbol{C}\subset\mathbb{R}\times\mathcal{C}$ *is said to be a center-invariant unstable manifold of* $\mathcal{E}$*-class for the solutions to Eq.* (2.2) *if there exists a family of Lipschitz continuous mappings*

$$f_t\colon\ \operatorname{Im}\widetilde{P}_1(t)\to\operatorname{Im}\Big(\widetilde{P}_2(t)+\widetilde{P}_3(t)\Big)$$

with projections $\big\{\widetilde{P}_j(t)\big\}$, $t\in\mathbb{R}$, $j=1,2,3$, *defined as in Eq.* (3.1), *and Lipschitz constants being independent of* t *such that* $\boldsymbol{C}_t=\operatorname{graph}(f_t)$ *has the following properties*:

(i) $\boldsymbol{C}_t$ *is homeomorphic to* $\operatorname{Im}\widetilde{P}_1(t)$ *for all* $t\in\mathbb{R}$.

(ii) *To each* $t_0\in\mathbb{R}$, $\phi\in\boldsymbol{C}_{t_0}$ *there corresponds one and only one solution* $u(t)$ *to Eq.* (2.2) *on* $(-\infty,t_0]$ *satisfying* $e^{-\gamma(t_0+\theta)}Fu_{t_0+\theta}=\phi(\theta)$ *for* $\theta\in[-r,0]$ *and*

$$z(t)=\begin{cases}e^{-\gamma(t+\cdot)}u_t(\cdot) & \textit{for}\quad t\leq t_0,\\ 0 & \textit{for}\quad t>t_0,\end{cases}\qquad t\in\mathbb{R},\quad \textit{belongs to}\quad \mathcal{E},\quad \textit{where}\quad \gamma=\frac{\beta+\alpha}{2}.$$

Moreover, for any two solutions $u(t)$ *and* $v(t)$ *to Eq.* (2.2) *corresponding to different functions* ϕ, $\psi\in\boldsymbol{C}_{t_0}$, *we have the estimate*

$$\|u_t-v_t\|_{\mathcal{C}}\leq C_\mu e^{(\gamma-\mu)(t_0-t)}\Big\|(\widetilde{P}_1(t_0)\phi)(0)-(\widetilde{P}_1(t_0)\psi)(0)\Big\|\quad \textit{for}\quad t\leq t_0,$$

where μ, C_μ *are positive constants independent of* t_0, $u(\cdot)$, *and* $v(\cdot)$.

(iii) $\boldsymbol{C}$ *is positively* F*-invariant under Eq.* (2.2) *in the sense that, if* $u(t)$, $t\leq t_0$, *is the solution to Eq.* (2.2) *satisfying the conditions that the function* $e^{-\gamma(t_0+\cdot)}\tilde{u}_{t_0}(\cdot)\in\boldsymbol{C}_{t_0}$ *and*

$$z(t)=\begin{cases}e^{-\gamma(t+\cdot)}u_t(\cdot) & \textit{for}\quad t\leq t_0,\\ 0 & \textit{for}\quad t>t_0,\end{cases}\qquad t\in\mathbb{R},\quad \textit{belongs to}\quad \mathcal{E},$$

then the function $e^{-\gamma(t+\cdot)}\tilde{u}_t(\cdot)\in\boldsymbol{C}_t$ *for all* $t\leq t_0$, *where* $\tilde{u}_t(\theta)=Fu_{t_0+\theta}$ *for all* $-r\leq\theta\leq 0$.

We come to our second main result. It proves the existence of a center-unstable manifold of $\mathcal{E}$-class for solutions of Eq. (2.2).

Theorem 3.1. *Let the evolution family* $(U(t,s))_{t\geq s}$ *have an exponential trichotomy with the trichotomy projections* $\{P_j(t)\}$, $t\in\mathbb{R}$, $j=1,2,3$, *and constants* N, α, β *given as in Definition* 3.1. *Assume Standing Hypothesis* 1.1 *and let the functions* φ, h_ν, e_ν, *and the operators* F, f *be determined as in Theorem* 2.1 *and* $e_\nu=e^{-\nu|t|}$ *for all* $t\in\mathbb{R}$. *Set* $q:=\sup\{\|P_j(t)\|: t\in\mathbb{R},\ j=1,3\}$, $N_0:=\max\{N,2Nq\}$, $\nu_0=\dfrac{\beta-\alpha}{2}$ *and*

$$\widetilde{k}:=(1+H)N_0e^{\nu_0 r}\|h_{\nu_0}\|_E.$$

Then, if

$$\max\left\{\frac{N_0(1+H)e^{\nu_0 r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-\|\Psi\|},\ \frac{N_0^2N_1(1+H)e^{\nu_0 r}\|e_{\nu_0}\|_E\|\varphi\|_{E'}}{1-\widetilde{k}-\|\Psi\|}\right\}<1$$

for each fixed $\beta>\alpha$, *there exists a center-invariant unstable manifold of* $\mathcal{E}$*-class for the solutions to Eq.* (2.2).

Proof. Set $P(t) := P_1(t)$ and $Q(t) := P_2(t) + P_3(t) = Id - P(t)$ for $t \in \mathbb{R}$. We have that $P(t)$ and $Q(t)$ are projections complemented to each other on X. Then we define the families of projections $\{\widetilde{P}_j(t)\}$, $t \in \mathbb{R}$, $j = 1,2,3$, on $\mathcal{C}$ as in Eq. (3.1). Setting $\widetilde{P}(t) = \widetilde{P}_1(t)$ and $\widetilde{Q}(t) = \widetilde{P}_2(t) + \widetilde{P}_3(t)$, $t \in \mathbb{R}$, we obtain that $\widetilde{P}(t)$ and $\widetilde{Q}(t)$ are complemented projections on $\mathcal{C}$ for each $t \in \mathbb{R}$. We consider the following rescaling evolution family:

$$\widetilde{U}(t,s) = e^{-\gamma(t-s)}U(t,s) \quad \text{for all} \quad t \geq s, \quad \text{where} \quad \gamma := \frac{\beta+\alpha}{2}.$$

We now prove that the evolution family $\widetilde{U}(t,s)$ has an exponential dichotomy with dichotomy projections $P(t)$, $t \in \mathbb{R}$. Indeed,

$$P(t)\widetilde{U}(t,s) = e^{-\gamma(t-s)}P_1(t)U(t,s) = e^{-\gamma(t-s)}U(t,s)P_1(s) = \widetilde{U}(t,s)P(s).$$

Since $U(t,s)|_{\operatorname{Im} P_j(s)}$ is a homeomorphism from $\operatorname{Im} P_j(s)$ onto $\operatorname{Im} P_j(t)$ for $t \geq s$, $j = 2,3$, and $\operatorname{Im}(P_2(t) + P_3(t)) = \operatorname{Ker} P(t)$ for all $t \in \mathbb{R}$, we have that $\widetilde{U}(t,s)|_{\operatorname{Ker} P(s)}$ is also a homeomorphism from $\operatorname{Ker}P(s)$ onto $\operatorname{Ker}P(t)$, and we denote $\widetilde{U}(s,t)_| := (\widetilde{U}(t,s)|_{\operatorname{Ker} P(s)})^{-1}$ for $s \leq t$. By the definition of exponential trichotomy we obtain

$$\left\|\widetilde{U}(t,s)P(s)x\right\| \leq e^{-(\beta+\gamma)(t-s)}\|P(s)x\| \quad \text{for all} \quad t \geq s.$$

On the other hand,

$$\left\|\widetilde{U}(s,t)_|Q(t)x\right\| = e^{-\gamma(t-s)}\left\|U(s,t)_|(P_2(t)+P_3(t))x\right\| \leq$$

$$\leq Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_2(t)x\| + e^{\alpha(t-s)}\|P_3(t)x\|) =$$

$$= Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_2(t)Q(t)x\| + e^{\alpha(t-s)}\|P_3(t)Q(t)x\|)$$

for all $t \geq s$ and $x \in X$.

Putting $q := \sup\{\|P_j(t)\|,\ t \in \mathbb{R},\ j = 2,3\}$, we finally get the following estimate:

$$\left\|\widetilde{U}(s,t)_|Q(t)x\right\| \leq 2Nqe^{-\frac{\beta-\alpha}{2}(t-s)}\|Q(t)x\|.$$

Therefore, $\widetilde{U}(t,s)$ has an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N_0 := \max\{N, 2Nq\}$, $\nu_0 := \dfrac{\beta-\alpha}{2}$.

Put $\widehat{u}(t) = e^{-\gamma t}u(t)$, and define the mapping $\widetilde{f}\colon \mathbb{R} \times \mathcal{C} \to X$ as follows:

$$\widetilde{f}(t,\phi) = e^{-\gamma t}f(t, e^{\gamma(t+\cdot)}\phi(\cdot)) \quad \text{for} \quad (t,\phi) \in \mathbb{R} \times \mathcal{C}.$$

Obviously, $\widetilde{f}$ is also φ-Lipschitz. Thus, we can rewrite Eq. (2.2) in the new form

$$F\widehat{u}_t = \widetilde{U}(t,s)F\widehat{u}_s + \int_s^t \widetilde{U}(t,\xi)\widetilde{f}(\xi,\widehat{u}_\xi)d\xi \quad \text{for all} \quad t \geq s, \qquad (3.2)$$

$$\widehat{u}_s(\cdot) = e^{-\gamma(s+\cdot)}\phi(\cdot) \in \mathcal{C}.$$

Hence, by Theorem 2.2, we obtain that if

$$\max\left\{\frac{N_0(1+H)e^{\nu_0 r}(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-\|\Psi\|},\ \frac{N_0^2N_1(1+H)e^{\nu_0 r}\|e_{\nu_0}\|_E\|\varphi\|_{E'}}{1-\widetilde{k}-\|\Psi\|}\right\}<1,$$

then there exists an invariant unstable manifold of $\mathcal{E}$-class $\mathbf{U}$ for the solutions to Eq. (3.2). Returning to Eq. (2.2) by using the relation $u(t):=e^{\gamma t}\widehat{u}(t)$ and Theorems 2.1, 2.2, we can easily verify the properties of $\mathbf{C}$ which are stated in (i), (ii) and (iii) in Definition 3.2. Thus, $\mathbf{C}$ is a center-invariant unstable manifold of $\mathcal{E}$-class for the solutions of Eq. (2.2).

4. Examples.

***Example* 4.1.** Consider the finite delayed heat equation for a material with memory which has formula

$$\frac{\partial}{\partial t}u(t,x)=m(t)\frac{\partial^2}{\partial x^2}\left[u(t,x)+\int_{t-1}^{t}(t-s)(t-s-1)u(s,x)ds\right]+$$
$$+\int_{t-1}^{t}[-2(t-s)+1]u(s,x)ds+a(t)\int_{t-1}^{t}\ln(1+|u(s,x)|)ds,$$
$$u(t,0)=u(t,\pi)=0,\quad t\ge s,\tag{4.1}$$
$$u_s(\theta,x)=u(s+\theta,x)=\psi(\theta,x),\quad x\in[0,\pi],\quad \theta\in[-1,0],$$

where $a(t)$ is defined by $a(t)=|l|e^{-\eta|t|}$, $\eta>1$ and $l\neq 0$, the given function ψ is continuous. The function $m(\cdot)\in L_{1,\mathrm{loc}}(\mathbb{R})$ and satisfies the condition $m_1\ge m(t)\ge m_0>0$ for fixed constants m_0, m_1 and a.e. $t\in\mathbb{R}$.

We choose the Hilbert space $X=L_2[0,\pi]$, and let $A\colon X\to X$ be defined by

$$A(v)=v''$$

with the domain $D(A)=\{v\in W^{2,2}[0,\pi]: v(0)=v(\pi)=0\}$.

Also, for $\mathcal{C}=C([-1,0],X)$, we define the difference and delay operators F and f as

$$F\colon\mathcal{C}\to X,\quad F(v)=v(0)+\int_{-1}^{0}b(-\theta)v(\theta)d\theta,$$

$$f\colon\mathbb{R}\times\mathcal{C}\to X,\quad f(t,\phi)=|l|e^{-\eta|t|}\int_{-1}^{0}\ln(1+|(\phi(\theta))(x)|)d\theta,\quad t\in\mathbb{R},\quad \theta\in[-1,0].\tag{4.2}$$

It is obvious that

$b(t)=t(t-1)$ satisfies $b(0)=b(1)=0$,

$F=\delta_0+\Psi$, here $\Psi(\cdot)=-\int_{-1}^{0}b(-\theta)\cdot(\theta)d\theta$ with $\|\Psi\|=\int_{-1}^{0}|\theta(\theta-1)|d\theta=\dfrac{5}{6}<1.$

Note that the fact that f takes value in $X = L_2[0,\pi]$ can be easily seen by using the Minkowskii inequality.

Putting now $A(t) = m(t)A$, $u(t) = u(t,\cdot)$, $t \in \mathbb{R}$ and $\phi(\theta) = \psi(\theta,\cdot)$; $\theta \in [-1,0]$ the Eq. (4.1) can now be rewritten as

$$\frac{\partial}{\partial t} F u_t = A(t) F u_t + f(t, u_t), \quad t \geq s, \quad t, s \in \mathbb{R},$$

$$u_s = \phi \in \mathcal{C} := C([-1,0], X).$$

From the definition of A, it can easily seen that A is the generator of an analytic semigroup $(T(t))_{t\geq 0} = \left(e^{tA}\right)_{t\geq 0}$ with $\sigma(A) = \left\{-1, -4, \ldots, -n^2, -(n+1)^2, \ldots\right\}$ and $\sigma(A) \cap i\mathbb{R} = \varnothing$. Applying the spectral mapping theorem for analytic semigroups, we get

$$\sigma(T(t)) = e^{t\sigma(A)} = \left\{e^{-t}, e^{-4t}, \ldots, e^{-n^2 t}, \ldots\right\}$$

and $\sigma(T(t)) \cap \{z \in \mathbb{C} \colon |z| = 1\} = \varnothing$ for all $t > 0$. Therefore, for fixed $t_0 > 0$, the spectrum of operator $T(t_0)$ splits into two disjoint sets σ_0, σ_1, where $\sigma_0 \subset \{z \in \mathbb{C} \colon |z| < 1\}$, $\sigma_1 \subset$ $\subset \{z \in \mathbb{C} \colon |z| > 1\}$.

Next, we choose $P = P(t_0)$ to be the Riesz projections corresponding to spectral set σ_0, and $Q = Id - P$. Clearly, P and Q commute with $T(t)$ for all $t \geq 0$. We denote by $T_Q(t) = T(t)Q$ the restriction of $T(t)$ on $\operatorname{Im} Q$. As known Semigroup Theory, in this case, the semigroup $(T(t))_{t\geq 0}$ is called hyperbolic (or having an exponential dichotomy) and restriction $T_Q(t)$ is invertible. Moreover, there are positive constants N, γ such that

$$\|T(t)_{|PX}\| \leq N e^{-\gamma t}, \tag{4.3}$$

$$\|T_Q(-t)\| \leq \|T_Q(t)^{-1}\| \leq N e^{-\gamma t} \tag{4.4}$$

for all $t \geq 0$.

Clearly, the family $(A(t))_{t\in\mathbb{R}} = (m(t)A)_{t\in\mathbb{R}}$ generates the evolution family $(U(t,s))_{t\geq s}$ defined by the formula

$$U(t,s) = T\left(\int_s^t m(\tau)d\tau\right) \quad \text{for all} \quad t \geq s.$$

From the dichotomy estimates of $(T(t))_{t\geq 0}$ in (4.3), it is straightforward to check that evolution family $(U(t,s))_{t\geq s}$ has an exponential dichotomy with projection P and constants N, $\nu = \gamma\, m_0$ by the following estimates:

$$\|U(t,s)_{|PX}\| = \|T(t-s)_{|PX}\| \leq N e^{-\nu(t-s)},$$

$$\|U(s,t)_{|}\| = \|\left(U(t,s)_{|\operatorname{Ker} P}\right)^{-1}\| = \|T_Q(-(t-s))_{|}\| \leq N e^{-\nu(t-s)}$$

for all $t \geq s$.

We now take $E = L_p(\mathbb{R})$, $1 \leq p \leq +\infty$, the delay operator $f : \mathbb{R} \times \mathcal{C} \to X$ defined as in (4.2) and check that f is φ-Lipschitz with $\varphi(t) = |l| e^{-\eta|t|} \in E' = Lq(\mathbb{R})$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Indeed, the condition (i) in Definition 2.2 is evident. To verify the condition (ii) in that definition we use Minkowskii inequality and the fact that $\ln(1+h) \le h$ for all $h \ge 0$. Then

$$\left\| f(t,\phi_1)(x) - f(t,\phi_2)(x) \right\|_2 = |l|\, e^{-\eta|t|} \left(\int_0^{\pi} \left(\int_{-1}^{0} \ln \frac{1+|(\phi_1(\theta))(x)|}{1+|(\phi_2(\theta))(x)|} d\theta \right)^2 dx \right)^{\frac{1}{2}} \le$$

$$\le |l|\, e^{-\eta|t|} \int_{-1}^{0} \left(\int_0^{\pi} \ln^2 \frac{1+|(\phi_1(\theta))(x)|}{1+|(\phi_2(\theta))(x)|} dx \right)^{\frac{1}{2}} d\theta =$$

$$= |l|\, e^{-\eta|t|} \int_{-1}^{0} \left(\int_0^{\pi} \ln^2 \left(1 + \frac{|(\phi_1(\theta))(x)| - |(\phi_2(\theta))(x)|}{1+|(\phi_2(\theta))(x)|} \right) dx \right)^{\frac{1}{2}} d\theta \le$$

$$\le |l|\, e^{-\eta|t|} \int_{-1}^{0} \left(\int_0^{\pi} |(\phi_1(\theta))(x) - (\phi_2(\theta))(x)|^2 dx \right)^{\frac{1}{2}} d\theta =$$

$$= |l|\, e^{-\eta|t|} \int_{-1}^{0} \|\phi_1(\theta) - \phi_2(\theta)\|_2 \, d\theta \le$$

$$\le |l|\, e^{-\eta|t|} \sup_{\theta\in[-1,0]} \|\phi_1(\theta) - \phi_2(\theta)\|_2.$$

Hence, f is φ-Lipschitz. In the space $L_p(\mathbb{R})$, the constants $N_1,\ N_2$ are defined by $N_1 = N_2 = 1$. We have

$$\|\varphi\|_{E'} = |l| \left(\int_{-\infty}^{+\infty} e^{-\eta q|t|} dt \right)^{\frac{1}{q}} = |l| \left(\frac{2}{\eta q} \right)^{\frac{1}{q}}.$$

Also, the function $h_\nu(\cdot)$ can be computed by

$$h_\nu(t) = \left\| e^{-\nu|t-\cdot|} \varphi(\cdot) \right\|_{L_q} = |l| \left(\frac{e^{-\nu q|t|} - e^{-\eta q|t|}}{(\eta-\nu)q} + \frac{e^{-\eta q|t|} + e^{-\nu q|t|}}{(\eta+\nu)q} \right)^{\frac{1}{q}} \quad \text{for} \quad t \in \mathbb{R}.$$

Therefore, $h_\nu \in L_p$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and

$$\|h_\nu\|_{L_p} \le |l| \left(\frac{2\eta}{q(\nu+\eta)(\eta-\nu)} \right)^{\frac{1}{q}} \left(\frac{4}{\nu p} \right)^{\frac{1}{p}}.$$

We have $\|e_\nu\|_{L_p} = \left(\dfrac{2}{\nu p} \right)^{\frac{1}{p}}$ and $\Lambda_1 \varphi(t) = \displaystyle\int_t^{t+1} \varphi(\tau) d\tau$. Thus, $\|\Lambda_1 \varphi\|_\infty \le \dfrac{|l|(e^\eta - 1)}{\eta}$.

By Theorem 2.2, we obtain that if

$$12N(1+H)e^{\nu r}\times$$

$$\times\max\left\{\frac{e^{\eta}-1}{\eta},\frac{N}{(\nu p)^{\frac{1}{p}}(\eta q)^{\frac{1}{q}}\left(1-6N(1+H)|l|\left(\frac{2\eta}{q(\nu+\eta)(\nu-\eta)}\right)^{\frac{1}{q}}\left(\frac{4}{\nu p}\right)^{\frac{1}{p}}\right)}\right\}<1,$$

then there exists an unstable manifold of $\mathcal{E}$-class $\mathbf{U}$ for the mild solutions to problem (4.1), and this manifold has the attraction property given in Theorem 2.3.

***Example* 4.2.** Consider the above Example 4.1, in Eq. (4.1) we replace the boundary condition by

$$u'_x(t,0)=u'_x(t,\pi)=0,\quad t\geq s.$$

Then we choose the Hilbert space $X=L_2[0,\pi]$, and let $A\colon X\to X$ be defined by

$$A(v)=v''$$

with the domain $D(A)=\left\{v\in W^{2,2}[0,\pi]\colon v'(0)=v'(\pi)=0\right\}$.

Putting now $A(t)=m(t)A$, $u(t)=u(t,\cdot)$, $t\in\mathbb{R}$, and $\phi(\theta)=\psi(\theta,\cdot)$, $\theta\in[-1,0]$, the Eq. (4.1) can now be rewritten as

$$\begin{aligned}&\frac{\partial}{\partial t}Fu_t=A(t)Fu_t+f(t,u_t),\quad t\geq s,\quad t,s\in\mathbb{R},\\&u_s=\phi\in\mathcal{C}:=C([-1,0],X).\end{aligned}\tag{4.5}$$

From the definition of A, it can easily seen that A is the generator of an analytic semigroup $(T(t))_{t\geq0}=\left(e^{tA}\right)_{t\geq0}$ with $\sigma(A)=\left\{0,-1,-4,\ldots,-n^2,-(n+1)^2,\ldots\right\}$. Applying the spectral mapping theorem for analytic semigroups we get

$$\sigma(T(t))=e^{t\sigma(A)}=\left\{e^{-t},e^{-4t},\ldots,e^{-n^2t},\ldots\right\}\cup\{1\}.$$

Therefore, for fixed $t_0>0$, the spectrum of operator $T(t_0)$ splits into three disjoint sets σ_1, σ_2, σ_3, where $\sigma_1\subset\{z\in\mathbb{C}\colon|z|<1\}$, $\sigma_2\subset\{z\in\mathbb{C}\colon|z|>1\}$, $\sigma_3\subset\{z\in\mathbb{C}\colon|z|=1\}$.

Next, we choose $P_1=P_1(t_0)$, $P_2=P_2(t_0)$, $P_3=P_3(t_0)$ to be the Riesz projections corresponding to spectral set $\sigma_1,\sigma_2,\sigma_3$. Clearly, P_1,P_2,P_3 commute with $T(t)$ for all $t\geq0$. We can see that $P_1+P_2+P_3=Id$ and $P_iP_j=0$ $\forall i\neq j$. Moreover, there exist are positive constants M, δ such that

$$\|T(t)_{|P_1X}\|\leq Me^{-\delta t}\quad\forall t\geq0.$$

We denote $Q:=P_2+P_3=Id-P_1$ and consider the semigroup on $\operatorname{Im}Q$ such that $T_Q(t)=T(t)Q$ the restriction of $T(t)$ on $\operatorname{Im}Q$. Because $\sigma_2\cup\sigma_3=\sigma(T_Q(t_0))$ implies $(T_Q(t))_{t\geq0}$ can be extended into group $(T_Q(t))_{t\in\mathbb{R}}$ in $\operatorname{Im}Q$. Moreover, there exist positive constants K, ϵ_0, and ϵ_1, $\epsilon_0<\epsilon_1$, such that

$$\|T_Q(-t)_{|P_2X}\|=\|\left(T_Q(t)_{|P_2X}\right)^{-1}\|\leq Ke^{-\epsilon_1t}\quad\forall t\geq0,$$

$$\|T_Q(t)_{|P_3X}\|\leq Ke^{\epsilon_0|t|}\quad\forall t\in\mathbb{R}.$$

Thus, the semigroup $(T(t))_{t\geq 0}$ having an exponential trichotomy with the trichotomy projections $\{P_j\},\ j=1,2,3,$ and constants $N,\ \epsilon_0,\ \beta_0$ satisfies

$$\begin{gathered}\|T(t)_{|P_1X}\| \leq Ne^{-\beta_0 t},\\ \|T(-t)_{|P_2X}\| = \|\big(T(t)_{|P_2X}\big)^{-1}\| \leq Ne^{-\beta_0 t},\\ \|T(t)_{|P_3X}\| \leq e^{\epsilon_0 t},\end{gathered} \tag{4.6}$$

where $N:=\max\{K,M\},\ \beta_0:=\min\{\delta,\epsilon_1\}$.

Clearly, the family $(A(t))_{t\in\mathbb{R}}=(m(t)A)_{t\in\mathbb{R}}$ generates the evolution family $(U(t,s))_{t\geq s}$ defined by the formula

$$U(t,s)=T\left(\int\limits_s^t m(\tau)d\tau\right) \quad \text{for all} \quad t\geq s.$$

From the trichotomy estimates of $(T(t))_{t\geq 0}$ in (4.2), it is straightforward to check that evolution family $(U(t,s))_{t\geq s}$ has an exponential trichotomy with projections $P_k,\ k=1,2,3,$ and trichotomy constants $N,\beta:=\epsilon_1 m_0,\ \alpha:=\epsilon_0 m_0$ by the following estimates:

$$\begin{gathered}\|U(t,s)_{|P_1X}\| = \|T(t-s)_{|P_1X}\| \leq Ne^{-\beta(t-s)},\\ \|U(s,t)_{|}\| = \|\big(U(t,s)_{|P_2X}\big)^{-1}\| \leq Ne^{-\beta(t-s)},\\ \|U(t,s)_{|P_3X}\| = \|T(t-s)_{|P_3X}\| \leq Ne^{\alpha(t-s)}\end{gathered}$$

for all $t\geq s$.

Set $q:=\sup\{\|P_j(t)\|: t\in\mathbb{R},\ j=1,3\},\ N_0:=\max\{N,2Nq\},\ \nu_0=\dfrac{\beta-\alpha}{2}$. By Theorem 3.1 and result in the Example 4.1, we obtain that if

$$12N_0(1+H)e^{\nu_0 r}\times$$

$$\times\max\left\{\frac{e^\eta-1}{\eta},\ \frac{N_0}{(\nu_0 p)^{\frac{1}{p}}(\eta q)^{\frac{1}{q}}\left(1-6N_0(1+H)|l|\left(\dfrac{2\eta}{q(\nu_0+\eta)(\nu_0-\eta)}\right)^{\frac{1}{q}}\left(\dfrac{4}{\nu_0 p}\right)^{\frac{1}{p}}\right)}\right\}<1,$$

then there exists a center-invariant unstable manifold of $\mathcal{E}$-class **C** for the mild solutions to problem (4.5).

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